Tracking Control of a Twin-rotor Helicopter Model under Thrust Constrains Using State-Dependent Gain-Scheduling and Reference Management

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***Abstract*— In this paper, we design a control law for a twin-rotor helicopter model with considering actuator constraints. The controller is composed of a state-dependent gain-scheduled feedback control law and a reference management device. We show that the proposed control law achieves higher tracking performance as compared to the standard constant feedback control law through experimental results.**

I. Introduction

In most of practical control systems such as flight control systems, there exists saturation limitation on controller outputs[3], [9]. If a feedback controller designed without considering such limitation is utilized, the closed-loop system may be unstable in the case where large external signal is added. One way to deal with such a problem is to design a low-gain controller which does not violate input constraints for all external signals that will be injected. However, it is clear that this method results in conservative control performance.

Recently, for this problem, several control methods that exploit on-line optimization have been proposed [2], [11], [12]. The state-dependent gain-scheduled control scheme [11], [12] is one of such methods. In this scheme, a control law which has a structure that a high-gain control law and a low-gain control law are interpolated by a scheduling parameter is utilized. The scheduling parameter is determined by solving a convex optimization problem on-line. The control law of [11], [12] is designed based on the polytopic representation of a saturation function of [7]. As a result, the control law can achieve large region of attraction even if the plant is unstable. This method is extended to tracking control problems [13]. However, effectiveness of these methods are evaluated only through numerical examples of linear systems whose dimensions are small, and have not been confirmed by experiments. In actual systems, there exist disturbances, nonlinearities, unmodeled dynamics, and computational delay. These factors may have seriously harmful effects on control performance. Therefore, to evaluate the effectiveness of the methods of [11], [12], [13] by experiments is quite important to put the methods to practical use.

In this paper, based on the method of [13], we design a tracking control law for the twin-rotor helicopter model(see Fig.1)[8], and evaluate the effectiveness of the method through experiments. The twin-rotor helicopter model has dynamics which resembles the dynamics of a VTOL aircraft and can be used to provide a basis for constructing control system design methodologies for more general class of aircrafts. Since the dynamics of this system has nonlinear characteristic and the system is multi-input, it is generally difficult to control. Also, since the dynamics is unstable, it is required to implement with a relatively small sampling period. Hence, it is appropriate for evaluating the effectiveness of the proposed method which requires on-line optimization. In this paper, for the twin-rotor helicopter model, under thrust constraints, we design a control law that stabilizes both altitude and attitude of the system, and makes the difference between the velocity in the horizontal direction and the reference velocity as small as possible. Further, we evaluate the effectiveness of the method through experiments.

Notations: For a diagonal matrix $A = \mathrm{diag}[a_1, \cdots, a_m] > 0$ and a vector $u \in \mathcal{R}^m$, we define the multivariable saturation function as $\Phi_A(u) := (\phi_{a_1}(u_1), \cdots, \phi_{a_m}(u_m))^T$, where $\phi_{a_i}(u_i) := \mathrm{sgn}(u_i)\min\{|u_i|, \ a_i\}$. If $A = I$, the subscript will be omitted. For a vector $v \in \mathcal{R}^n$, we denote its Euclidean norm as $\|v\|_2 := (v^T v)^{1/2}$. For a positive definite matrix $P \in \mathcal{R}^{n\times n}$, we denote $\mathcal{E}(P,\eta) := \left\{x \in \mathcal{R}^n : x^T P x \le \eta\right\}$. For $F \in \mathcal{R}^{m\times n}$, we denote the ith row of F as $F^{(i)}$. Furthermore, we define $\mathcal{L}(F,\rho) := \left\{x \in \mathcal{R}^n : |F^{(l)}x| \le \rho_l, l = 1, \cdots, m\right\}$, where $\rho = \mathrm{diag}[\rho_1, \cdots, \rho_m]$.

II. Preliminary

In this section, we introduce a polytopic model of a saturation function of [7]. Let $\mathcal{V}$ be the set of $m\times m$ diagonal matrices whose elements are either 1 or 0. There are 2^m elements in $\mathcal{V}$. Suppose that each element of $\mathcal{V}$ is labeled as $\mathbf{E}_j, j = 1, 2, \cdots, 2^m$, and denote $\mathbf{E}_j^- := I - \mathbf{E}_j$. Clearly, $\mathbf{E}_j^-$ is also an element of $\mathcal{V}$.

Lemma 1: ([7]) Let $u, v \in \mathcal{R}^m$. Suppose that $|v_j| \le 1, \forall j \in [1, m]$, then $\Phi(u)$ can be represented as $\Phi(u) = \sum_{j=1}^{2^m} \lambda_j(\mathbf{E}_j u + \mathbf{E}_j^- v)$, where $0 \le \lambda_j \le 1$, $\sum_{j=1}^{2^m} \lambda_j = 1$.

III. Tracking Control Law for a Input Constrained System

In this section, we introduce the tracking control algorithms for input constrained systems of [13].

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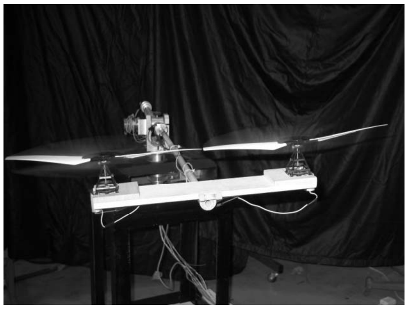

Fig. 1. Twin-rotor helicopter model

A. Problem Formulation

Let us consider the system described by

$$x(t+1) = Ax(t) + B\Phi(u(t)) + Ew(t) \quad (1)$$
$$z(t) = Cx(t) + D\Phi(u(t)) + D_w w(t) \quad (2)$$

where $x \in \mathcal{R}^n, u \in \mathcal{R}^m, w \in \mathcal{R}^p, z \in \mathcal{R}^q$. $w(t)$ represents a reference signal. $z(t)$ represents a tracking error.

