Design of a Static Anti-windup Compensator that Optimizes $\mathcal{L}_2$ Performance: An LMI Based Approach

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Abstract – **In this paper, we propose a design method of a static anti-windup compensator that guarantees closed-loop stability and optimizes performance criterion proposed by Teel and Kapoor. Further, we extend the method to the robust performance problem. We provide design procedures based on linear matrix inequality (LMI) representation.**

I. INTRODUCTION

Most of practical control systems involve plants whose actuators are limited by inherent physical constraints. It is well-known that such limitation can produce significant performance degradation called windup phenomena [1]. An anti-windup scheme [8] is one way to design a controller that counteracts such undesirable phenomena. Recently, several systematic design methods of the anti-windup compensator that guarantees closed-loop stability and $\mathcal{L}_2$ gain performance (e.g., [10, 11, 13, 16]).

On the other hand, in [14], a rigorous and useful performance criterion for anti-windup control systems was proposed. The criterion of [14] utilizes the $\mathcal{L}_2$ norm of the output deviation between the fictitious linear system and anti-windup control system as a measure for estimating anti-windup performance. It is referred to as the $\mathcal{L}_2$ performance criterion in this paper. An important feature of the criterion is that it enables us to consider rigorously the behavior of the anti-windup control systems for the exogenous inputs that do not belong to $\mathcal{L}_2$ (e.g., step signal, sinusoidal signal). So far, in [2, 5, 7, 14, 15], several design methods of the dynamic anti-windup compensator were proposed based on the criterion. In the case of the static anti-windup compensator, a design method has been presented in [12]. However, since the method of [12] is based on the line search, it is not easy to apply to the systems with the high order controller and/or multivariable systems.

In this paper, we present a design method of the static anti-windup compensator that optimizes the performance criterion of [14]. We show that the design problem can be formulated as a quasi-convex optimization problem referred to as the generalized eigenvalue problem (GEVP) [3]. Hence, it can readily be solved by using the existing optimization algorithm [3]. Furthermore, we extend the method to the robust performance problem. The proposed methods are applicable to the systems with the high order controller and/or multivariable systems. The following two numerical examples are provided to illustrate the effectiveness of the proposed method: 1) the control problem of the ill-conditioned multivariable system, 2) the angular velocity control problem of the 2-mass-spring system.

Notations. We define the saturation function $\phi_a(\cdot)$ and deadzone function $\psi_a(\cdot)$ as follows.

$$\phi_a(v) \triangleq \begin{cases} a \cdot \mathrm{sgn}(v), & |v| > a \\ v, & |v| \le a \end{cases}, \ \psi_a(v) \triangleq v - \phi_a(v)$$

If $a = 1$, we shall omit the subscript a. Further, for a given diagonal matrix $A = \mathrm{diag}(a_1, \cdots, a_n), a_i > 0$, we define the multivariable saturation function $\Phi_A(v) \triangleq (\phi_{a_1}(v_1), \cdots, \phi_{a_n}(v_n))^T$ and multivariable deadzone function $\Psi_A(v) \triangleq (\psi_{a_1}(v_1), \cdots, \psi_{a_n}(v_n))^T$. If $A = I$, we shall omit the subscript A. For a vector $v \in \mathcal{R}^n$, we denote its Euclidean norm as $\|v\|_2 \triangleq (v^T v)^{1/2}$. For a signal $v(t)$ defined on $[0, \infty)$, we define its $\mathcal{L}_2$ norm as $\|v\|_{\mathcal{L}_2} \triangleq (\int_0^\infty v(t)^T v(t) dt)^{1/2}$. A signal $v(t)$ is said to belong to $\mathcal{L}_2$, i.e., $v \in \mathcal{L}_2$, if $\|v\|_{\mathcal{L}_2} < \infty$. $\mathcal{F}_l(S, \Delta) \triangleq S_{11} + S_{12}\Delta(I - S_{22}\Delta)^{-1}S_{21}$, where S_{ij} denotes a part of S that is partitioned according to the size of Δ. Further, we shall use

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \triangleq C(sI - A)^{-1}B + D$$

II. PROBLEM FORMULATION

Let us consider the following feedback system without saturation (see Fig.1).

$$y_{lin} = P \begin{bmatrix} w \\ u_{lin} \end{bmatrix}, \ u_{lin} = K \begin{bmatrix} r \\ y_{lin} \end{bmatrix} \qquad (1)$$

