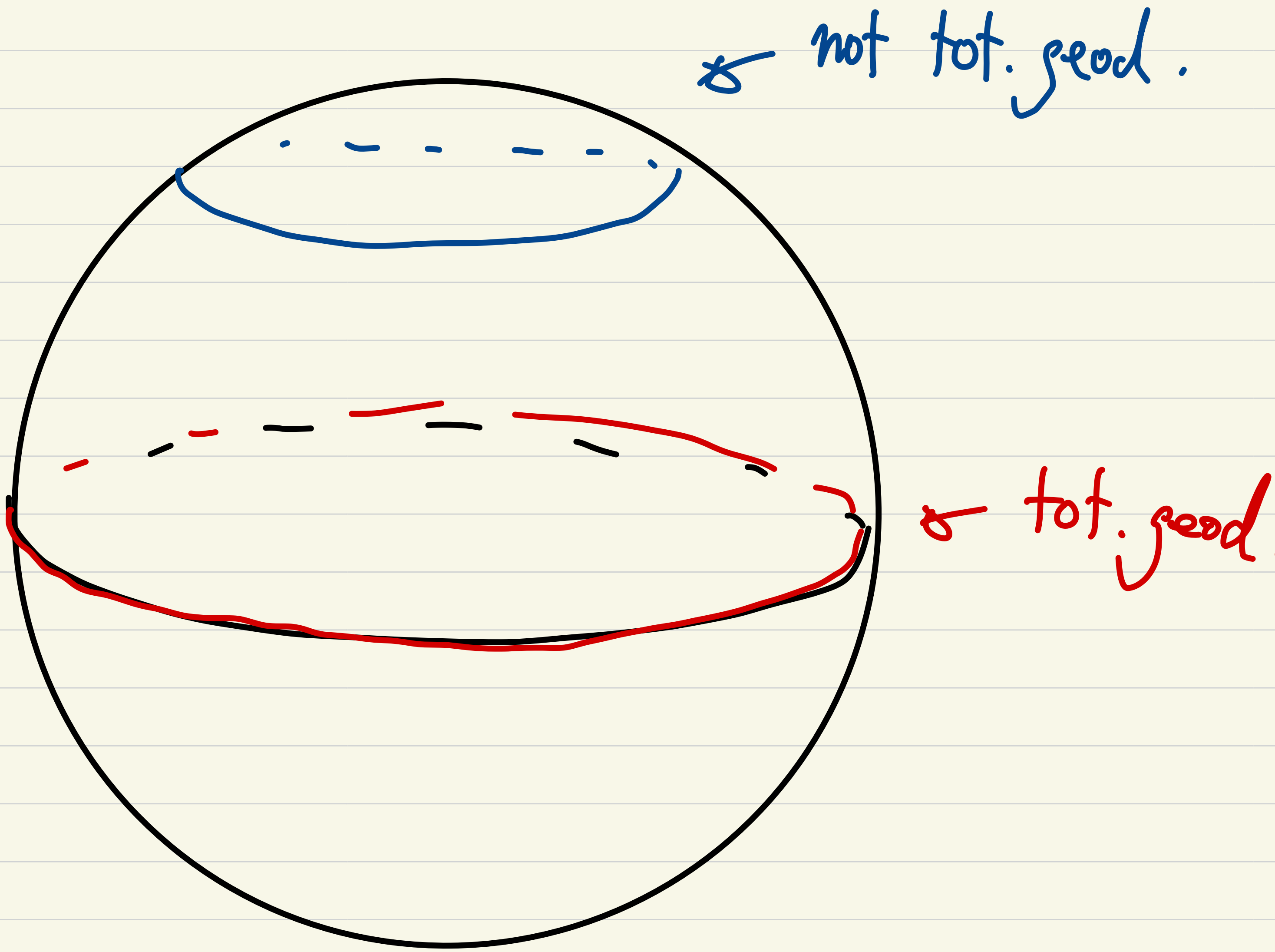
§ 8 Totally geodesic submanifolds and homogeneous spaces of reductive type