In Section III-B, we first consider the following problem.

Problem 1: Consider the system (1) and (2). Suppose that $w(t)$ is generated by

$$r(t+1) = Sr(t) \quad (3)$$
$$w(t) = r(t) \quad (4)$$

Further, we assume that the system (3) is neutrally stable and $\|r(t)\|_2 \le r_{\max}, \forall t \ge 0$. Design a feedback control law

$$u(t) = F(t)x(t) + M(t)w(t) \quad (5)$$

that achieves fast convergence of the signal $z(t)$ and large region of attraction.

In Section III-B, we show a state-dependent gain-scheduled feedback control law that achieves the control objectives of Problem 1. Then, in Section III-C, we extend the results of Section III-B to the case where $r(t)$ is an arbitrary time-varying signal.

B. Tracking Control for a Reference Signal Generated by an Exo-system

We initially introduce the following theorem.

Theorem 1: Consider the system (1)–(4). We suppose that there exist matrices $\Pi \in \mathcal{R}^{n\times p}, \Gamma \in \mathcal{R}^{m\times p}$ that satisfy

$$\Pi S = A\Pi + B\Gamma + E \quad (6)$$
$$0 = C\Pi + D\Gamma + D_w \quad (7)$$

Further, we suppose that $\max_{t\ge 0} |\Gamma^{(l)} r(t)| < 1, \forall l \in [1, m]$. For given positive scalars η, γ_0, γ_1 such that $\gamma_0 < \gamma_1$ and a matrix $\mathbf{R} > 0$, assume that there exist matrices $Q_i, Y_i, Z_i, (i = 0, 1)$ that satisfy

$$\begin{bmatrix} Q_i & * & * \\ \begin{bmatrix} CQ_i \\ \mathbf{R}^{\frac{1}{2}} Y_i \end{bmatrix} + \mathbf{D}(\mathbf{E}_j Y_i + \mathbf{E}_j^- Z_i) & \gamma_i I & * \\ AQ_i + B(\mathbf{E}_j Y_i + \mathbf{E}_j^- Z_i) & 0 & Q_i \end{bmatrix} > 0$$
$$\forall i \in [0,1],\ \forall j \in [1, 2^m] \quad (8)$$
$$\begin{bmatrix} Q_i & * \\ Z_i^{(l)} & \frac{\rho_l^2}{\eta} \end{bmatrix} \ge 0, \qquad \forall i \in [0,1],\ \forall l \in [1, m] \quad (9)$$
$$Q_0 < Q_1 \quad (10)$$

where $\rho_l := 1 - \max_{t\ge 0} |\Gamma^{(l)} r(t)|$, $\mathbf{D} := [D^T, 0]^T$ and the symbol $*$ stands for symmetric block in matrix inequalities. Further, for some constant $\alpha \in [0, 1]$, we suppose that $\xi(0) \in \mathcal{E}(P(\alpha), \eta)$ where $P(\alpha) := Q(\alpha)^{-1}, Q(\alpha) := (1-\alpha)Q_0 + \alpha Q_1$, $\xi := x - \Pi w$. Then, by applying the feedback control law

$$u(t) = F(\alpha)x(t) + M(\alpha)w(t) \quad (11)$$

where $F(\alpha) = Y(\alpha)Q(\alpha)^{-1}$, $Y(\alpha) := (1-\alpha)Y_0 + \alpha Y_1$ and $M(\alpha) = \Gamma - F(\alpha)\Pi$ to the system (1)–(4), the relations $\xi(t) \in \mathcal{E}(P(\alpha), \eta), \forall t \ge 0$, $\lim_{t\to\infty} z(t) = 0$ and $J := \sum_{t=0}^{\infty} \|\mathbf{z}(t)\|_2^2 < \gamma(\alpha)\eta$, where $\mathbf{z} := [z^T, u_e^T \mathbf{R}^{1/2}]^T$, $u_e := u - \Gamma w$, $\gamma(\alpha) := (1-\alpha)\gamma_0 + \alpha\gamma_1$ hold.

Proof: See Appendix I. ■

In this paper, based on Theorem 1, we design a gain $F(1) = Y_1 Q_1^{-1}$ which makes the region $\mathcal{E}(P(1), \eta)$ large and a gain $F(0) = Y_0 Q_0^{-1}$ which achieves fast convergence of the state variable in $\mathcal{E}(P(0), \eta)$ by suitably choosing the parameters γ_0, γ_1 and $\mathbf{R}$. Then we construct a control law (11) by interpolating the obtained gains.

Remark 1: Equations (6) and (7) are the conditions for the output regulation problem is solvable in the case of linear systems (see e.g., [10]).

Remark 2: We can impose a penalty on all state variables by replacing the first term of the (2,1) element of eq.(8) with $[Q_i C^T, Y_i^T \mathbf{R}^{1/2}, Q_i \mathbf{S}^{1/2}]^T$ and $\mathbf{D}$ with $\mathbf{D} = [D, 0, 0]^T$, where $\mathbf{S} = \mathbf{S}^T > 0$. In this case, the cost function becomes $J = \sum_{t=0}^{\infty} \{\|\mathbf{z}(t)\|_2^2 + \xi(t)^T \mathbf{S} \xi(t)\}$.

In this section, we show a gain-scheduling algorithm of the control law (11) which achieves fast convergence of $z(t)$.

Algorithm 1:

Step 0: Set $t = 0$.

Step 1: Measure $x(t)$ and $w(t)$.

Step 2: Solve $\min_{\alpha \in [0,1]} \alpha$, s.t. $\xi(t)^T Q(\alpha)^{-1} \xi(t) \le \eta$. Then, set $\alpha(t) = \alpha$.

Step 3: Apply $u(t) = F(\alpha(t))x(t) + M(\alpha(t))w(t)$ to the plant (1), (2).

