ある Cell polarization model の数学的解析について

０．研究の背景

１．細胞極性モデル

２．極限方程式について

３．数値計算による大域的分岐図と安定性の結果

４．大域的分岐シートとその性質

５．大域的分岐シート作成のアイデア

６．大域的分岐シートの性質の数学的証明

研究の背景

Y. Mori, A. Jilkine and L. Edelstein-Keshet,
Asymptotic and bifurcation analysis of wave-pinning in a reaction-diffusion model for cell polarization
SIAM J, Appl. Math 71(2011), 1401-27.

モデルの提示
非線形項を双安定な
３次式に置き換え,
極限方程式（D→∞）
の数値計算による
分岐構造の研究

K. Kuto and T. Tsujikawa,
Bifurcation structure of steady-states for bistable equations with nonlocal constraint
DCDS Supplement 2013, 467-476.

定常極限方程式の分岐構造の
数学的研究

分岐点および特異極限の近傍での
解形状と分岐曲線の
概要に関する数学的結果

森竜樹，久藤衡介，辻川亨，四ツ谷晶二，
日本数学会2013 年度秋季総合分科会
函数方程式論分科会アブストラクト.

全ての解の表示式を求め,
数値的に,
二次分岐以降も含む,
大域的分岐構造を解明した

↑ 変数変換 Cahn-Hilliard方程式

S. Kosugi, Y. Morita and S. Yotsutani,
Stationary solutions to the one-dimensional Cahn-Hilliard equation: Proof by the complete elliptic integrals,
Discrete and Continuous, Dynamical Systems 19(2007), 609-629.

研究の目的

- 細胞運動モデルの極限方程式の，表示式を用いて
  1次分岐のみならず2次分岐以上等のすべてを含む,
  大域的分岐構造を数学的に証明したい.

- 極限方程式の分岐構造を元に拡散係数が十分大のときの
  細菌運動モデルの定常解の極限形状を数学的に証明したい.

- 各分岐曲線の安定性を数学的に証明したい.

細胞極性モデル

Y. Mori, A. Jilkine and L. Edelstein-Keshet,
Asymptotic and bifurcation analysis of wave-pinning in a reaction-diffusion model for cell polarization
SIAM J, Appl. Math 71(2011), 1401-27.

非線形項を双安定な３次式に置き換え,

極限方程式（D→∞）の数値計算による分岐構造の研究

Fig.1 in Y. Mori, A. Jilkine and L. Edelstein-Keshet,
Biophys. J., 94 (2008), pp. 3684-97.

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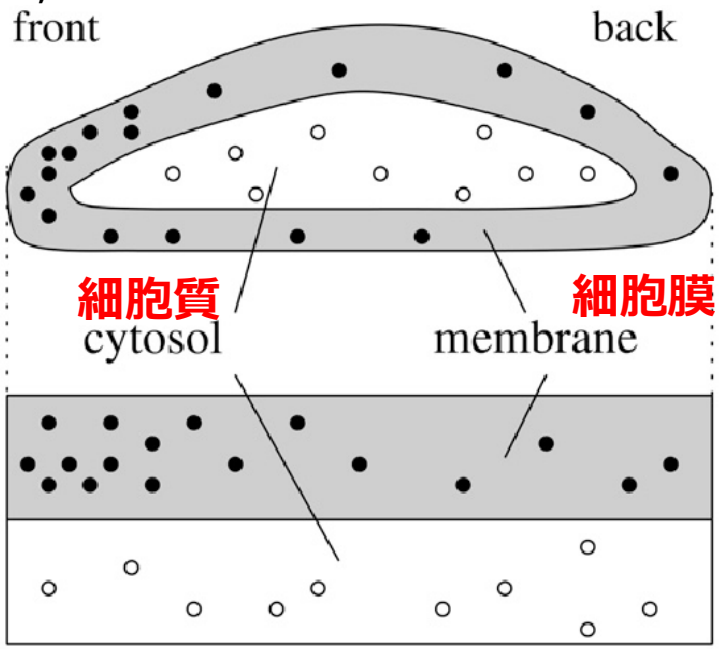

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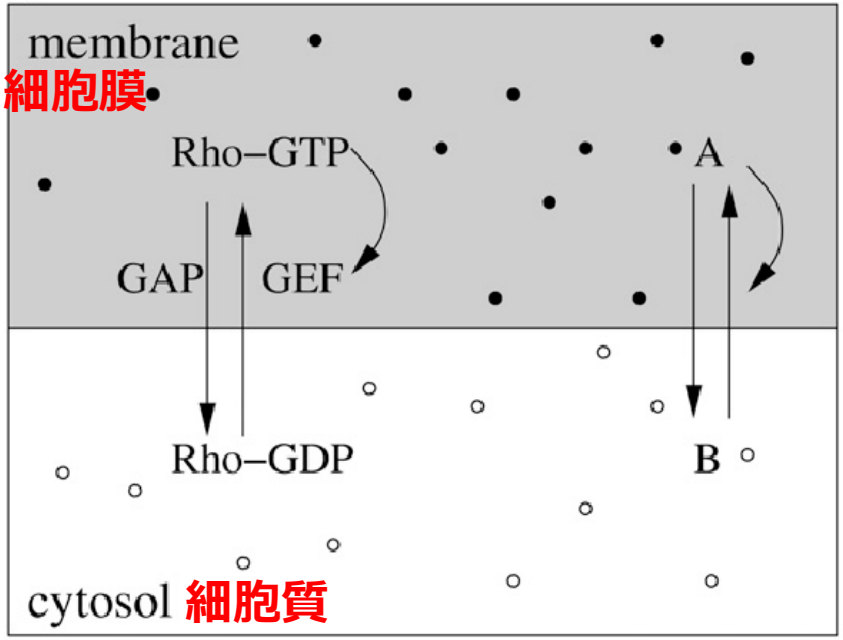

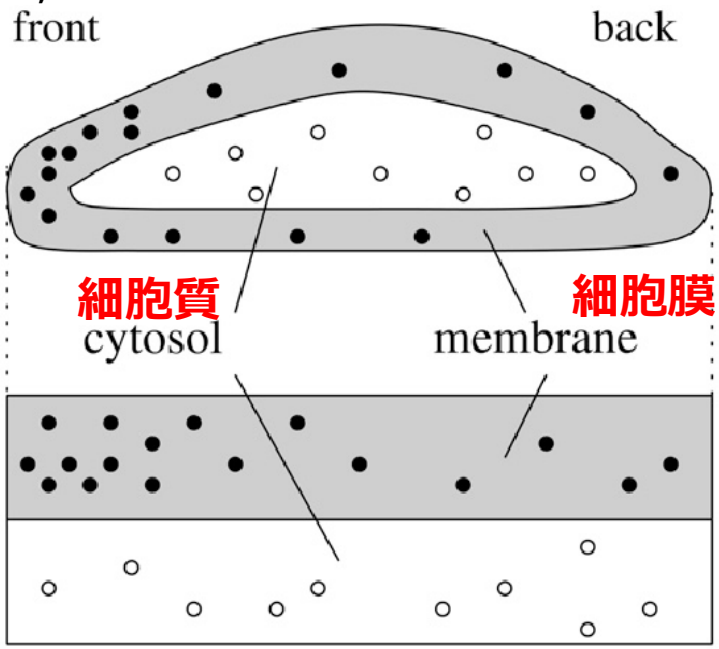

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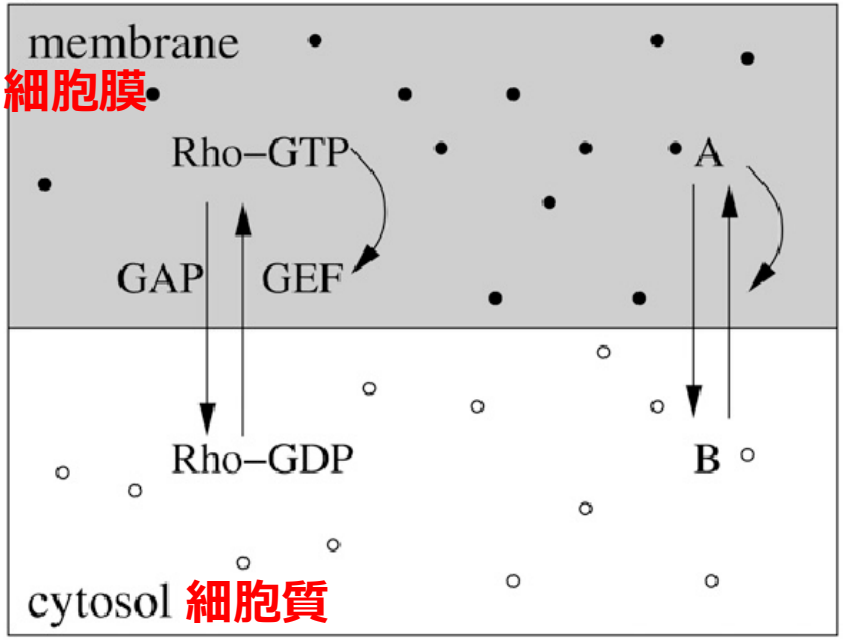

