§7 : Homogeneous spaces of reductive types

§7.1 : Linear reductive Lie groups

For each $N \in \mathbb{N}$,

We put

$$GL(N;\mathbb{R}) := \{ g \in M(N;\mathbb{R}) \mid \det g \neq 0 \}.$$

(a non-compact Lie group)

Let us define

$$\theta : GL(N;\mathbb{R}) \to GL(N;\mathbb{R}),$$
$$g \mapsto {}^t g^{-1}.$$

(A Cartan involution on $GL(N;\mathbb{R})$)

Def 7.1.1. In this course, we say that a Lie group G is a linear reductive

if $\#\{\text{conn. comp of } G\} \lneq \infty$

and $\exists N \in \mathbb{N}$,

$\exists G_0$: a closed subgroup of $GL(N;\mathbb{R})$

with $\theta(G_0) = G_0$

s.t. $G \underset{\text{Lie group isom.}}{\simeq} G_0$.

§7.2 : Homogeneous spaces of reductive type

Let G be a linear reductive Lie group.

Def 7.2.1 : A closed subgroup H of G

is said to be reductive in G

if $\#\{\text{conn. comp of } H\} \lneq \infty$

and $\exists N \in \mathbb{N}$

$\exists G_0 \subset GL(N;\mathbb{R})$:

$\exists H_0 \subset G_0$: closed subgroups

with $\theta(G_0) = G_0$, $\theta(H_0) = H_0$

s.t. $(G, H) \simeq (G_0, H_0)$

Def 7.2.2 : A homogeneous G - space X is said to be a reductive type if the isotropies are reductive in G.

Ex 7.2.3 :

$O(p+1, q)$: a linear reductive Lie group

$X(p, q)$: a homogeneous $O(p+1, q)$ - space of reductive type.

Theorem 7.2.4

Any homogeneous G-space of reductive type

admits a G-invariant pseudo-Riem str.

Ex 7.2.5 :

$X(p,q)$ admits a $O(p+1,q)$-inv (p,q)-metric with constant sectional curvature

(a space form)

§7.3 : Riemannian symmetric spaces.

Let G be a linear reductive Lie group

Theorem 7.3.1 : Let K be a subgroup of G.

(i) K is a maximal compact subgroup of G

$\Updownarrow$

(ii) $\exists N \in \mathbb{N}$.

$\exists G_0 < GL(N:\mathbb{R})$

$K_0 := \{ g \in G_0 \mid \theta(g) = g \} < G_0$

s.t. $(G, K) \simeq (G, K_0)$

Def 7.3.2 : In this course,

we say that a homogeneous G-space M is a Riemannian symmetric space if its isotropies are maximal compact subgroup of G.

Prop 7.3.3: Let M be a Riem. symm. sp.

(1) M is diffeomorphic to an Euclidean space.

(2) M admits a G-invariant Riem. str.

(not unique but

the Levi-Civita connection is unique)