Step 4: $t \leftarrow t + 1$ and go to Step 1.

The optimization problem of Step 2 in Algorithm 1 is an LMI optimization problem (see e.g., [6]). Hence, the problem can be solved by the interior point method. Alternatively, the problem can be solved as a simpler eigenvalue problem as follows. By the Schur complement, $\xi(t)^T Q(\alpha)^{-1} \xi(t) \le \eta$ is equivalent to $Q(\alpha) - \frac{1}{\eta}\xi(t)\xi(t)^T \ge 0$. Further, This condition can be rewritten as $\alpha I \ge \mathbf{Q}(\xi(t))$ where $\mathbf{Q}(\xi(t)) := (Q_1 -$

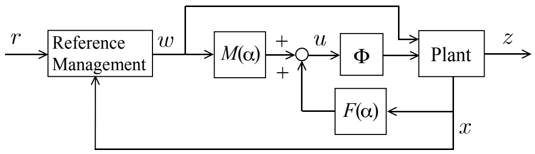

Fig. 2. Feedback System with a Reference Management Device

$Q_0)^{-1/2}\left[\frac{1}{\eta}\xi(t)\xi(t)^T - Q_0\right](Q_1 - Q_0)^{-1/2}$. Hence, with considering $\alpha \geq 0$, the solution of the optimization problem of Step 2 can be obtained as $\alpha = \max[0,\ \lambda_{\max}(\mathbf{Q}(\xi(t)))]$.

The following theorem can be stated.

Theorem 2: Consider the system (1), (2). Assume that there exist matrices Π and Γ that satisfy (6) and (7). Further, assume that $\max_{t\geq 0}|\Gamma^{(l)}r(t)| < 1, \forall l \in [1, m]$. Moreover, for given positive scalars η, γ_0 and γ_1, assume that there exist matrices Q_i, Y_i, Z_i that satisfy (8)–(10). Moreover, assume that $\xi(0) \in \mathcal{E}(P(1), \eta)$. Then by applying Algorithm 1 to the system (1), (2), $z(t)$ converges to zero as $t \to \infty$.

Proof: See Appendix II. ■

C. Control Law for Arbitrary Time Varying Reference Signals

In this section, we extend the method of the previous section to the case where $r(t)$ is an arbitrary time-varying signal. For example, in human-machine systems [1], [3], [4], [9], the reference signal needs to be considered as an arbitrary time-varying signal. In this case, if Algorithm 1 is carried out with $w(t) = r(t)$, both feasibility of the algorithm and closed-loop stability may not be guaranteed. Hence, in this section, we extend the previous control algorithm so that any time-varying reference signal can be applied. In this case, it is difficult to guarantee strict asymptotic convergence of the tracking error. Hence, in this section, we show a control algorithm that makes the tracking error as small as possible at each time and guarantees asymptotic convergence in the case where the reference signal becomes constant after a finite time. In order to guarantee that the error signal converges to zero when the reference signal is constant, we make the following assumption.

Assumption 1: For $S = I$, there exists a matrix Π that satisfies (6),(7) and $\Gamma = 0$.

This assumption is satisfied if the plant has an integrator. Further, we make the following assumption.

Assumption 2: For a matrix $\mathbf{R} > 0$ and positive scalars η, γ_0, γ_1, where $\gamma_0 < \gamma_1$, there exist Q_i, Y_i, Z_i that satisfy (8)–(10).

We assume that the control law (11) has been designed by using the matrices that satisfy the above assumptions.

In this section, we assume that $r(t) \in \mathcal{R}^p$ is an arbitrary time-varying signal. If we simply set $w(t) = r(t)$ and apply Algorithm 1 to the system, feasibility of the algorithm and stability of the closed-loop system may not be guaranteed. To avoid such a situation, we introduce a reference management mechanism that computes a modified reference signal $w(t)$ from the signal $r(t)$ (see Fig.2). In the following, we show a control algorithm that includes the reference management and the state-dependent gain-scheduling.

Algorithm 2:

Step 0: Set $t = 0$ and $\alpha(-1) = 1$.

Step 1: Measure $x(t)$ and $r(t)$.

Step 2: If $x(t) - \Pi r(t) \in \mathcal{E}(P(\alpha(t-1)), \eta)$, then set $w(t) = r(t)$ and go to Step 4. Otherwise, go to Step 3.

Step 3: Solve $\min_{\tilde{w}\in\mathcal{R}^p}\|r(t) - \tilde{w}\|_2^2$, s.t. $[x(t) - \Pi\tilde{w}]^T Q(\alpha(t-1))^{-1}[x(t) - \Pi\tilde{w}] \leq \eta$. Then, set $\alpha(t) = \alpha(t-1),\ w(t) = \tilde{w}$ and go to Step 5.

Step 4: Solve $\min_{\alpha\in[0,1]}\alpha$, s.t. $\xi(t)^T Q(\alpha)^{-1}\xi(t) \leq \eta$. Then, set $\alpha(t) = \alpha$.

Step 5: Apply $u(t) = F(\alpha(t))x(t) + M(\alpha(t))w(t)$ to the plant (1), (2).

Step 6: $t \leftarrow t + 1$ and go to Step 1.

In the above algorithm, Step2 and Step 3 represent the reference management mechanism that compute the modified reference signal $w(t)$ from the original reference signal $r(t)$.

Remark 3: The optimization problem of Step 3 in Algorithm 2 is a quadratic optimization problem with respect to $\tilde{w}$. Hence, this problem can be easily solved.

Theorem 3: Consider the system (1), (2). Assume that there exists $\tilde{w} \in \mathcal{R}^p$ that satisfies $x(0) - \Pi\tilde{w} \in \mathcal{E}(P(1), \eta)$. Then by applying Algorithm 2 to the system (1), (2), feasibility of the algorithm is guaranteed for all times. Moreover, if $r(t) = \bar{r}, \forall t \geq T_r$, $\lim_{t\to\infty} w(t) = \bar{r}$ and $\lim_{t\to\infty} z(t) = 0$ hold.