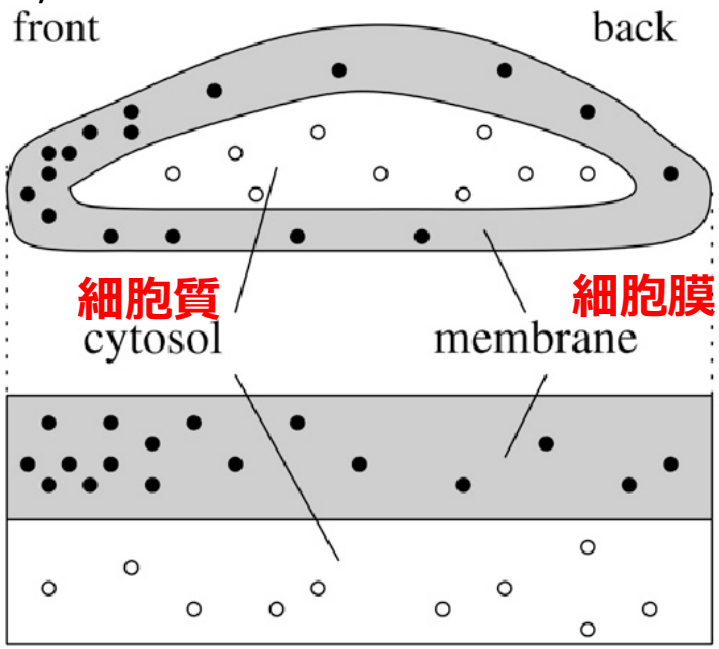

A： 活性たんぱく質 (Rho-GTP, ●)
B：不活性たんぱく質 (Rho-GDP, ○)

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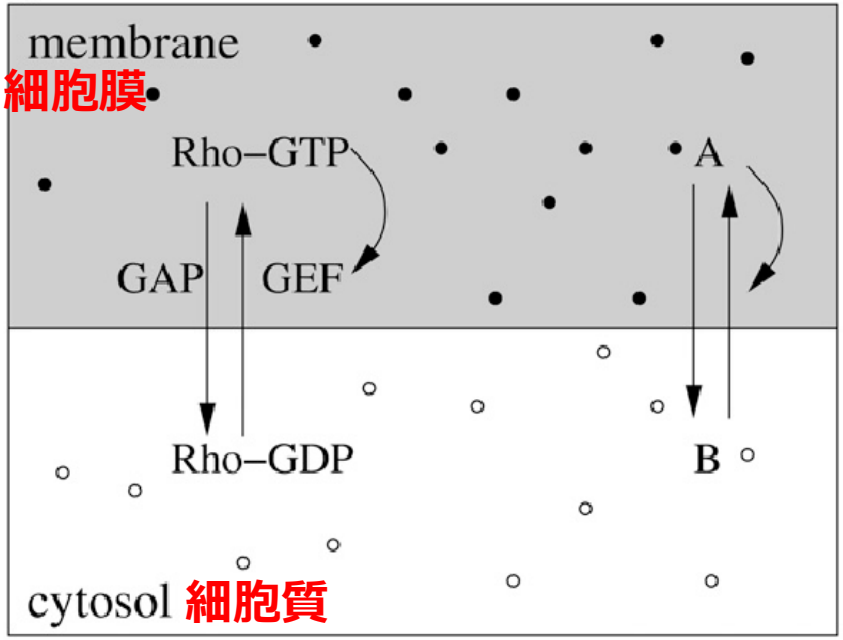

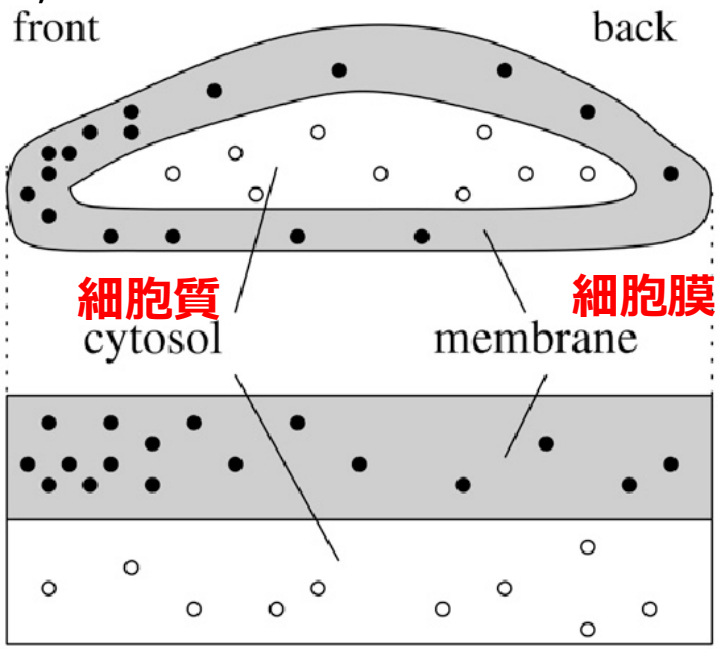

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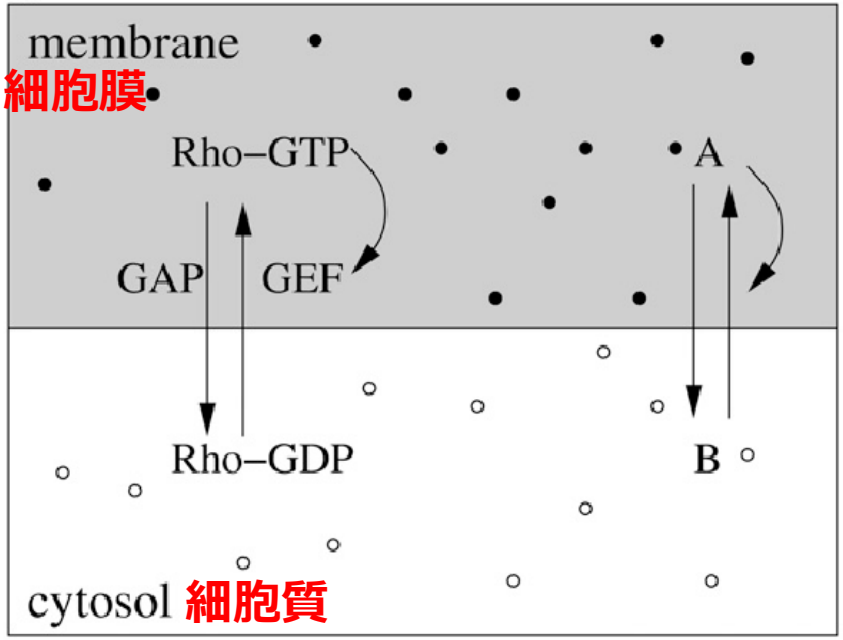

