§5 Proper actions on a conjugacy class of closed homogeneous subsets

§5.1 : Homogeneous subsets

Setting G : a group

M : a G-set

Def 5.1.1 : A subset $D \subset M$ is said to be homogeneous if

$H_D := \{ g \in G \mid gD = D \}$ acts on D transitively

§ 5.2 : A conjugacy classes of closed homogeneous subsets

Setting

G : a locally compact Hausdorff group

M : a homogeneous G-space

D : a closed subset of M

Prop. 5.2.1 :

$$H_D := \{ g \in G \mid gD = D \}$$

is a closed subgroup of G

We put $X_D := \{ gD \subset M \mid g \in G \}$

(a G - conjugacy class of D)

Prop 5.2.2 X_D is a homogeneous G - space and the isotropy of G at $D \in X_D$ is H_D.

Setting

G : a locally compact Hausdorff group

M : a homogeneous G-space with compact isotropies.

D : a closed homogeneous subset of M

L : a closed subgroup of G

Theorem 5.2.3

The following two conditions on (D, L) are equivalent:

(i) $L \curvearrowright X_D$ is proper

(ii) ${}^\forall C_M \subset M$: compact

$$L(C_M : X_D) := \{ g \in L \mid {}^\exists D' \in X_D \text{ s.t. } C_M \cap D' \neq \emptyset,\ gC_M \cap D' \neq \emptyset \}$$

is compact

Cor 5.2.4

Assume that

$$\forall x, y \in X, \ \exists D' \in X_D \text{ s.t. } \{x, y\} \subset X_D.$$

Then the following conditions on L are equivalent:

(i) $L \curvearrowright X_D$ is proper

(ii) L is compact

Hint $L(c_M ; X_D) = L$ for any c_M.

§ 5.3 Proof of Thm 5.2.3

Lemma 5.3.1 Let L be a topological group,

X a Hausdorff space

$L \curvearrowright X$ a continuous action.

Then for any compact $S \subset X$,

$$L_S := \{ g \in L \mid gS \cap S \neq \emptyset \}$$

is closed in L.

In the setting of Thm 5.2.3,

we only need to show the following :

Prop 5.3.2 :

(1) $\forall C_X \subset X_D$: compact,

$\exists C_M \subset M$: compact s.t. $L_{C_X} \subset L(C_M ; X_D)$

(2) $\forall C_M \subset M$: compact

$\exists C_X \subset X_D$: compact s.t. $L_{C_X} = L(C_M : X_D)$

Lemma 5.3.3 :

Let Y, Z be locally compact Hausdorff topological spaces,

and $f : Y \twoheadrightarrow Z$ a continuous open surjective map

Then ${}^{\forall} C_Z \subset Z$: compact

${}^{\exists} C_Y \subset Y$: compact s.t. $f(C_Y) = C_Z$

Lemma 5.3.4

Let G be a group, X a G-set.

Fix $p \in X$.

Let D_1, D_2 are both homogeneous subsets in X

s.t. $p \in D_1 \cap D_2$.

Assume that $\exists g \in G$ s.t. $gD_1 = D_2$.

Then $\exists k \in G^p$ s.t. $kD_1 = D_2$.

The isotropy at p

$\boxed{\text{Added slides at 16 July}}$

We can apply the following for Thm 5.2.3

Setting : X, M, Z : locally compact Hausdorff spaces

$L \curvearrowright X, M, Z$: continuous actions

$\varphi : Z \twoheadrightarrow X$, $\psi : Z \twoheadrightarrow M$: L-equivariant continuous open surjective maps

Assume ψ is a proper map

Notation : For each $x \in X$, we put

$$D_x := \psi(\varphi^{-1}(\{x\})) \subset M$$

Theorem 5.3.5

The following conditions are equivalent:

(i) $L \curvearrowright X$ is proper

(ii) $\forall C_M \subset M$: compact

$$L(C_M ; \{D_x\}_{x \in X}) := \left\{ g \in L \;\middle|\; \begin{array}{l} {}^{\exists} x \in X \\ C_M \cap D_x \neq \emptyset \\ g C_M \cap D_x \neq \emptyset \end{array} \right\}$$

is compact