Proof: See Appendix III. ■

IV. Tracking Control of a Twin-rotor Helicopter Model

In this section, based on the method of the previous section, we design a tracking controller for the twin-rotor helicopter model, and verify the effectiveness of the method by experiments. In Section IV-A, we show the equation of motion of the twin-rotor helicopter model. Then, in Section IV-B, we design a tracking controller. In Section IV-C, we show the experimental results.

A. Mathematical Model

The equation of motion of the system shown in Figs.3–4 is described by

$$\dot{x} = \mathsf{F}(x) + \mathsf{G}(x)\tilde{\mathsf{u}} \quad (12)$$

$$\tilde{\mathsf{u}} = \Xi_1(x)\mathsf{u} - \mathsf{g} \quad (13)$$

$$\mathsf{u} = \Xi_2\tilde{f} \quad (14)$$

where $x := [\mathsf{v}_x, \mathsf{y}_1, \mathsf{y}_2, \theta_1, \theta_2]^T$, $\mathsf{g} := [\frac{Mg}{L_2}\cos\mathsf{y}_1, 0]^T$, $\tilde{\mathsf{u}}, \mathsf{u}, \tilde{f} \in \mathcal{R}^2$ and

$$\mathsf{F}(x) := \begin{bmatrix} \frac{L_1}{L_2}Mg\tan\theta_1\cos\mathsf{y}_1 \\ \mathsf{y}_2 \\ 0 \\ \theta_2 \\ -\sin\theta_1\cos\theta_1(\mathsf{v}_x^2 - y_2^2) \end{bmatrix}, \mathsf{G}(x) := \begin{bmatrix} L_1\tan\theta_1 & 0 \\ 0 & 0 \\ L_2 & 0 \\ 0 & 0 \\ 0 & L_3 \end{bmatrix}$$

$$\Xi_1(x) := \begin{bmatrix} \cos\theta_1 & 0 \\ 0 & 1 \end{bmatrix},\ \Xi_2 := \begin{bmatrix} 1 & 1 \\ l_r & -l_r \end{bmatrix}$$

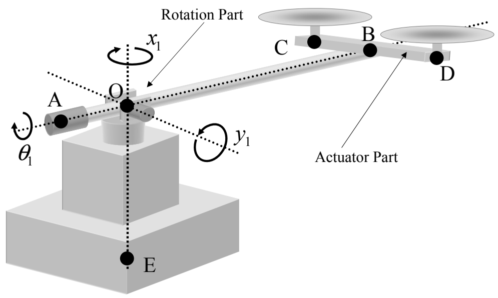

Fig. 3. Experimental apparatus of the twin-rotor helicopter system

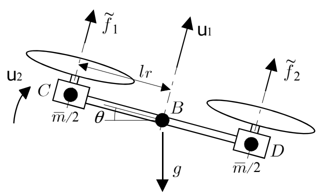

Fig. 4. Actuator Part

x_1[rad] denotes the angle of the actuator part in the horizontal direction, $\mathsf{v}_x(:= \dot{\mathsf{x}}_1)$[rad/s] denotes the velocity in the horizontal direction, y_1[rad] denotes the angle in the vertical direction, θ_1[rad] is the pitch angle, u_1[N] is the lift force, u_2[N] is the rotary force (see Fig.4). $\tilde{f}$[N] denotes the thrust generated by the rotors, and is given by

$$\tilde{f} = a\tilde{\mathsf{u}}_v \tag{15}$$

where $\tilde{\mathsf{u}}_v$[V] is the input signal to the motor driver and $a = \mathrm{diag}[a_1, a_2]$ is the conversion factor. The signal $\tilde{\mathsf{u}}_v$ is computed by

$$\tilde{\mathsf{u}}_v = \Phi_{v_{\max}I}(\mathsf{u}_v) \tag{16}$$

where u_v[V] is the control signal, $v_{\max}$ is the maximum value of $\tilde{\mathsf{u}}_v$. The parameters of the experimental apparatus are $L_1 = 2.6553\mathrm{kg}^{-1}\mathrm{m}^{-1}$, $L_2 = 2.7018\mathrm{kg}^{-1}\mathrm{m}^{-1}$, $L_3 = 179.0718\mathrm{kg}^{-1}\mathrm{m}^{-2}$, $M = 0.4739\mathrm{m}^{-1}$, $\bar{m}$ =0.158kg, l_r =0.188m, $v_{\max} = 5.5$V, $a_1 = 0.2729$N/V, $a_2 = 0.2846$N/V.

Note that, in the above mathematical model, the friction force of the rotation part and the air resistance are omitted.

B. Controller Design

In this section, for the mathematical model derived in the previous section, we design a controller that stabilizes in y and θ directions, and makes the difference between the reference signal r and v_x as small as possible. In the following, firstly, we show a method of reducing the controlled system (12)–(16) to the standard form (1),(2). Then, we design a controller based on the method of the previous section.