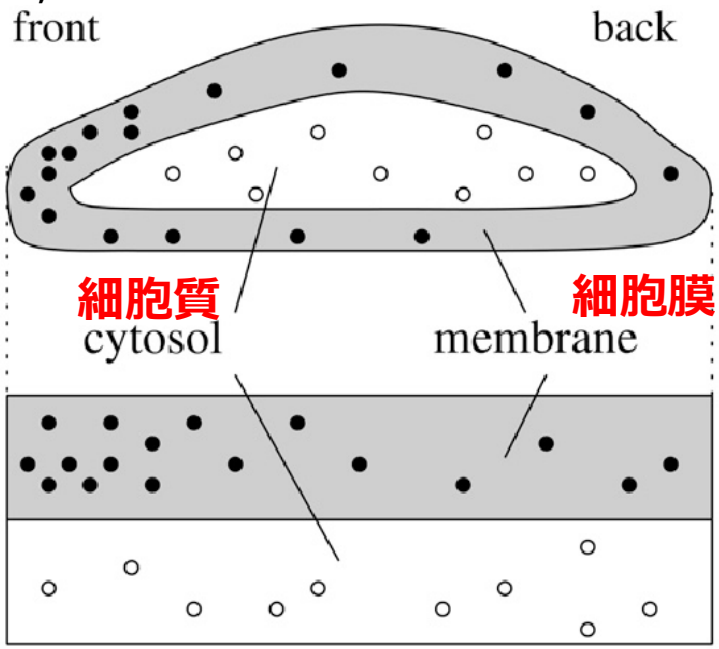

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不活性たんぱく質の拡散係数は,活性たんぱく質よりも,
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不活性たんぱく質の濃度は,ほぼ一様と見られる.

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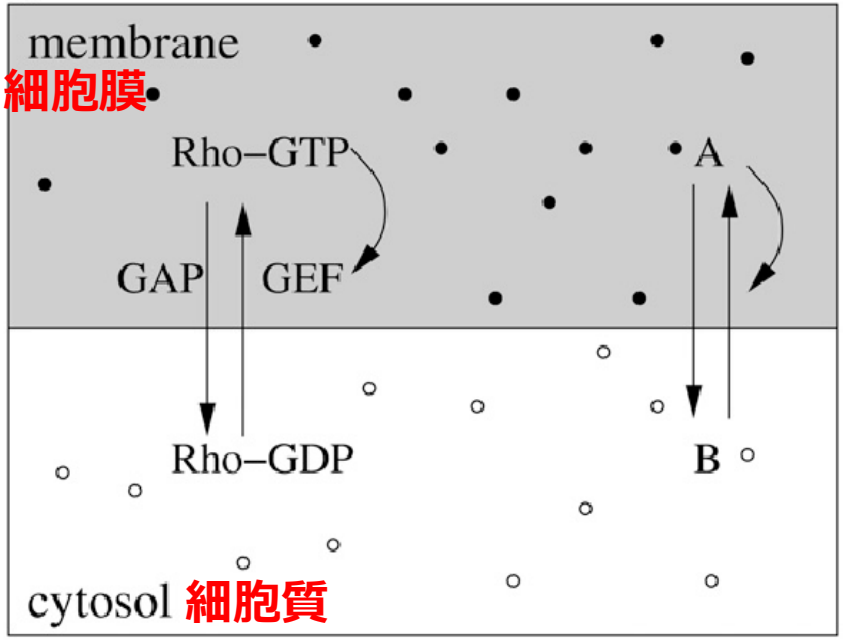

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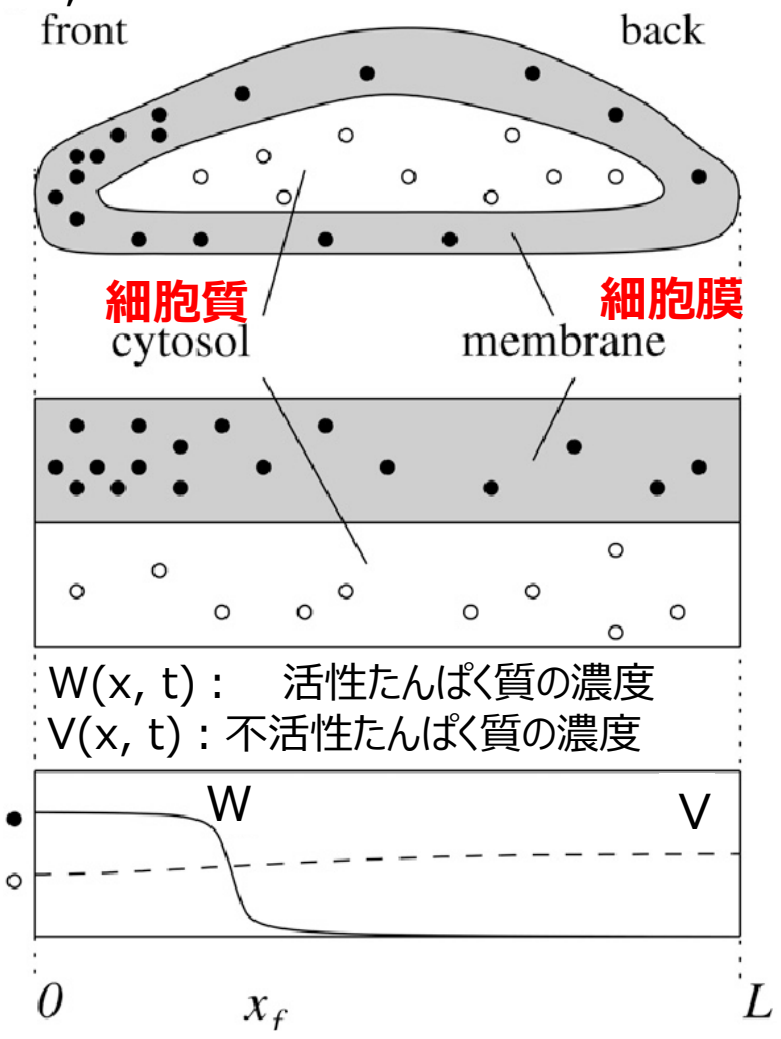

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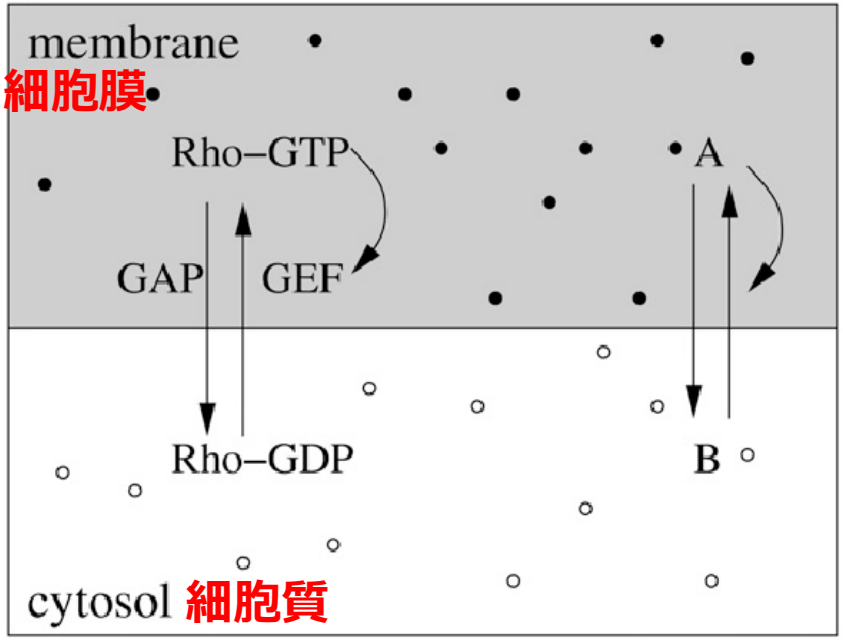