§8.1 Totally geodesic submanifolds

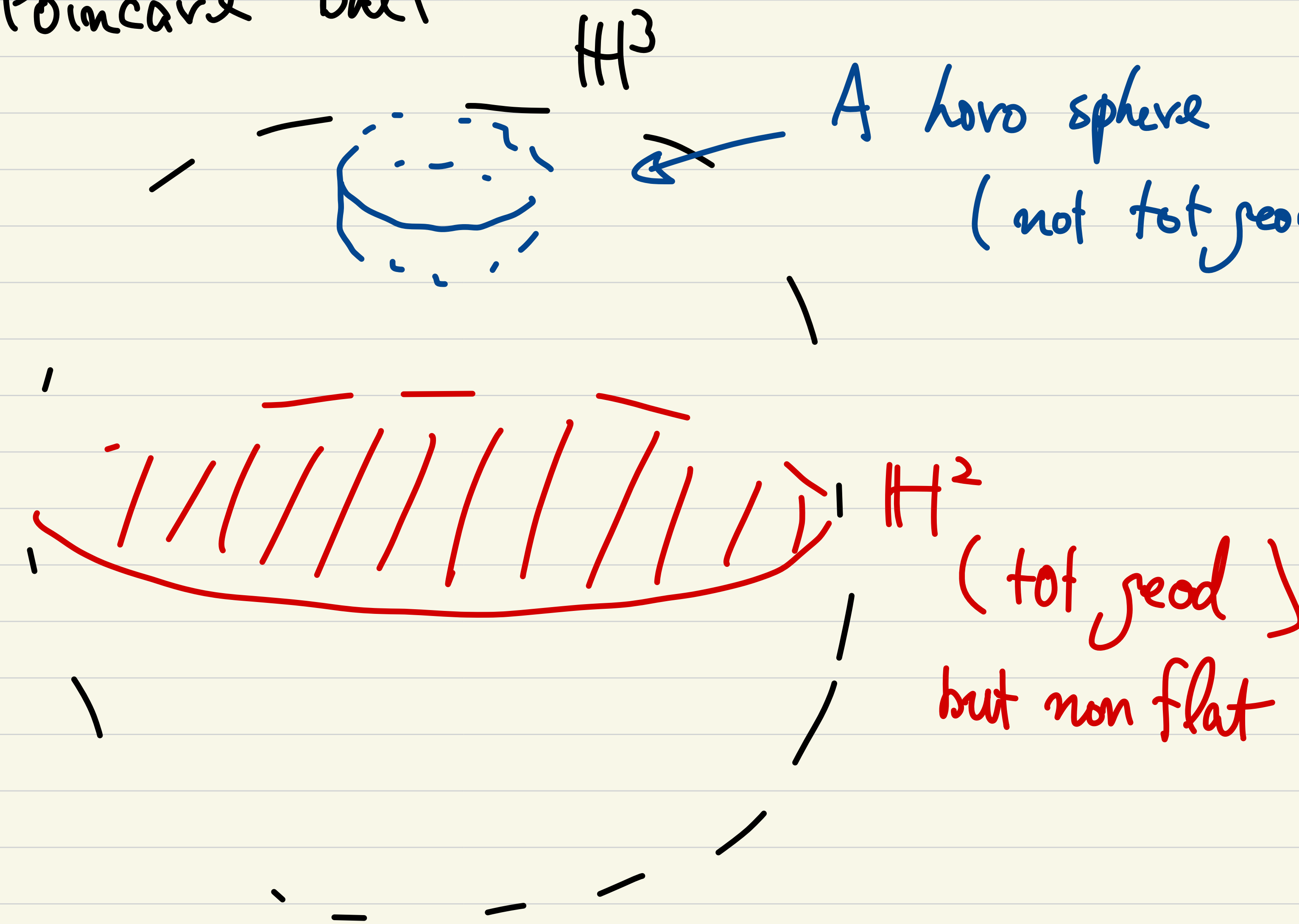
Setting : M a Riem. mfd.

Def 8.1.1 : A closed complete submanifold D in M is said to be totally geodesic if any geodesic in D is also a geodesic in M

Ex



Ex Poincáre ball



§8.2 Homogeneous spaces of reductive type realized as families of totally geodesic submanifolds

Setting : G : a linear reductive Lie group

M : a Riem. symmetric space equipped with a G-inv. Riem. str.

Theorem 8.2.1

Let D be a totally geodesic submfd in M.

Then

$$H_D := \{ g \in G \mid gD = D \}$$

is reductive in G

and

D is homogeneous

For each tot. geod D in M

we put $X_D := \{ gD \mid g \in G \}$.

Then X_D is a homogeneous G-space

of reductive type

Theorem 8.2.2

Let X be a homogeneous G-space of reductive type.

Then

$\exists\, D \subset M$: tot. geod.

$\exists\, f : X \twoheadrightarrow X_D$: $\overset{\text{conti. surjective}}{\text{proper}}$ G-equivariant

Prop 8.2.3

L : a topological group

X, X' : a topological space

$L \curvearrowright X \curvearrowright X'$: continuous actions

Assume $\exists f : X \twoheadrightarrow X'$: conti proper L-equiv map

Then (i) $L \curvearrowright X$: proper

$\Updownarrow$

(ii) $L \curvearrowright X'$: proper

We obtained the following

Setting : G : a linear reductive

M : a Riem. symm space

X : a homog space of red. type

D : a tot. geod in M

with $^\exists f : X \twoheadrightarrow X_D$: proper G-eq.

Theorem 8.2.4

Let L be a closed subsp of G

(i) $L \curvearrowright X$: proper

$\Updownarrow$

(ii) $L \curvearrowright X_D$: proper

$\Updownarrow$

(iii) ${}^\forall C_u \subset M$: compact

$$L(C_u ; X_D) := \left\{ g \in L \;\middle|\; \begin{array}{c} {}^\exists D' \in X_D \\ \text{s.t.} \\ C_u \cap D' \neq \emptyset \\ g C_u \cap D' \neq \emptyset \end{array} \right\}$$

is compact.