We linearize (12) at $x = 0$, and discretize the linearized dynamics with the sampling period $T = 5$ms, and obtain

$$x(t+1) = A_d x(t) + B_d \tilde{\mathsf{u}}(t) \tag{17}$$

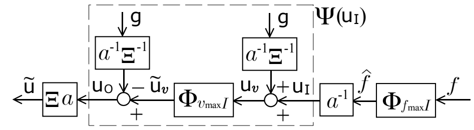

Fig. 5. Equivalent Transformation

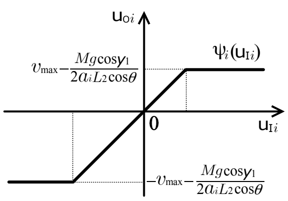

Fig. 6. Nonlinear element $\Psi(\cdot)$

where $A_d := \exp(A_c T)$, $B_d := \int_0^T \exp(A_c\tau)d\tau B_c$ and $A_c := \partial\mathsf{F}/\partial x|_{x=0}$, $B_c := \mathbf{G}(0)$. We define u_v as

$$\mathsf{u}_v \;:=\; a^{-1}\left(\hat{f} + \Xi(x)^{-1}\mathsf{g}\right) \tag{18}$$

$$\hat{f} \;:=\; \Phi_{f_{\max}I}(f) \tag{19}$$

where $\Xi(x) := \Xi_1(x)\Xi_2$, $f_{\max} := \min[f_{\max,1}, f_{\max,2}]$, $f_{\max,i} := a_i\left[v_{\max} - Mg/(2a_iL_2\cos\theta_{\max})\right]$ and $f \in \mathcal{R}^2$. $\theta_{\max}$ is the maximum value of $|\theta|$, f is the desired thrust of the rotor. We make the following assumption.

Assumption 3: For $v_{\max}$ and $\theta_{\max}$, the relation $Mg/(2a_iL_2\cos\theta_{\max}) \leq v_{\max}$ holds.

The above assumption guarantees that the rotors of the twin-rotor system have enough capacity for flying. We set $\theta_{1\max} = 0.578$rad. In this case, the following relation holds.

$$\tilde{\mathsf{u}} = \Xi(x)\hat{f} \tag{20}$$

In the following, we explain about this. We define $\mathsf{u}_o := \tilde{\mathsf{u}}_v - a^{-1}\Xi(x)^{-1}\mathsf{g}, \mathsf{u}_I := a^{-1}\hat{f}$, $\Psi(\mathsf{u}_I) := \Phi_{V_{\max}I}(\mathsf{u}_I + a^{-1}\Xi(x)^{-1}\mathsf{g}) - a^{-1}\Xi(x)^{-1}\mathsf{g}$. Then, the input/output relationship between f and $\tilde{\mathsf{u}}$ (which is described by eqs.(13)–(16), (18)–(19)) can be rewritten as

$$\tilde{\mathsf{u}} = \Xi(x)a\mathsf{u}_o,\ \mathsf{u}_o = \Psi(\mathsf{u}_I) \tag{21}$$

$\mathsf{u}_I := a^{-1}\hat{f}$ and eq.(19) (see.Fig.5). On the other hand, from the relation $a^{-1}\Xi(x)^{-1}\mathsf{g} = a^{-1}Mg\cos\mathsf{y}_1/(2L_2\cos\theta_1)[1,1]^T$ and Assumption 3, the nonlinear element $\Psi(\mathsf{u}_I)$ becomes the asymmetric saturation nonlinearity as shown in Fig.6. Therefore, when we use $f_{\max}$ defined above, the signal u_I always remains in the linear region of $\Psi(\mathsf{u}_I)$. As a result, since the relation $\mathsf{u}_o = \mathsf{u}_I$ holds, we can conclude that eq.(20) holds. Eq.(19) can be rewritten as

$$\hat{f} = f_{\max}\tilde{u},\ \tilde{u} = \Phi(u),\ u = f_{\max}^{-1}f \tag{22}$$

From the above discussion, we choose the coefficient matrices of (1) as $A = A_d$, $B = B_d\Xi(0)f_{\max}$, $E = 0$. Further, since we consider the tracking control problem of v_x, we choose $z := w - y$, $y := \mathsf{v}_x$. For this system, we apply the design method of Section III-A. The solutions of

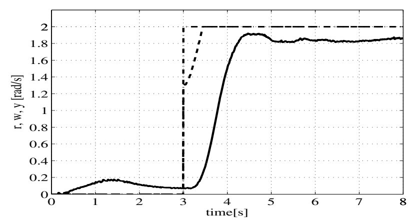

Fig. 7. r:dash-dot, w:dashed, y:solid (gain-scheduled FB)

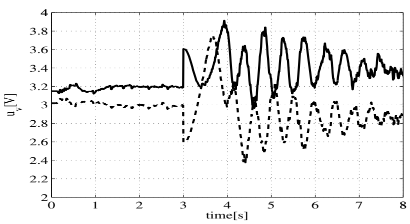

Fig. 8. u_{v1}:solid, u_{v2}:dashed (gain-scheduled FB)

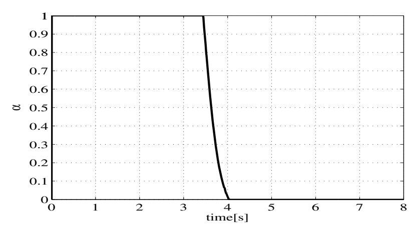

Fig. 9. Scheduling parameter α

eqs.(6), (7) are $\Pi = [1, 0, 0, 0, 0]^T$ and $\Gamma = [0, 0]^T$. We obtain the matrices that satisfy the LMI conditions (8)–(10) with $\gamma_0 = 16, \gamma_1 = 260, \mathbf{R} = 0.1I, \eta = 1, \mathbf{S} = \mathrm{diag}[0.01, 1.5, 0.1, 1, 0.1]$ numerically. We utilize the extended cost function in Remark 2. The control algorithm is implemented in the digital computer (Intel Core2 3GHz, 2GB RAM) by using Matlab/xPC Target.

Remark 4: Note that the actual control signal u_v is computed from (18), (19) and the third equation of (22).

C. Experimental Results

Figs.7–9 show the experimental results in the case where Algorithm 2 is carried out. The reference signal is $r(t) =$ 2rad/s, $\forall t \geq$ 3s. It can be seen that the modified reference signal $w(t)$ is generated just after the step reference signal is added. This implies that, if Algorithm 1 is utilized, the algorithm becomes infeasible at $t = 3$s. From Fig.7, it can be seen that $y(t)$ tracks $r(t)$, although the relatively small steady state error occurs. This steady state error may occur due to the friction force of the rotation part and the air resistance which are disregarded at the controller design. Figs.10–11 show the results in the case where the constant low gain feedback $u(t) = F(1)x(t) + M(1)w(t)$ is utilized (namely, in the case where Algorithm 2 is performed with $\alpha = 1$). In this case, the larger tracking error occurs as compared with the case of the gain-scheduled feedback.