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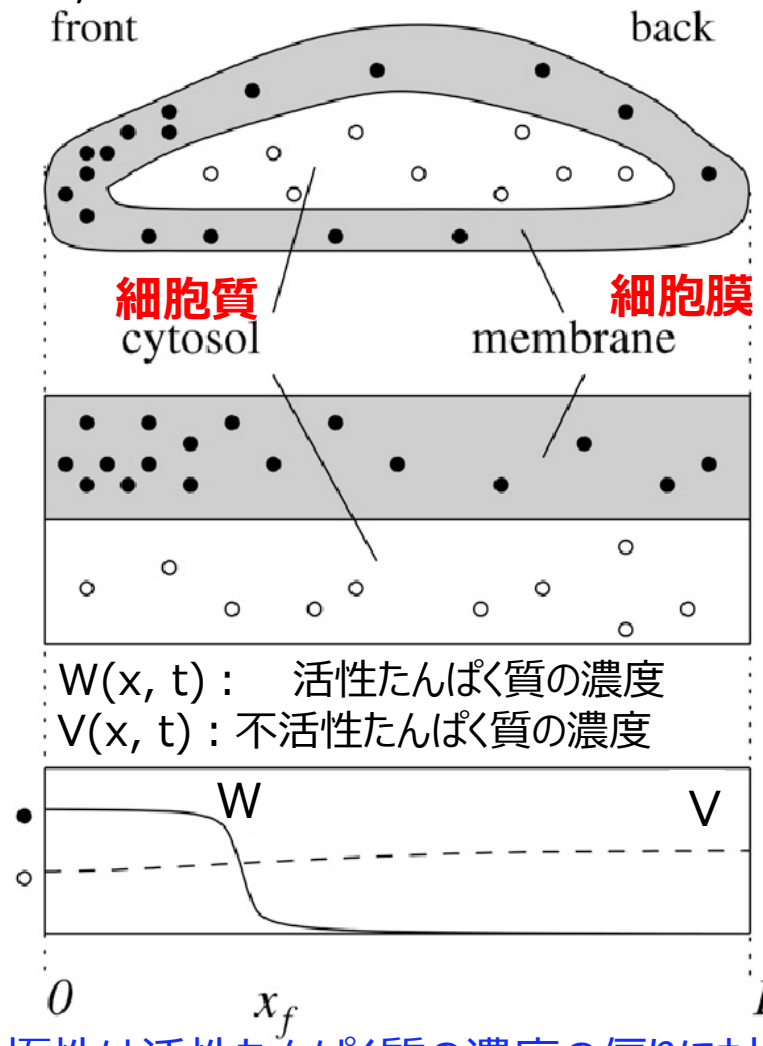

極性は活性たんぱく質の濃度の偏りに対応.

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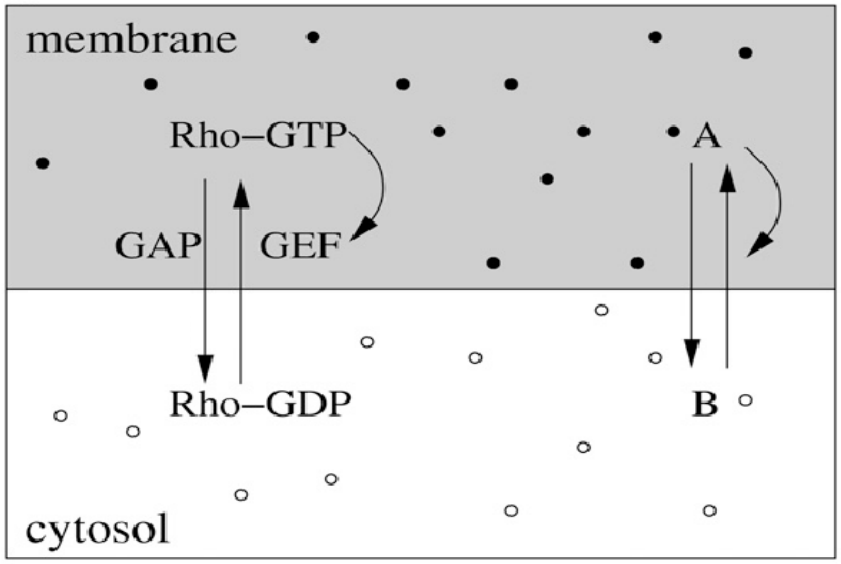

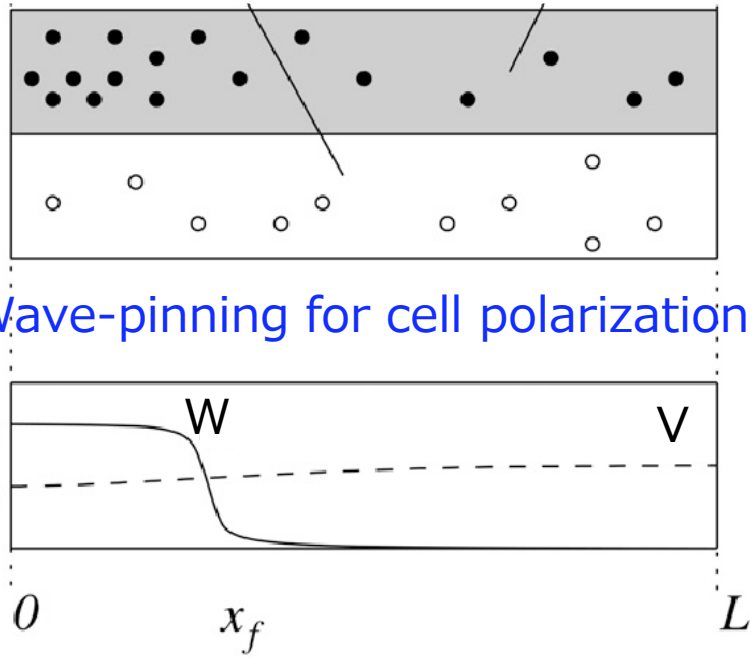

Wave-pinning for cell polarization

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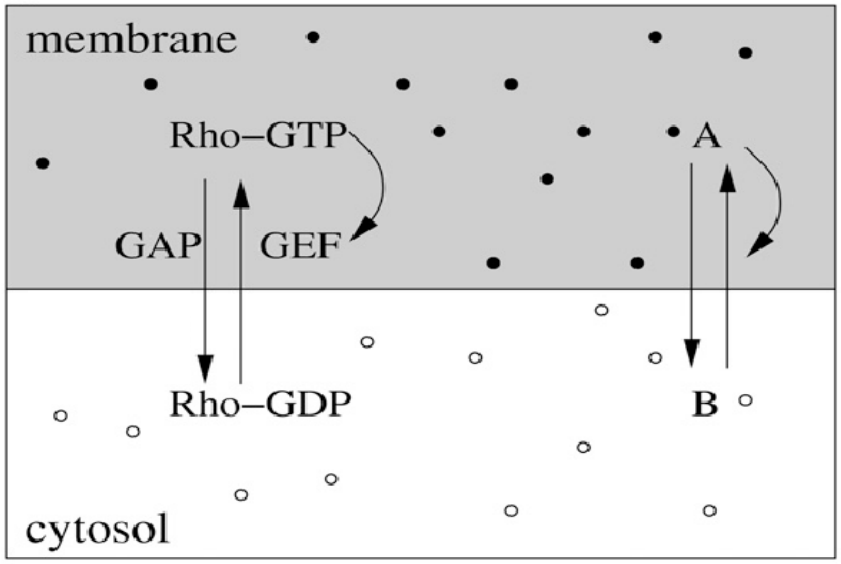

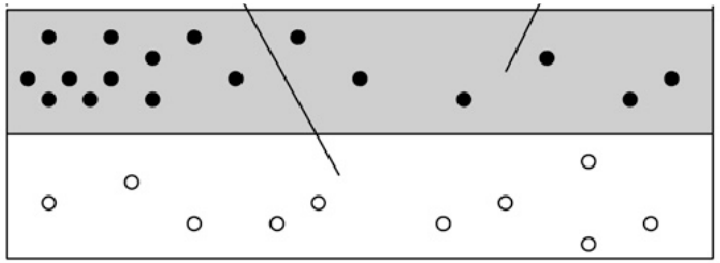

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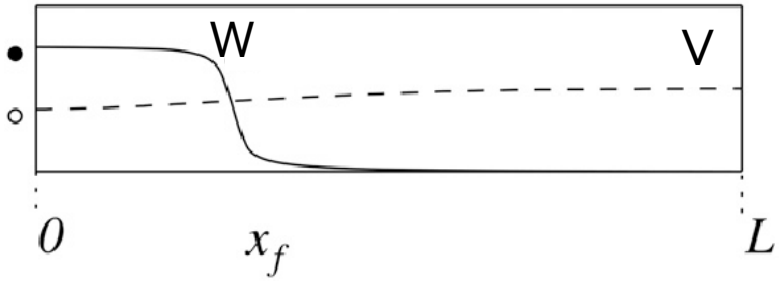