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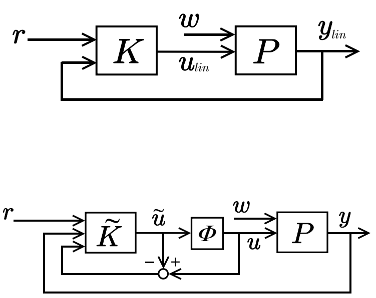

Fig. 2: Anti-windup control system

where $u_{lin} \in \mathcal{R}^{n_u}, y_{lin} \in \mathcal{R}^{n_y}$, $r \in \mathcal{R}^{n_r}$ and $w \in \mathcal{R}^{n_w}$. $P(s)$ denotes a plant and is described by

$$P \triangleq [P_1, P_2] = \left[\begin{array}{c|cc} A_p & B_{p1} & B_{p2} \\ \hline C_p & D_{p1} & 0 \end{array}\right] \tag{2}$$

In this paper, we assume that $P(s)$ is asymptotically stable. $K(s)$ denotes a controller and is described by

$$K \triangleq [K_1, K_2] = \left[\begin{array}{c|cc} A_c & B_{c1} & B_{c2} \\ \hline C_c & D_{c1} & D_{c2} \end{array}\right] \tag{3}$$

In the following, we assume that $K(s)$ has been already designed so that it guarantees closed-loop stability of the system (1) and satisfies certain performance specifications. Next let us consider the following feedback system with anti-windup compensation (see Fig.2).

$$y = P\begin{bmatrix} w \\ u \end{bmatrix}, \quad u = \Phi(\tilde{u}), \quad \tilde{u} = \tilde{K}\begin{bmatrix} r \\ y \\ u - \tilde{u} \end{bmatrix} \tag{4}$$

where $u, \tilde{u} \in \mathcal{R}^{n_u}$ and $y \in \mathcal{R}^{n_y}$. The state equation of $\tilde{K}(s)$ is given by

$$\tilde{K} \triangleq [K_1, K_2, K_3] = \left[\begin{array}{c|ccc} A_c & B_{c1} & B_{c2} & \Lambda_1 \\ \hline C_c & D_{c1} & D_{c2} & \Lambda_2 \end{array}\right] \tag{5}$$

We assume that the transfer functions K_1 and K_2 of (5) are equal to those of (3). The constant matrices Λ_1 and Λ_2 are introduced to attenuate the windup phenomena and referred to as the static anti-windup compensator. This type of two parameters anti-windup compensator was first proposed in [8]. In this paper, we consider the following problem.

Problem 1 *Consider the linear control system (1) and the anti-windup control system (4). Find the matrices* Λ_1 *and* Λ_2 *that minimize* γ *under the following inequality constraint.*

$$\|y - y_{lin}\|_{\mathcal{L}_2} \le \gamma \|\Psi(u_{lin})\|_{\mathcal{L}_2} \tag{6}$$

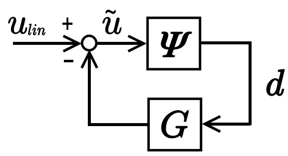

Fig. 3: Subsystem for closed-loop stability

It should be noted that $\|\Psi(u_{lin})\|_{\mathcal{L}_2}$ is bounded when u_{lin} of the system (1) converges to the linear region of the saturation function. Hence, in this case, from (6), the plant output $y(t)$ of the anti-windup control system (4) converges to the plant output $y_{lin}(t)$ of the linear system (1) as $t \to \infty$. Moreover, note that $\|\Psi(u_{lin})\|_{\mathcal{L}_2}$ can be bounded even in the case where the reference signal $r(t)$ does not belong to $\mathcal{L}_2$ (e.g., step signal, sinusoidal signal). Eq.(6) was first introduced in [14] as a performance criterion for anti-windup control systems, and it is referred to as the $\mathcal{L}_2$ anti-windup performance criterion in this paper. In the next section, we will explicitly derive γ of (6).

III. UPPERBOUND OF $\mathcal{L}_2$ PERFORMANCE

The error system between the linear system (1) and the anti-windup control system (4) can be described by

$$y - y_{lin} = -P_2(G + I)d \tag{7}$$
$$\tilde{u} - u_{lin} = -Gd, \quad d = \tilde{u} - \Phi(\tilde{u}) = \Psi(\tilde{u}) \tag{8}$$

where $G(s) \triangleq (I - K_2P_2)^{-1}(K_2P_2 + K_3)$. Then we introduce the following lemma.