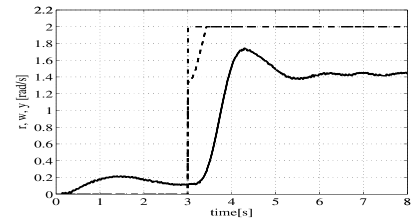

Fig. 10. r:dash-dot, w:dashed, y:solid (constant low-gain FB)

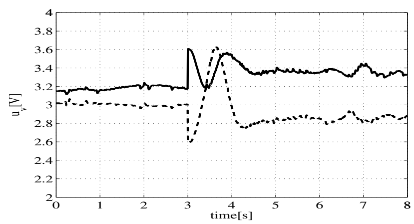

Fig. 11. u_{v1}:solid, u_{v2}:dashed (constant low-gain FB)

V. Conclusions

In this paper, we have designed a tracking control law for the twin-rotor helicopter model based on the method of [13]. We have explained the input transformation to reduce the dynamics of the system to the standard form. Further, we have implemented the control algorithm in the digital computer, and evaluated the effectiveness of the method by experiments. As a result, we have confirmed that the proposed control algorithm can be applied to this class of a mechanical system even if the modeling error and computational delay exist. Further, we have confirmed that the gain-scheduled control law can achieve higher tracking control performance as compared with the constant low-gain feedback.

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APPENDIX I
PROOF OF THEOREM 1

From (1), (3), (4), (6), (11), we obtain

$$\xi(t+1) = A\xi(t) + B\Psi(F(\alpha)\xi(t)) \tag{23}$$

where $\Psi(F(\alpha)\xi) := \Phi(F(\alpha)\xi + \Gamma w) - \Gamma w$. In the following, we first show that if $\xi \in \mathcal{L}(H(\alpha), \rho)$ and $\max_{t\geq 0}|\Gamma^{(l)}r(t)| < 1, \ \forall l \in [1,m]$, $\Psi(F(\alpha)\xi)$ can be represented as $\Psi(F(\alpha)\xi) = \sum_{j=1}^{2^m}\lambda_j\{\mathbf{E}_jF(\alpha) + \mathbf{E}_j^-H(\alpha)\}\xi$, where $H(\alpha) := Z(\alpha)Q(\alpha)^{-1}$, $Z(\alpha) := (1-\alpha)Z_0 + \alpha Z_1$ and $\rho := \mathrm{diag}[\rho_1, \cdots, \rho_m]$. If $\xi \in \mathcal{L}(H(\alpha), \rho)$ and $\max_{t\geq 0}|\Gamma^{(l)}r(t)| < 1, \ \forall l \in [1,m]$, then $|H(\alpha)^{(l)}\xi + \Gamma^{(l)}w| \leq 1, \ \forall l \in [1,m]$. Hence, in this case, the relation $\Phi(F(\alpha)\xi + \Gamma w) = \sum_{j=1}^{2^m}\lambda_j\{\mathbf{E}_j(F(\alpha)\xi + \Gamma w) + \mathbf{E}_j^-(H(\alpha)\xi + \Gamma w)\}$ holds (see Lemma 1 in Appendix). Therefore, we can show that $\Psi(F(\alpha)\xi) = \sum_{j=1}^{2^m}\lambda_j\{\mathbf{E}_jF(\alpha) + \mathbf{E}_j^-H(\alpha)\}\xi$. By using this relation, if $\xi(t) \in \mathcal{L}(H(\alpha), \rho)$ and $\max_{t\geq 0}|\Gamma^{(l)}r(t)| < 1, \ \forall l \in [1,m]$, the close-loop system (1), (3) and (11) can be rewritten as

$$\xi(t+1) = \mathcal{A}(\lambda(t))\xi(t) \tag{24}$$

where $\mathcal{A}(\lambda) := \sum_{j=1}^{2^m}\lambda_j\mathcal{A}_j, \ \mathcal{A}_j := A + B\{\mathbf{E}_jF(\alpha) + \mathbf{E}_j^-H(\alpha)\}$. On the other hand, from the definition, the signal $\mathbf{z}$ can be rewritten as

$$\mathbf{z} = \mathbf{C}x + \mathbf{D}\Phi(u) + \mathbf{D_w}w \tag{25}$$

where $\mathbf{C} := [C^T, F(\alpha)^T\mathbf{R}^{\frac{1}{2}}]^T, \ \mathbf{D_w} := [D_w^T, (M(\alpha) - \Gamma)^T\mathbf{R}^{\frac{1}{2}}]^T$. It can be verified that the matrices Π and Γ that satisfy (6) and (7) also satisfy

$$\mathbf{C}\Pi + \mathbf{D}\Gamma + \mathbf{D_w} = 0. \tag{26}$$

From (25), (26) and (11) if $\xi(t) \in \mathcal{L}(H(\alpha), \rho)$ and $\max_{t\geq 0}|\Gamma^{(l)}r(t)| < 1, \ \forall l \in [1,m]$, the signal $\mathbf{z}(t)$ can be represented as

$$\mathbf{z}(t) = \mathcal{C}(\lambda(t))\xi(t) \tag{27}$$

where $\mathcal{C}(\lambda) := \sum_{j=1}^{2^m}\lambda_j\mathcal{C}_j, \ \mathcal{C}_j := \mathbf{C} + \mathbf{D}\{\mathbf{E}_jF(\alpha) + \mathbf{E}_j^-H(\alpha)\}$. From (24) and (27), in the ξ-coordinate system, the feedback system can be regarded as the system without exogenous input. In the following, we prove Theorem 1 based on this representation.