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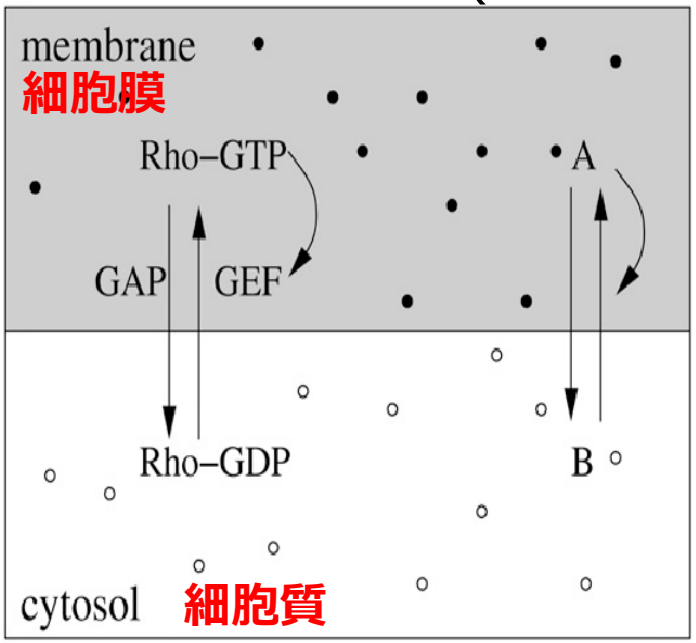

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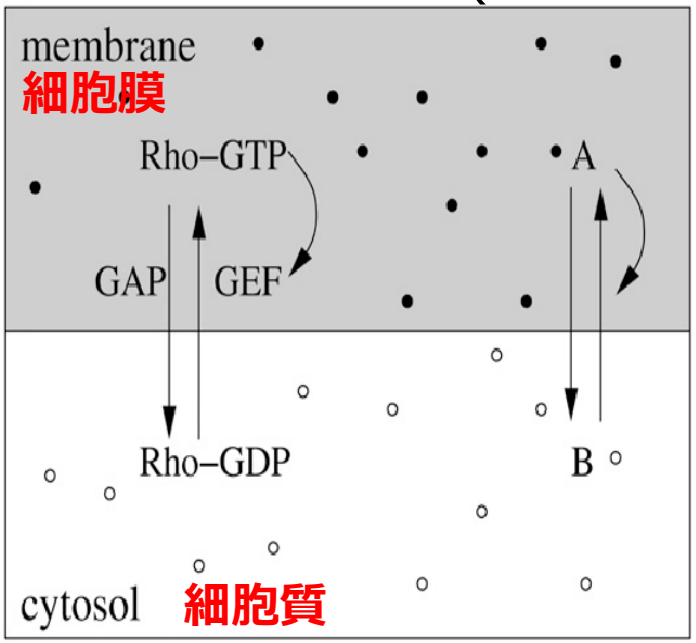

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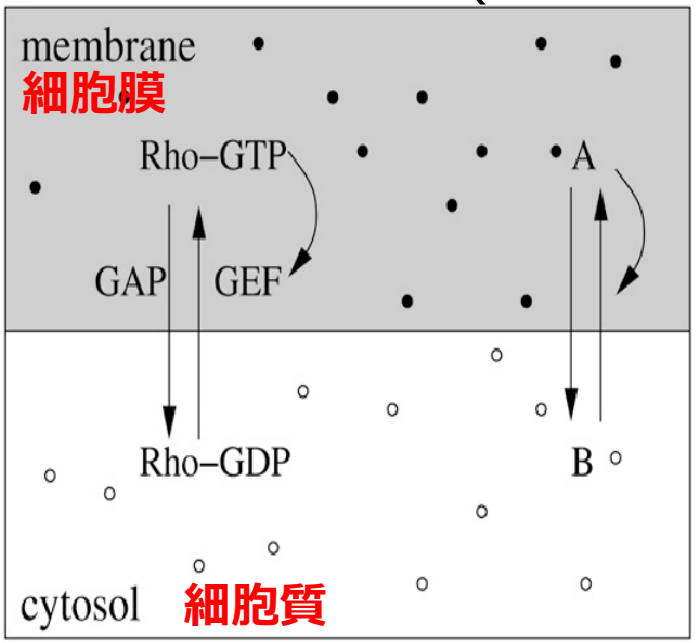

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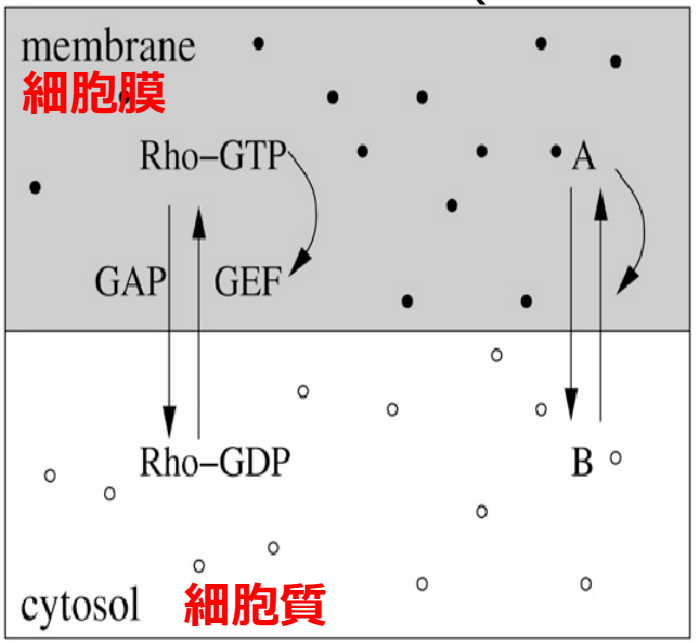

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$f(W,V)$

＝(活性化速度)・V － (不活性化速度)・W

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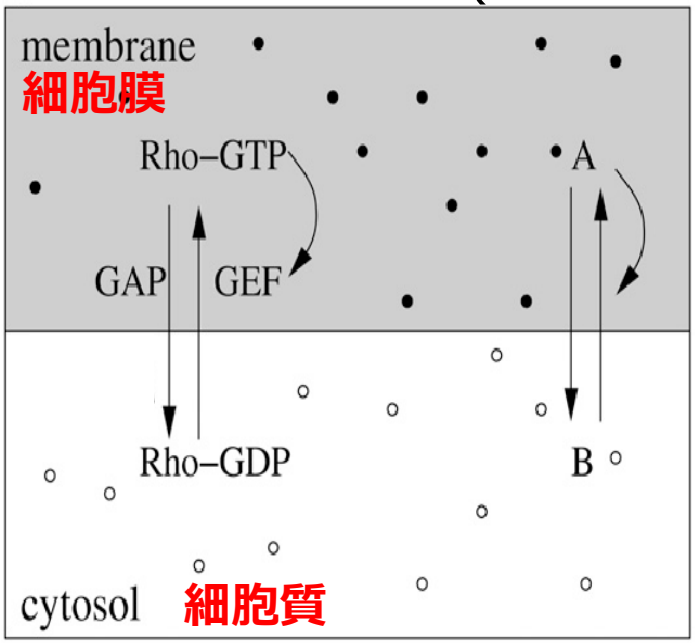

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(第 1 項について V → W)　GEFs で活性化される速度.
生物学的には十分に解読されていないが、
Vが多ければ多いほどVがWに変化するというプラスのフィードバックがある
ということが経験的に知られている.
活性化速度はプラスのフィードバックがあるとしてヒル関数と仮定する.