Lemma 1 *[5] For any* $F = diag[f_1, \cdots, f_{n_u}] > 0$ *and* $x, y \in \mathcal{R}^n$, *where* $\Psi_F(x), y \in \mathcal{L}_{2[0,T)}$, *the following inequality holds .*

$$\|\Psi_F(x + y)\|_{\mathcal{L}_{2[0,T)}} \le \|\Psi_F(x)\|_{\mathcal{L}_{2[0,T)}} + \|y\|_{\mathcal{L}_{2[0,T)}} \tag{9}$$

Proof) For any $f_i > 0$ *and* $x_i, y_i \in \mathcal{R}$, *the following inequality clearly holds.*

$$|\psi_{f_i}(x_i + y_i)| \le |\psi_{f_i}(x_i)| + |y_i| \tag{10}$$

Then, squaring both sides of the inequality (10) and adding them for $i = 1, \cdots, n_u$ *and applying Schwartz's inequality, we obtain*

$$\|\Psi_F(x + y)\|_2^2 \le \|\Psi_F(x)\|_2^2 + \|y\|_2^2 + 2\|\Psi_F(x)\|_2\|y\|_2 \tag{11}$$

Furthermore, integrating both sides of (11) over $[0, T)$ *and applying Schwartz's inequality again, (9) can be obtained.* □

Then, the following theorem can be derived by applying the technique used to prove the theorem in [5] to the control system of this paper.

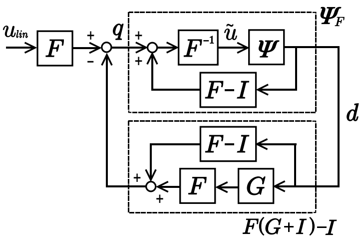

Fig. 4: Equivalent system

Theorem 1 *Consider the system (7), (8). For a given* $F = diag[f_1, \cdots, f_{n_u}] > 0$*, if there exist* Λ_1 *and* Λ_2 *that satisfy*

$$\|F(G+I) - I\|_\infty < 1 \tag{12}$$

Further, provided that $\Psi(u_{lin}) \in \mathcal{L}_2$*, then the following inequality holds.*

$$\|y - y_{lin}\|_{\mathcal{L}_2} \le \frac{\|F\|\|P_2(G+I)\|_\infty}{1 - \|F(G+I) - I\|_\infty}\|\Psi(u_{lin})\|_{\mathcal{L}_2} \tag{13}$$

Proof) The closed-loop system (8) (depicted in Fig.3) can be transformed to that of Fig.4 by performing equivalent transformations on the block diagram. Now we define $q \triangleq Fu_{lin} - \{F(G+I)-I\}d$*. Then, by using* $d = \Psi(\tilde{u}) = \Psi_F(q)$ *and applying Lemma 1 to the system of Fig.4, we have*

$$\begin{aligned}\|d\|_{\mathcal{L}_{2[0,T)}} &= \|\Psi_F(Fu_{lin} - \{F(G+I) - I\}d)\|_{\mathcal{L}_{2[0,T)}} \\ &\le \|\Psi_F(Fu_{lin})\|_{\mathcal{L}_{2[0,T)}} + \|\{F(G+I) - I\}d\|_{\mathcal{L}_{2[0,T)}} \\ &\le \|\Psi_F(Fu_{lin})\|_{\mathcal{L}_{2[0,T)}} + \|F(G+I) - I\|_\infty\|d\|_{\mathcal{L}_{2[0,T)}} \\ &\le \frac{\|F\|}{1 - \|F(G+I) - I\|_\infty}\|\Psi(u_{lin})\|_{\mathcal{L}_{2[0,T)}} \end{aligned} \tag{14}$$

To derive the last inequality of (14), we used the inequality (12) and the fact $\Psi_F(Fu_{lin}) = F\Psi(u_{lin})$*. Since the inequality (14) still holds even if* $T \to \infty$ *and* $\Psi(u_{lin}) \in \mathcal{L}_2$*, the left hand side of the inequality (14) is bounded even if* $T \to \infty$*. Finally, letting* $T \to \infty$ *in (14) and using (7), we obtain (13).* □

From (13), γ of (6) can be obtained as $\gamma_n \triangleq \|F\|\|P_2(G+I)\|_\infty/(1 - \|F(G+I) - I\|_\infty)$. Since the dead zone function $\Psi_F(\cdot)$ lies inside the sector $[0, 1]$, the global asymptotic stability of the system of Fig.4 is guaranteed in the case where (12) holds. This implies that the condition that A_p must be Hurwitz is necessary to obtain a solution that satisfies (12).