In the following, we first show that the condition (9) implies that $\mathcal{E}(P(\alpha), \eta) \subseteq \mathcal{L}(H(\alpha), \rho)$. From (9), we have

$$\begin{bmatrix} Q(\alpha) & * \\ Z(\alpha)^{(l)} & \frac{\rho_l^2}{\eta} \end{bmatrix} \geq 0, \ \forall l \in [1,m] \tag{28}$$

Then, by substituting $Z(\alpha)^{(l)} = H(\alpha)^{(l)}Q(\alpha)$ for (28) and performing a congruence transformation with block-diag$[Q(\alpha)^{-1}, 1]$ and substituting $Q(\alpha)^{-1} = P(\alpha)$, and applying Schur complement, we have

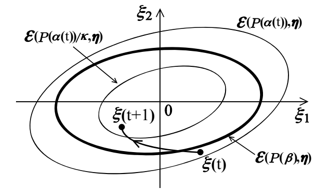

Fig. 12. Invariant set

$$\frac{1}{\rho_l^2}H(\alpha)^{(l)T}H(\alpha)^{(l)} \leq \frac{1}{\eta}P(\alpha), \ \forall l \in [1,m] \tag{29}$$

Equation (29) implies that $\mathcal{E}(P(\alpha), \eta) \subseteq \mathcal{L}(H(\alpha), \rho)$.

Then, we show that the relations $\xi(t) \in \mathcal{E}(P(\alpha), \eta), \forall t \geq 0$ and $\lim_{t\to\infty} z(t) = 0$ and $J < \gamma(\alpha)\eta$ hold. By carrying out the similar procedures used to derive (28) to (8), and substituting $Z(\alpha) = H(\alpha)Q(\alpha)$ and $Y(\alpha) = F(\alpha)Q(\alpha)$ for the resulting inequality, and performing a congruence transformation with block-diag$[Q(\alpha)^{-1}, I, I]$, and multiplying the resulting inequality by $\lambda_j(t)$, and summing them up for $j = 1, \cdots, 2^m$, we have

$$\begin{bmatrix} P(\alpha) & * & * \\ \mathcal{C}(\lambda(t)) & \gamma(\alpha)I & * \\ \mathcal{A}(\lambda(t)) & 0 & P(\alpha)^{-1} \end{bmatrix} > 0 \tag{30}$$

By applying Schur complement to (30), and multiplying the resulting inequality from the left by $\xi(t)^T$ and from the right by $\xi(t)$, and using (24) and (27), we have

$$V(\xi(t+1)) - V(\xi(t)) < -\frac{1}{\gamma(\alpha)}\|\mathbf{z}(t)\|_2^2 \tag{31}$$

where $V(\xi) := \xi^TP(\alpha)\xi$. From (31), we can conclude that if $\xi(0) \in \mathcal{E}(P(\alpha), \eta)$ then

$$V(\xi(t)) < V(\xi(0)) \leq \eta, \ \ \forall t \geq 0 \tag{32}$$

Equation (32) implies that $\xi(t) \in \mathcal{E}(P(\alpha), \eta), \ \forall t \geq 0$. On the other hand, the nonlinearity $\Psi(F(\alpha)\xi(t))$ can be represented as $\Psi(F(\alpha)\xi(t)) = \sum_{j=1}^{2^m}\lambda_j(t)\{\mathbf{E}_jF(\alpha) + \mathbf{E}_j^-H(\alpha)\}\xi(t)$ if $\xi(t) \in \mathcal{L}(H(\alpha), \rho)$ and $\max_{t\geq 0}|\Gamma^{(l)}r(t)| < 1, \ \forall l \in [1,m]$. From (29) and (32), we can state that if the conditions in Theorem 1 hold, the relation $\xi(t) \in \mathcal{L}(H(\alpha), \rho), \ \forall t \geq 0$ holds. From (31), since $\xi(t) \to 0, (t \to \infty)$, $z(t) \to 0, (t \to \infty)$ holds. Moreover, from (31) and (32), $\sum_{t=0}^{\infty}\|\mathbf{z}(t)\|_2^2 < \gamma(\alpha)\eta$ holds.

APPENDIX II
PROOF OF THEOREM 2

In the following, we initially show that by applying Algorithm 1 $\alpha(t)$ monotonically decreases until the condition $\alpha(t) \leq \epsilon$ holds. We assume that at time t the optimization problem of Step 2 in Algorithm 1 is feasible. In this case, it is clear that $\xi(t) \in \mathcal{E}(P(\alpha(t)), \eta)$ holds. When the control signal $u(t) = F(\alpha(t))x(t) + M(\alpha(t))w(t)$ is applied to the system (1), $\xi(t)^T P(\alpha(t))\xi(t) > \xi(t+1)^T P(\alpha(t))\xi(t+1)$ holds from Theorem 1. Hence, for some scalar $\kappa < 1$, $\xi(t+1) \in \mathcal{E}(P(\alpha(t))/\kappa, \eta)$ holds (see Fig.12). In the following, we show that the relation $\mathcal{E}(P(\alpha(t))/\kappa, \eta) \subset \mathcal{E}(P(\beta), \eta) \subset \mathcal{E}(P(\alpha(t)), \eta)$ holds for a scalar β such that $\kappa\alpha(t) < \beta < \alpha(t)$.

