Section 9 Kobayashi's properness criterion.

§9.1 : A geometric translation of Kobayashi's properness criterion

Setting : G : a linear reductive

M : a Riem. symm space

X : a homog space of red. type

D : a tot. geod in M

with $^{\exists} f : X \twoheadrightarrow X_D$: proper G-eq.

Theorem 9.1.1 (Kobayashi's properness criterion)

Let L be a closed subgp of G. Then

(iii) $\Leftrightarrow$ (i) $\Leftrightarrow$ (ii) in Thm 8.2.4.

$\Updownarrow$

(iv) $\forall p \in M$, $\forall C_M \subset M$: compact with $p \in C_M$. ⊛ below holds.

$\Updownarrow$

(v) $\exists p \in M$, $\forall C_M \subset M$: compact with $p \in C_M$ ⊛ below holds.

$\boxed{\text{Non-trivial implication: (v)} \Rightarrow \text{(iii)}}$

$$L(p, C_n ; X_D)$$

$$:= \{ g \in L \mid {}^{\exists} D' \in X_D ,\ p \in D' ,\ g C_n \cap D' \neq \emptyset \}$$

is compact.

Cor 9.2. Let Γ be a torsion-free discrete subgroup.

We write $\pi : M \to \Gamma \backslash M$ for the quotient map.

Then

(1) $\Gamma \curvearrowright X$ is properly-discontinuous

(2) $\forall p \in \Gamma \backslash X$, $\forall r > 0$, ☆☆ below holds

(3) $\exists p \in \Gamma \backslash X$, $\forall r > 0$. ☆☆ below holds

(★★) $^{\exists} l_0 \in \mathbb{R}_{>0}$ s.t.

$$\neg \Big({}^{\exists}\text{geodesic } \gamma \text{ in } M,\ {}^{\exists} l > l_0$$

s.t. $^{\exists} D' \in X_D$ with γ is a geodesic in D',

$\pi(\gamma(0)) = p$, and

$$d_{p\backslash M}(\pi(\gamma(l)), p) < r \Big)$$

The dist. fct on $p\backslash M$