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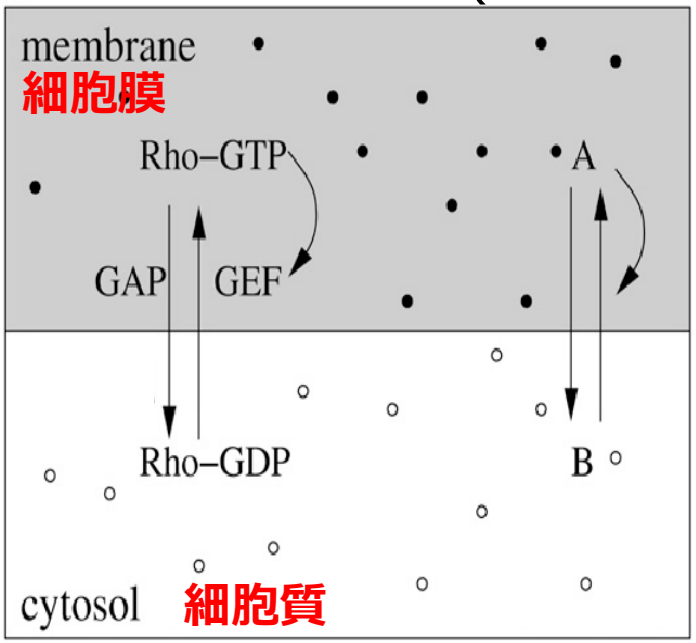

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(第 2 項について, W → V) GAPsで不活性化される速度は単純に定数とする.

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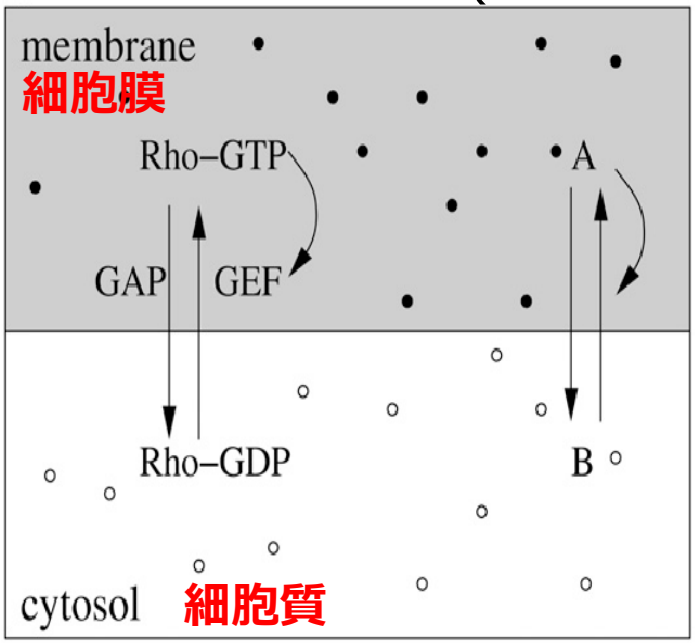

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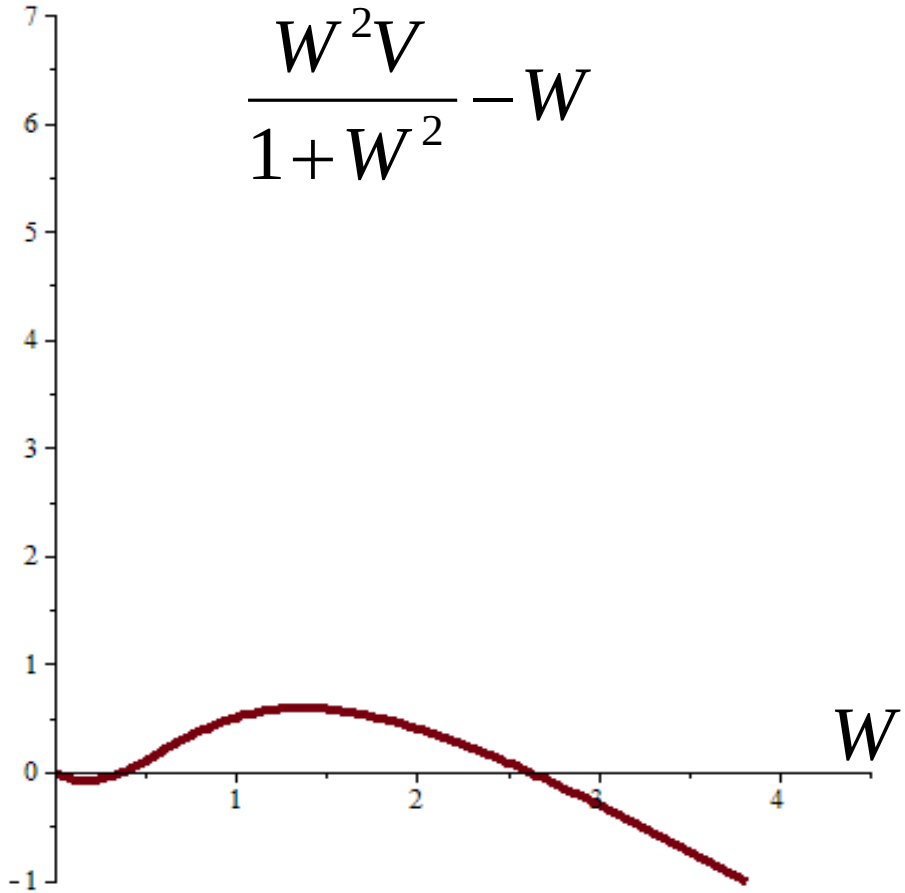