- $\mathcal{E}(P(\beta), \eta) \subset \mathcal{E}(P(\alpha(t)), \eta)$:
  Since $Q_0 < Q_1$ and $\beta < \alpha(t)$ hold from the assumption, we obtain $0 < (\alpha(t) - \beta)(Q_1 - Q_0)$. This implies that $\mathcal{E}(P(\beta), \eta) \subset \mathcal{E}(P(\alpha(t), \eta)$.
- $\mathcal{E}(P(\alpha(t))/\kappa, \eta) \subset \mathcal{E}(P(\beta), \eta)$:
  From the assumption, $\kappa\alpha(t) < \beta < \alpha(t)$ holds. Further, since $\kappa < 1$ and $\alpha(t) \leq 1$, $(1-\kappa)\alpha(t) \leq (1-\kappa)$ holds. Hence, $\alpha(t) \leq \kappa\alpha(t) + (1-\kappa)$ holds. Therefore, we obtain $\kappa\alpha(t) < \beta < \kappa\alpha(t) + (1-\kappa)$. From this relation and $Q_0 < Q_1$, we have $0 < [(1-\kappa) - (\beta - \kappa\alpha(t))]Q_0 + [\beta - \kappa\alpha(t)]Q_1$. This implies that $\mathcal{E}(P(\alpha(t))/\kappa, \eta) \subset \mathcal{E}(P(\beta), \eta)$.

From the above discussion, we can conclude that for a scalar β such that $\kappa\alpha(t) < \beta < \alpha(t)$, the relation $\mathcal{E}(P(\alpha(t))/\kappa, \eta) \subset \mathcal{E}(P(\beta), \eta) \subset \mathcal{E}(P(\alpha(t)), \eta)$ holds (see Fig.12). Then we set $\alpha(t+1) = \beta$. In this case, it is clear that $\xi(t+1) \in \mathcal{E}(P(\alpha(t+1)), \eta)$ holds. Namely, the optimization problem of Step 3 in Algorithm 1 is feasible at $t+1$, and the solution $\alpha(t+1)$ satisfies $\alpha(t+1) < \alpha(t)$. The same arguments also hold for $t+2, t+3, \cdots$. Therefore, $\alpha(t)$ decreases monotonically. Further, $\alpha(t)$ is bounded from below by zero. Hence, there exists some time T such that the condition $\alpha(T) = 0$ holds. It can be verified that a contradiction occurs if there is not such a time T. After the time T, the control law $u(t) = F(\alpha(T))x(t) + M(\alpha(T))w(t)$ is applied to the system (1), (2). In this case, from Theorem 1, $\xi(t)$ converges to zero as $t \to \infty$. As a result, $z(t)$ converges to zero as $t \to \infty$

Appendix III
Proof of Theorem 3

We first show that feasibility of Algorithm 2 is guaranteed for all time. In Algorithm 2, if the condition $x(0) - \Pi r(0) \in \mathcal{E}(P(1), \eta)$ holds, the optimization problem of Step 4 is solved to update α. It is clear that there exists a solution α that satisfies $\alpha \leq 1$. Otherwise, if the condition $x(0) - \Pi r(0) \in \mathcal{E}(P(1), \eta)$ does not hold, the optimization problem of Step 3 is solved to compute the modified reference signal $\tilde{w}$. The existence of the solution $\tilde{w}$ is guaranteed from the assumption. Hence, we can conclude that there exists a pair of solutions $\alpha(0)$ and $w(0)$.

By applying $u(0) = F(\alpha(0))x(0) + M(\alpha(0))w(0)$ with $\alpha(0), w(0)$ obtained from Step3 or Step4 to (1), (2), the inequality $(x(1) - \Pi w(0))^T P(\alpha(0))(x(1) - \Pi w(0)) \leq (x(0) - \Pi w(0))^T P(\alpha(0))(x(0) - \Pi w(0))$ holds from Theorem 2. Therefore, $x(1) - \Pi w(0) \in \mathcal{E}(P(\alpha(0)), \eta)$ holds. This implies that there exists a pair of solutions $\alpha(1), w(1)$ at $t = 1$. The same arguments also hold for $t \geq 2$. Therefore, we can conclude that feasibility of Algorithm 2 is guaranteed for all time.

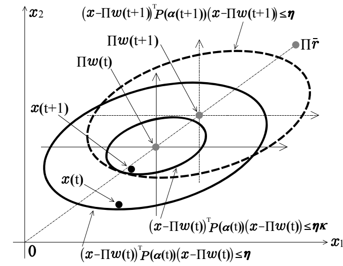

Fig. 13. Graphical Interpretation of Algorithm 2 in case of $n = 2$ and $p = 1$

Then we show that if $r(t) = \bar{r}, \forall t \geq T_r$, the relations $\lim_{t\to\infty} w(t) = \bar{r}$ and $\lim_{t\to\infty} z(t) = 0$ hold. Let us consider the case where the condition $x(t) - \Pi w(t) \in \mathcal{E}(P(\alpha(t)), \eta)$ holds but the equality $w(t) = \bar{r}$ does not hold for a time $t \geq T_r$ (Note that $z(t)$ converges to zero if the equality $w(t) = \bar{r}$ holds from Theorem 2). In this case, at Step3, a modified reference signal $w(t) \neq \bar{r}$ is computed and the scheduling parameter is chosen as $\alpha(t) = \alpha(t-1)$. By applying $u(t) = F(\alpha(t))x(t) + M(\alpha(t))w(t)$ to the system, the relation $x(t+1) - \Pi w(t) \in \mathcal{E}(P(\alpha(t)), \eta\kappa)$ holds for some positive scalar $\kappa < 1$ (see Fig.13). Then, at time $t+1$, if the equality $w(t+1) = \bar{r}$ does not hold, a modified reference signal $w(t+1) \neq \bar{r}$ is computed and the scheduling parameter is chosen as $\alpha(t+1) = \alpha(t)$ at Step 3. In this case, since $x(t+1) - \Pi w(t) \in \mathcal{E}(P(\alpha(t)), \eta\kappa)$, $\Pi w(t+1)$ can be chosen so that $\|\Pi\bar{r} - \Pi w(t)\|_2$ decreases (see Fig.13). Hence, by repeating this process, $x(t) - \Pi\bar{r} \in \mathcal{E}(P(\alpha(t)), \eta)$ holds for some time. As a result, $\lim_{t\to\infty} w(t) = \bar{r}$ and $\lim_{t\to\infty} z(t) = 0$ hold.