Goal : Kobayashi's Properness criterion on homogeneous spaces of reductive type

① Proper actions on families of homog subsets.

(Wed)

② Homog sps of red. type $\doteqdot$ Families of totally geod. submfds in Riem. symm. sps

(Thu)

③ Geometries of geodesics in Riem. symm. sps.

(Fri)

§6 : Clifford-Klein forms

§6.1 : Homogeneous space of Lie groups

Reference :

- 小林・大島, "リー群と表現論", 岩波
- S. Helgason, "Differential Geometry, Lie groups and Symmetric spaces".
  GSM vol 34. AMS.

Def 6.1.1: Let G be topological group equipped with a smooth structure.

G is said to be a Lie group

if
$$\begin{cases} G \times G \to G, & (g,h) \mapsto gh \\ G \to G, & g \mapsto g^{-1} \end{cases}$$
are both smooth.

Prop 6.1.2 : Let G be a Lie group.

A C^∞-manifold X equipped

with transitive smooth G-action

is a homogeneous G-space

(In particular,

$$\pi_x : G \twoheadrightarrow X \quad \text{is open} \quad (\forall x \in X)$$
$$g \mapsto gx$$
)

This proposition follows from the following lemma.

Lemma 6.1.3 :

Let G be a loc. compact Hausdorff second countable group and X a loc. compact Hausdorff space equipped with a transitive continuous G-action.

Then $\pi_x : G \twoheadrightarrow X$ is open for any $x \in X$.

(Hint : Baire category theorem)

Theorem 6.1.4 (von-Neumann - Cartan)

Let G be a Lie group,
and H a closed subgroup of G.

Then H uniquely admits a smooth structure
s.t. $H \hookrightarrow G$ is smooth immersion.

(cf. Helgason Ch II §2)

Theorem 6.1.5

Let G be a Lie group, and
H a closed subgroup.
Then the homogeneous G-space G/H
admits a smooth structure
s.t. $G \curvearrowright G/H$ is smooth.

(cf. 小林-大島 §6
Helgason Ch II §4)

Cor 6.1.6

Let G be a Lie group, and X a G-set with closed isotropies.
Then X uniquely admits a smooth structure s.t. the G-action $G \curvearrowright X$ is smooth.

Ex. 6.1.7: $p, q \in \mathbb{Z}_{\geq 0}$. a symmetric bilinear form

Put $\mathbb{R}^{p+1,q} := \left(\mathbb{R}^{p+q+1}, \underset{\text{a symmetric bilinear form}}{(\ ,\ )_{p+1,q}} \right)$

where $(x, y)_{p+1,q} = \sum_{i=1}^{p+1} x_i y_i - \sum_{i=p+2}^{p+q+1} x_i y_i$

Define

$$O(p+1,q) := \left\{ g \in M(p+q+1, \mathbb{R}) \;\middle|\; \begin{array}{c} (gx, gy)_{p+1,q} \\ \parallel \\ (x,y)_{p+1,q} \\ \forall x, y \in \mathbb{R}^{p+1,q} \end{array} \right\}$$

Then $O(p+1,q)$ is a Lie group

$$\left(O(p+1,q) \underset{\substack{\text{closed}\\\text{regular}\\\text{submfd}}}{\subset} M(p+q+1,\mathbb{R}) \cong \mathbb{R}^{(p+q+1)^2} \right)$$

We put $X(p,q) := \{ x \in \mathbb{R}^{p+1,q} \mid \sum_{i=1}^{p+1} x_i^2 - \sum_{i=p+2}^{p+q+1} x_i^2 = 1 \}$

which is a closed regular submfd of $\mathbb{R}^{p+q+1}$.

$O(p+1,q)$ acts on $X(p,q)$ transitively and smoothly.

Thus $X(p,q)$ is a homogeneous $O(p+1,q)$-space with its isotropy $\cong O(p,q)$.

§ 6.2 Clifford-Klein forms

Let G be a Lie group and X a homogeneous G-space.

($\rightsquigarrow$ X is a smooth manifold and $G \curvearrowright X$ is smooth)

Def 6.2.1: A discrete subgroup Γ of G is said to be a discontinuous group for X if $\Gamma \curvearrowright X$ is properly-discontinuous.

Prop 6.2.2: Let Γ be a discontinuous group for X.

Then $\Gamma \backslash X$ uniquely admits a smooth structure s.t. $X \twoheadrightarrow \Gamma \backslash X$ is a smooth covering

Furthermore $\Gamma \backslash X$ is a (G, X)-manifold.

(Any G-invariant local structure on X induces a local structure on $\Gamma \backslash X$)

cf.

W.P. Thurston,

"Three-dimensional Geometry and Topology",

35 Princeton University Press, 1997.

Proposition 6.2.3

If the isotropies of X is compact, then any closed subgroup L of G acts on X properly.

Cor 6.2.4

If the isotropies of X is compact. then any torsion-free discrete subgroup Γ of G acts on X properly-discontinuously

Open problem:

For which (p,q),

the homogeneous $O(p+1,q)$-space $X(p,q)$

admits compact Clifford-Klein forms?

Conjecture

p	any	0	1	3	7
q	0	any	$2\mathbb{N}$	$4\mathbb{N}$	8

(Recent results: Y. Morita, Selecta Math. (N.S.) 23 (2017))