IV. DESIGN METHOD

In this section, we provide a method to solve Problem 1 based on the results of Theorem 1. Firstly, we introduce positive scalars δ, μ such that

$$\|F(G+I) - I\|_\infty < \delta, \quad \|P_2(G+I)\|_\infty < \mu \tag{15}$$

and $\delta < 1$. Then, by applying the bounded real lemma, (15) can be transformed into the following matrix inequality constraints.

$$\begin{bmatrix} \mathcal{A}Q + Q\mathcal{A}^T & Q\mathcal{C}^T & \mathcal{B} \\ \mathcal{C}Q & -\delta I & \mathcal{D} \\ \mathcal{B}^T & \mathcal{D}^T & -\delta I \end{bmatrix} < 0, \quad Q = Q^T > 0 \tag{16}$$

$$\begin{bmatrix} \mathcal{A}R + R\mathcal{A}^T & R\mathcal{C}_y^T & \mathcal{B} \\ \mathcal{C}_y R & -\mu I & 0 \\ \mathcal{B}^T & 0 & -\mu I \end{bmatrix} < 0, \quad R = R^T > 0 \tag{17}$$

where

$$\mathcal{A} \triangleq \begin{bmatrix} A_c & B_{c2}C_p \\ B_{p2}C_c & A_p + B_{p2}D_{c2}C_p \end{bmatrix}, \mathcal{B} \triangleq \begin{bmatrix} \Lambda_1 \\ B_{p2}(I + \Lambda_2) \end{bmatrix}$$

$$\mathcal{C} \triangleq \begin{bmatrix} FC_c & FD_{c2}C_p \end{bmatrix}, \qquad \mathcal{C}_y \triangleq \begin{bmatrix} 0 & C_p \end{bmatrix}$$

and $\mathcal{D} \triangleq F(\Lambda_2 + I) - I$. Note that (16) and (17) are LMIs with respect to the variables $\Lambda_1, \Lambda_2, Q, R, \delta, \mu$.

By using δ and μ of (15), the upperbound of γ_n can be derived as $\gamma_n < \|F\|\mu/(1-\delta)$. Moreover, we introduce a positive scalar λ such that $\|F\|\mu/(1-\delta) < \lambda$. Then, Problem 1 can be reduced to the following optimization problem.

Problem 2 *For a given* $F > 0$*, find* Λ_1 *and* Λ_2 *that minimize* λ *under the constraints* $\lambda(1-\delta) > \|F\|\mu$, $1 - \delta > 0$*, (16) and (17).*

Problem 2 is a quasi-convex optimization problem referred to as the generalized eigenvalue problem (GEVP) [3]. Hence, it can efficiently be solved by using the numerical optimization algorithm [6].

Note 1 *If we treat* F *as one of the decision variables, the inequality (16) becomes the bilinear matrix inequality constraint. Thus, in this paper, we treat* F *as the fixed parameter. In this note, we show a guideline for choosing* F*. In the scalar case, the condition (12) requires that the Nyquist plot of* $G(j\omega)$ *remains in the circle whose center is* $(1/f - 1, 0)$ *and diameter is* $1/f$*. By choosing a small value as* f*, the restriction on the size of* $G(j\omega)$ *can be relaxed. Hence we can expect that the upperbound of* γ_n *can be made small by choosing sufficiently small value as* f*. Similarly, in the multi-input case, we can expect that the upperbound of* γ_n *can be made small by choosing* $F = fI$, $f \ll 1$.