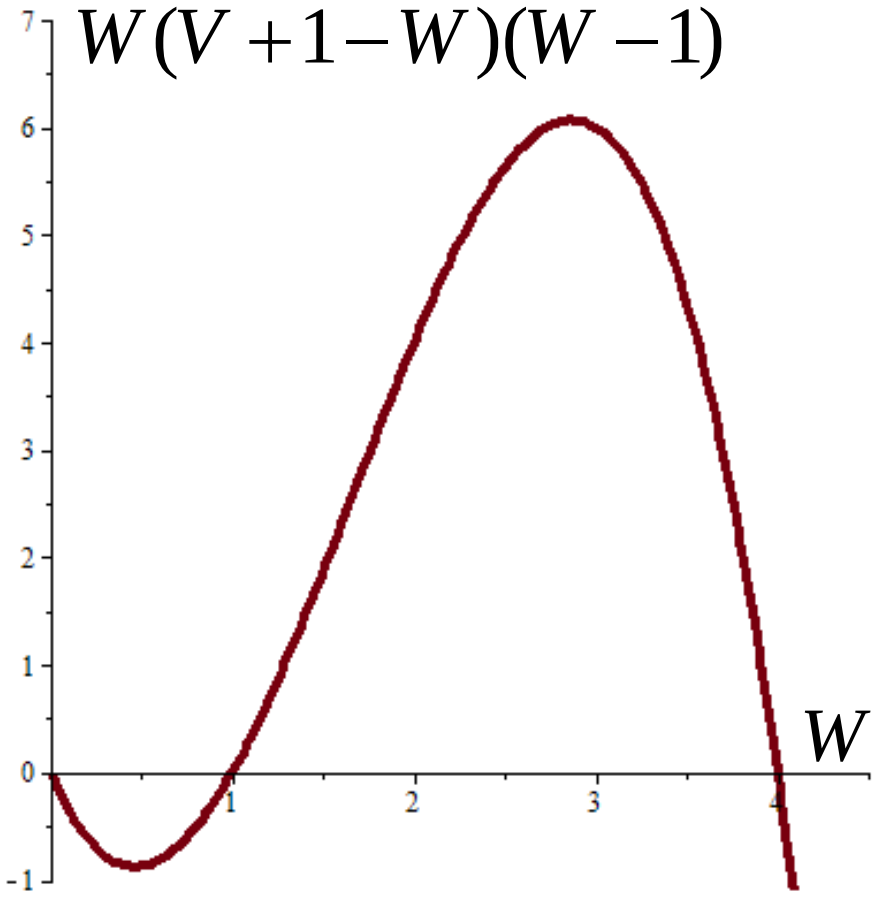

$\frac{W^2V}{1+W^2}-W$ ：双安定

$W(V+1-W)(W-1)$……数学的考察が行いやすい、多項式

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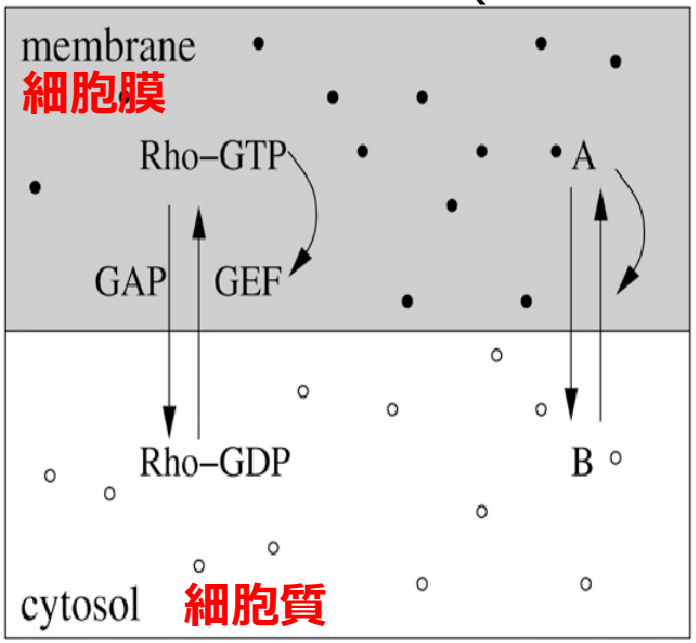

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数学的な解析のため
非線形項を置き換えた.

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Y.Mori, A. Jilkine and L. Edelstein-Keshet,
SIAM J, Appl. Math 71(2011), 1401-1427.

細胞極性モデル

$$(TP)\begin{cases} \varepsilon W_t = \varepsilon^2 \Delta W + W(V+1-W)(W-1), \\ \varepsilon V_t = D\Delta V - W(V+1-W)(W-1), \\ W_x(0,\,t) = W_x(1,\,t) = V_x(0,\,t) = V_x(1,\,t) = 0, \\ W(x,0) = W_0(x),\ V(x,\,0) = V_0(x). \end{cases}$$

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質量保存

$$\frac{d}{dt}\left\{\int_0^1 \big(W(x,t)+V(x,t)\big)\mathrm{dx}\right\} = 0$$

$$\therefore \int_0^1 \big(W(x,t)+V(x,t)\big)\mathrm{dx} = \int_0^1 \big(W_0(x)+V_0(x)\big)\mathrm{dx} = \mathrm{m}.$$

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$$(TP)\begin{cases} \varepsilon W_t = \varepsilon^2 \Delta W + W(V+1-W)(W-1), \\ \varepsilon V_t = D\Delta V - W(V+1-W)(W-1), \\ W_x(0,t) = W_x(1,t) = V_x(0,t) = V_x(1,t) = 0, \\ W(x,0) = W_0(x),\ V(x,0) = V_0(x). \end{cases}$$

$D \to \infty$の時間発展問題の極限方程式

$$(TLP)\begin{cases} \varepsilon \dfrac{\partial}{\partial t} W(x,t) = \varepsilon^2 \Delta W(x,t) + W(x,t)\left(\tilde{V}(t)+1-W(x,t)\right)\left(W(x,t)-1\right), \\ \varepsilon \dfrac{d}{dt}\tilde{V}(t) = -\displaystyle\int_0^1 W(x,t)\left(\tilde{V}(t)+1-W(x,t)\right)\left(W(x,t)-1\right)\mathrm{dx}, \\ W_x(0,t) = W_x(1,t) = 0, \\ W(x,0) = W_0(x),\ \tilde{V}(0) = \tilde{V}_0. \end{cases}$$

$$(CP)\begin{cases} \varepsilon W_t = \varepsilon^2 \Delta W + W(V+1-W)(W-1), & (1) \\ \varepsilon V_t = D\Delta V - W(V+1-W)(W-1), & (2) \\ W_x(0,t) = W_x(1,t) = V_x(0,t) = V_x(1,t) = 0, & (3) \\ W(x,0) = W_0(x),\ V(x,0) = V_0(x). & (4) \end{cases}$$

(2)式の両辺をxで積分する.

$$\varepsilon \frac{\partial}{\partial t}\int_0^1 V(x,t)\,\mathrm{dx} = D\int_0^1 \Delta V(x,t)\,\mathrm{dx} - \int_0^1 W(x,t)\big(V(x,t)+1-W(x,t)\big)\big(W(x,t)-1\big)\mathrm{dx}$$

$$\varepsilon \frac{\partial}{\partial t}\int_0^1 V(x,t)\,\mathrm{dx} = - \int_0^1 W(x,t)\big(V(x,t)+1-W(x,t)\big)\big(W(x,t)-1\big)\mathrm{dx}$$

$D\to\infty$ $V(x,t)\to\tilde{V}(t)$ とする.

$$\varepsilon \frac{d}{dt}\tilde{V}(t) = -\int_0^1 W(x,t)\big(\tilde{V}(t)+1-W(x,t)\big)\big(W(x,t)-1\big)\mathrm{dx}$$

このとき(1)式は

$$\varepsilon \frac{\partial}{\partial t}W(x,t) = \varepsilon^2 \Delta W(x,t) + W(x,t)\big(\tilde{V}(t)+1-W(x,t)\big)\big(W(x,t)-1\big)$$

以上より $D\to\infty$ の時間発展問題を得る.

時間発展問題の極限方程式（ $D \to \infty$ ）

$$
(TLP)\begin{cases}
\varepsilon \dfrac{\partial}{\partial t} W(x,t) = \varepsilon^2 \Delta W(x,t) + W(x,t)\left(\tilde{V}(t) + 1 - W(x,t)\right)\left(W(x,t) - 1\right), \\
\displaystyle\int_0^1 W(x,t)\,\mathrm{dx} + \tilde{V}(t) = \int_0^1 W_0(x)\,\mathrm{dx} + \tilde{V}_0 = \mathrm{m}, \\
W_x(0,\ t) = W_x(1,\ t) = 0, \\
W(x,0) = W_0(x),\ \ \tilde{V}(0) = \tilde{V}_0.
\end{cases}
$$

定常問題

$$
(SP)\begin{cases}
\varepsilon^2 \Delta W + W(V+1-W)(W-1) = 0, & x \in (0,1), \\
D\Delta V - W(V+1-W)(W-1) = 0, & x \in (0,1), \\
W_x(0) = W_x(1) = V_x(0) = V_x(1) = 0, & \\
W(x) > 0,\ V(x) > 0, & \\
\int_0^1 \big(W(x)+V(x)\big)\mathrm{dx} \left(= \int_0^1 \big(W_0(x)+V_0(x)\big)\mathrm{dx} \right) = m. &
\end{cases}
$$

定常極限方程式（ $D \to \infty$ ）

$$
(SLP)\begin{cases}
\varepsilon^2 \Delta W + W(\tilde{V}+1-W)(W-1) = 0, & x \in (0,1), \\
W_x(0) = W_x(1) = 0, & \\
W(x) > 0, \quad \tilde{V} > 0, & \\
\int_0^1 W(x)\,\mathrm{dx} + \tilde{V} \left(= \int_0^1 W_0(x)\,\mathrm{dx} + \tilde{V}_0 \right) = m. &
\end{cases}
$$