Note 2 *Problem 2 may produce a solution* Λ_2 *such that* $\det(I + \Lambda_2) \approx 0$*, which is problematic in practical situations, since* $\tilde{u} = (I + \Lambda_2)^{-1}[C_c x_c + D_{c1} r + D_{c2} y + \Lambda_2 u]$ *from (4), (5). In this note, we show a simple method for avoiding such a problem. We assume that the matrix* F *is chosen as* $F = fI$*. Then the condition (16) implies* $\mathcal{D}\mathcal{D}^T < \delta^2 I$*, which also implies that the eigenvalues of* $I + \Lambda_2$ *is located in the circle whose center is* $(1/f, 0)$ *and* *diameter is $1/f$ in the complex plane. Then we introduce the following LMI condition.*

$$(\Lambda_2 + I) + (\Lambda_2 + I)^T - 2\sigma I > 0 \qquad (18)$$

where $\sigma \geq 0$ is a scalar. The condition (18) guarantees that the eigenvalues of $I + \Lambda_2$ is located in $\Omega_c \triangleq \{z \in \mathcal{C} : Re\{z\} > \sigma\}$ where $\mathcal{C}$ denotes the complex plane, and ensures $\sigma^{n_u} < \det(I + \Lambda_2)$. Hence, the problem mentioned above can easily be avoided by solving Problem 2 with (18).

V. ROBUST PERFORMANCE CASE

In the preceding sections, we have treated the design problem for the case where the plant model has no uncertainty, namely, the nominal performance case. In this section, we extend the results to the case where the plant uncertainty exists. We assume that the perturbed plant model $\tilde{P}(s)$ is given by

$$\tilde{P}(s) \triangleq [\tilde{P}_1, \tilde{P}_2] = (I + W(s)\Delta(s))P(s) \qquad (19)$$

where $P(s)$ represents a nominal plant model, and whose state equation is given by (2). $\Delta(s) \in \mathcal{RH}_\infty$ represents multiplicative uncertainty that satisfies $\|\Delta\|_\infty \leq 1$. $W(s) \in \mathcal{RH}_\infty$ represents a weighting function. In the following, for simplicity, we will restrict the class of $W(s)$ to the constant matrix $W(s) = \epsilon^{-1} I$. But, it is possible to extend the following results to the general dynamic case. Moreover, we assume that the linear feedback system (1) with the perturbed plant (19) is robustly stable.

We formulate the robust performance design problem as follows.

Problem 3 *Consider the linear control system (1) and the anti-windup control system (4), where $P(s)$ is replaced with $\tilde{P}(s)$. Find the matrices Λ_1 and Λ_2 that minimize γ under the constraint (6).*

From Theorem 1, the following corollary can readily be derived.

Corollary 1 *For a given $F = diag[f_1, \cdots, f_{n_u}] > 0$, if there exist Λ_1, Λ_2 that satisfy $\|F(\tilde{G}+I) - I\|_\infty < 1$, where $\tilde{G}(s) \triangleq (I - K_2\tilde{P}_2)^{-1}(K_2\tilde{P}_2 + K_3)$. Further, provided that $\Psi(u_{lin}) \in \mathcal{L}_2$, then the condition (6) holds with $\gamma = \gamma_r \triangleq \|F\|\|\tilde{P}_2(\tilde{G}+I)\|_\infty/(1 - \|F(\tilde{G}+I) - I\|_\infty)$.*

In the following, we provide a method to solve Problem 3 based on Corollary 1. We introduce positive scalars δ, μ such that

$$\|F(\tilde{G}+I) - I\|_\infty < \delta, \;\; \|\tilde{P}_2(\tilde{G}+I)\|_\infty < \mu \qquad (20)$$

and $\delta < 1$. It is well-known that to obtain an exact solution to this type of robust performance conditions is not simple [18]. Hence, we derive sufficient conditions that

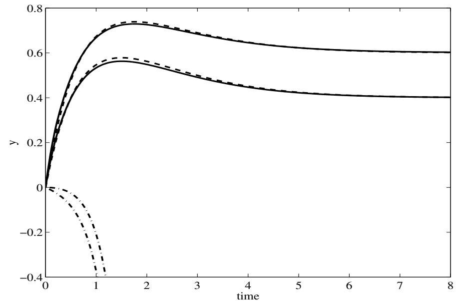

Fig. 5: $y(t)$ (dashed:unconstrained, solid:with AWC, dash-dot:w/o AWC)

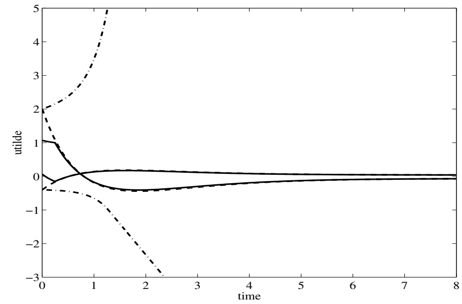

Fig. 6: $\tilde{u}(t)$ (dashed:unconstrained, solid:with AWC, dash-dot:w/o AWC)

can be expressed as LMIs. Firstly, (20) can be rewritten as

$$\|\mathcal{F}_l(M, \Delta)\|_\infty < \delta, \;\; \|\mathcal{F}_l(N, \Delta)\|_\infty < \mu \qquad (21)$$

where $M(s) \triangleq \mathsf{C_m}(sI - \mathsf{A})^{-1}\mathsf{B} + \mathsf{D_m}$, $N(s) \triangleq \mathsf{C_n}(sI - \mathsf{A})^{-1}\mathsf{B} + \mathsf{D_n}$ and

$$\mathsf{A} \triangleq \mathcal{A}, \quad \mathsf{B} \triangleq \begin{bmatrix} \Lambda_1 & \epsilon^{-1}B_{c2} \\ B_{p2}(I + \Lambda_2) & \epsilon^{-1}B_{p2}D_{c2} \end{bmatrix}$$

$$\mathsf{C_m} \triangleq \begin{bmatrix} FC_c & FD_{c2}C_p \\ 0 & C_p \end{bmatrix}, \quad \mathsf{C_n} \triangleq \begin{bmatrix} 0 & C_p \\ 0 & C_p \end{bmatrix}$$

$$\mathsf{D_m} \triangleq \begin{bmatrix} F(\Lambda_2 + I) - I & \epsilon^{-1}FD_{c2} \\ 0 & 0 \end{bmatrix}, \mathsf{D_n} \triangleq \begin{bmatrix} 0 & \epsilon^{-1}I \\ 0 & 0 \end{bmatrix}$$

Then, from the results on μ-analysis [18], sufficient conditions of (21) are given by

$$\left\| M \begin{bmatrix} \delta^{-1}I & 0 \\ 0 & I \end{bmatrix} \right\|_\infty < 1, \; \left\| N \begin{bmatrix} \mu^{-1}I & 0 \\ 0 & I \end{bmatrix} \right\|_\infty < 1 \qquad (22)$$

Then by applying the bounded real lemma and Schur complements, (22) can equivalently be converted to

$$\begin{bmatrix} \mathbb{A}_m + \delta\mathbb{B}_{m2}\mathbb{B}_{m2}^T & \mathbb{B}_{m1} \\ \mathbb{B}_{m1}^T & -\delta I \end{bmatrix} < 0, \; \mathbb{Q} = \mathbb{Q}^T > 0 \qquad (23)$$

$$\begin{bmatrix} \mathbb{A}_n + \mu\mathbb{B}_{n2}\mathbb{B}_{n2}^T & \mathbb{B}_{n1} \\ \mathbb{B}_{n1}^T & -\mu I \end{bmatrix} < 0, \quad \mathbb{R} = \mathbb{R}^T > 0 \qquad (24)$$

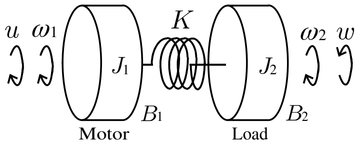

Fig. 7: 2-mass-spring system

where

$$\mathbb{A}_m(\mathbb{Q},\delta) \triangleq \begin{bmatrix} \mathsf{A}\mathbb{Q}+\mathbb{Q}\mathsf{A}^T & \mathbb{Q}\mathsf{C}_m^T \\ \mathsf{C}_m\mathbb{Q} & -\delta I \end{bmatrix}$$

$$\mathbb{A}_n(\mathbb{R},\mu) \triangleq \begin{bmatrix} \mathsf{A}\mathbb{R}+\mathbb{R}\mathsf{A}^T & \mathbb{R}\mathsf{C}_n^T \\ \mathsf{C}_n\mathbb{R} & -\mu I \end{bmatrix}$$

$$\begin{bmatrix} \mathbb{B}_{m1}(\Lambda_1,\Lambda_2), & \mathbb{B}_{m2} \end{bmatrix} \triangleq \begin{bmatrix} \mathsf{B} \\ \mathsf{D}_m \end{bmatrix}$$

$$\begin{bmatrix} \mathbb{B}_{n1}(\Lambda_1,\Lambda_2), & \mathbb{B}_{n2} \end{bmatrix} \triangleq \begin{bmatrix} \mathsf{B} \\ \mathsf{D}_n \end{bmatrix}$$

and $n_m \triangleq n_c + n_p + n_u + n_y$, $n_n \triangleq n_c + n_p + 2n_y$, $\mathbb{B}_{m1} \in \mathcal{R}^{n_m \times n_u}$, $\mathbb{B}_{m2} \in \mathcal{R}^{n_m \times n_y}$, $\mathbb{B}_{n1} \in \mathcal{R}^{n_n \times n_u}$, $\mathbb{B}_{n2} \in \mathcal{R}^{n_n \times n_y}$. Note that (23) and (24) are the LMIs with respect to the variables $\Lambda_1, \Lambda_2, \mathbb{Q}, \mathbb{R}, \mu, \delta$.

By using δ and μ of (20), the upperbound of γ_r can be derived as $\gamma_r < \|F\|\mu/(1-\delta)$. Moreover, we introduce a positive scalar λ such that $\|F\|\mu/(1-\delta) < \lambda$. Then Problem 3 can be reduced to the following optimization problem.

Problem 4 *For a given $F > 0$, find Λ_1 and Λ_2 that minimize λ under the constraints $\lambda(1-\delta) > \|F\|\mu$, $1-\delta > 0$, (23) and (24).*

Problem 4 is also a GEVP. Hence it can efficiently be solved by using the numerical optimization algorithm.

VI. NUMERICAL EXAMPLES

A. *Example 1 (Multivariable system)*

Let us consider the numerical example of [4]. $P(s)$ and $K(s)$ are given by

$$P_1 = 0, \quad P_2 = \frac{1}{10s+1}\begin{bmatrix} 15 & 40 \\ 12 & 30 \end{bmatrix}$$

$$K_1 = \frac{10s+1}{s}\begin{bmatrix} 1/3 & 0 \\ 0 & -1/10 \end{bmatrix}, \quad K_2 = -K_1$$

For the above system, we designed Λ_1 and Λ_2 by solving Problem 2 with the condition (18) where $\sigma = 0.5$ and $F = 0.01I$.

Fig.5 and Fig.6 show the responses of the system for $r(t) = [0.6, 0.4]^T, (t \geq 0)$. Although the plant outputs of the system without anti-windup compensation become unbounded (dash-dot), the outputs of the system with the proposed anti-windup compensator (solid) are fairly close to those of the linear system (dashed).

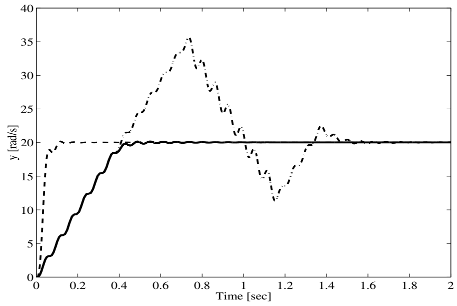

Fig. 8: $y(t)$ (dashed:unconstrained, solid:with AWC, dash-dot:w/o AWC)

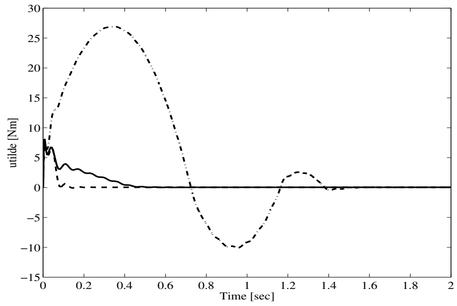

Fig. 9: $\tilde{u}(t)$ (dashed:unconstrained, solid:with AWC, dash-dot:w/o AWC)

B. *Example 2 (2-mass-spring system)*

Let us consider a 2-mass-spring system depicted in Fig.7. u is the motor torque [Nm], w is the disturbance torque [Nm], ω_1 and ω_2 are the angular velocity of the motor and that of the load respectively [rad/s]. J_1 and J_2 are the moment of inertia of the motor and that of the load respectively. B_1 and B_2 are the coefficients of viscous friction of the motor and that of the load respectively. K is the coefficient of elasticity of the spring. In this example, $J_1 = J_2 = 0.01$ [kgm^2], $B_1 = B_2 = 0.001$ [Nms/rad] and $K = 50$ [Nm/rad]. The state equation of the system in Fig.7 is given by

$$\dot{x}_p = \begin{bmatrix} 0 & 1 & -1 \\ -K/J_1 & -B_1/J_1 & 0 \\ K/J_2 & 0 & -B_2/J_2 \end{bmatrix} x_p + \begin{bmatrix} 0 \\ 0 \\ -1/J_2 \end{bmatrix} w + \begin{bmatrix} 0 \\ 1/J_1 \\ 0 \end{bmatrix} u \quad (25)$$

and $y = [0\ 0\ 1]x_p$, where $x_p = [\theta, \omega_1, \omega_2]^T$. The poles of the system (25) are $(-0.05 \pm 100j, -0.1)$. For this plant, we designed a two degree of freedom controller $K(s)$ by using the single step approach of [9] with the desired closed-loop transfer function $M_0(s)$ and the loop shaping weighting function $W(s)$ as $M_0(s) = 100^2/(s^2 + 200s + 100^2), W(s) = 10(0.01s+1)/s$. Moreover, we chose $\rho = 5$. Then we designed Λ_1 and Λ_2 by solving Problem 2 with the condition (18) where $\sigma = 0.5$ and $F = 0.01$.

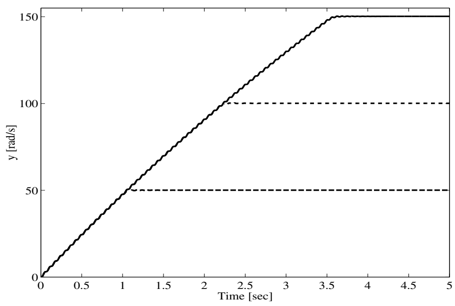

Fig. 10: $y(t)$ (solid: $r(t) = 150$, dash-dot: $r(t) = 100$, dashed: $r(t) = 50$)

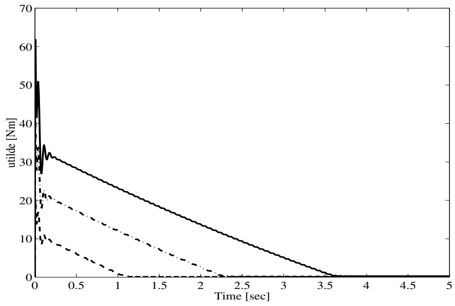

Fig. 11: $\tilde{u}(t)$ (solid: $r(t) = 150$, dash-dot: $r(t) = 100$, dashed: $r(t) = 50$)

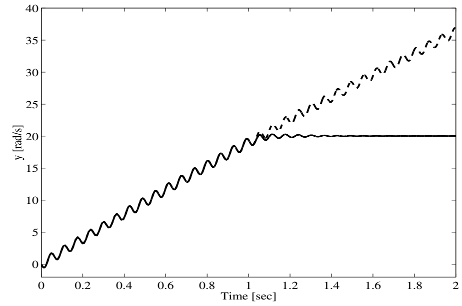

Fig. 12: $y(t)$ (solid:with AWC, dashed:w/o AWC)

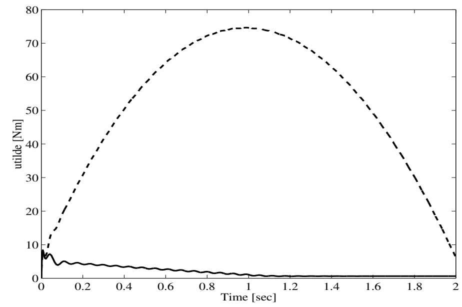

Fig. 13: $\tilde{u}(t)$ (solid:with AWC, dashed:w/o AWC)

Fig.8 and Fig.9 show the responses of the system for $r(t) = 20, (t \geq 0)$ and $w(t) = 0, (t \geq 0)$. Although the output $y(t)$ of the system without anti-windup compensation shows oscillatory behavior (dash-dot), that of the system with the proposed anti-windup compensator shows good performance (solid). Fig.10 and Fig.11 show the responses of the system with the proposed anti-windup compensator for $r(t) = 150$ (solid), $r(t) = 100$ (dash-dot) and $r(t) = 50$ (dashed). In all cases, the anti-windup compensator completely suppresses the overshoot, and the plant output $y(t)$ tracks each reference signal. Fig.12 and Fig.13 show the responses of the system for $r(t) = 20, (t \geq 0)$ and $w(t) = 0.6, (t \geq 0)$. We can see that the proposed anti-windup compensator effectively attenuates the destructive effect of the step disturbance (solid).

VII. CONCLUSION

In this paper, we have presented the design methods of the static anti-windup compensator that optimizes $\mathcal{L}_2$ performance. Both the nominal performance problem and robust performance problem can be reduced to the GEVP. Hence the problems can efficiently be solved by using the numerical optimization algorithm.

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