定常問題

$$
(SP)\begin{cases}
\varepsilon^2 \Delta W + W(V+1-W)(W-1) = 0, & x \in (0,1), \\
D\Delta V - W(V+1-W)(W-1) = 0, & x \in (0,1), \\
W_x(0) = W_x(1) = V_x(0) = V_x(1) = 0, & \\
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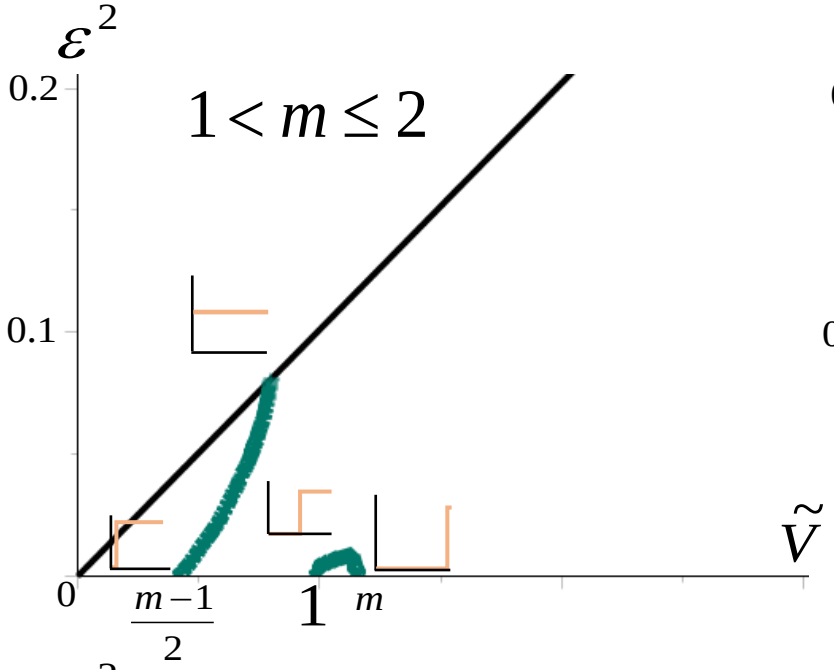
\int_0^1 \big(W(x)+V(x)\big)\mathrm{dx} \left(= \int_0^1 \big(W_0(x)+V_0(x)\big)\mathrm{dx} \right) = m.
\end{cases}
$$

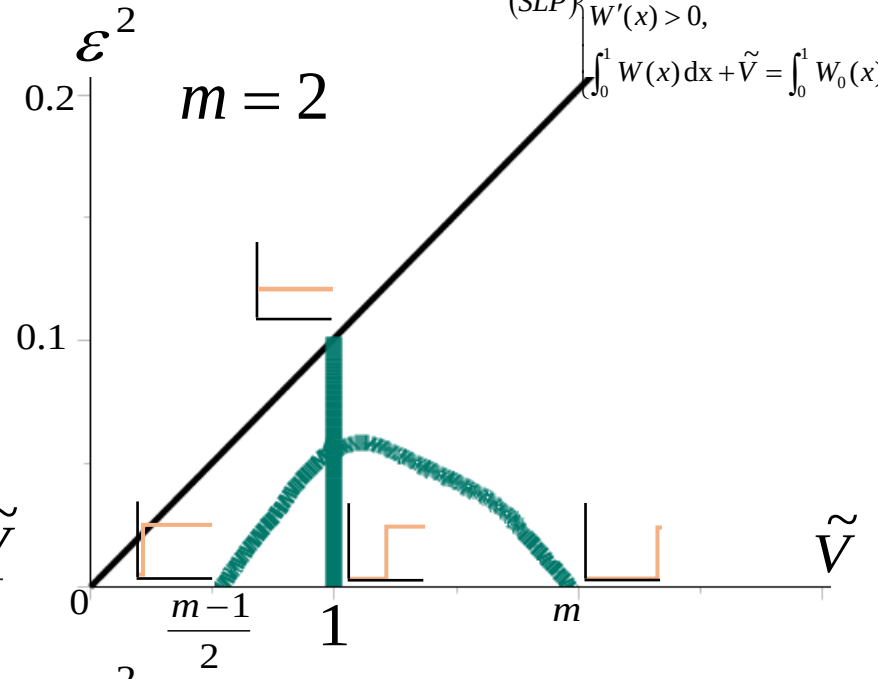
定常極限方程式（ $D \to \infty$ ）

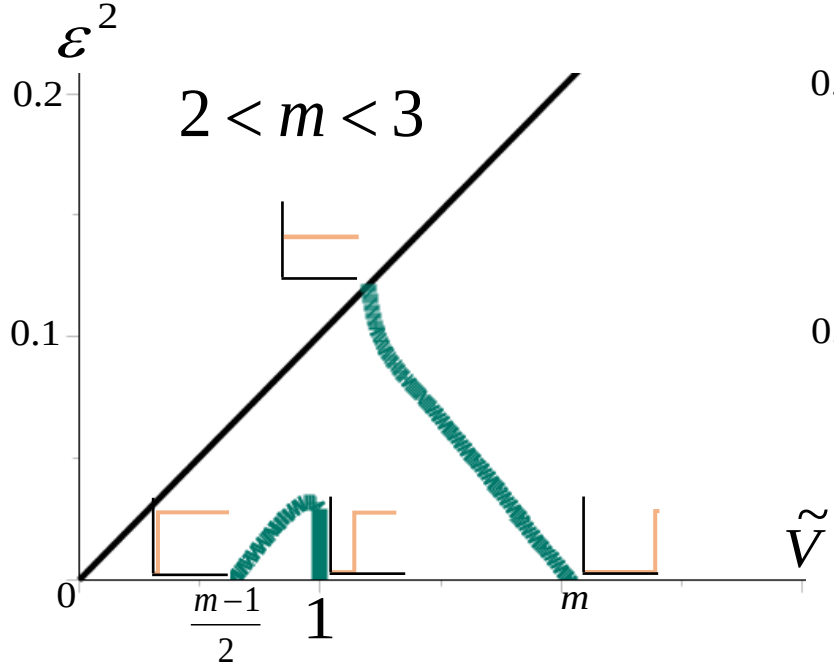
$$
(SLP)\begin{cases}
\varepsilon^2 \Delta W + W(\tilde{V}+1-W)(W-1) = 0, & x \in (0,1), \\
W_x(0) = W_x(1) = 0, & \text{簡単のため単調増加なものを考える} \\
W(x) > 0, \quad W'(x) > 0, \quad \tilde{V} > 0, & \\
\int_0^1 W(x)\,\mathrm{dx} + \tilde{V} \left(= \int_0^1 W_0(x)\,\mathrm{dx} + \tilde{V}_0 \right) = m.
\end{cases}
$$

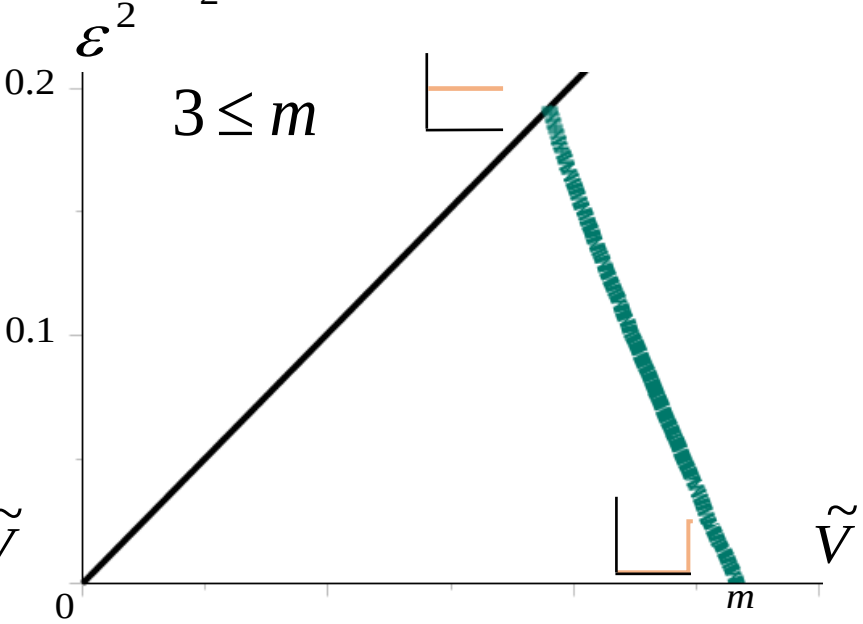
(SLP)の分岐曲線

$$(SLP)\begin{cases}\varepsilon^2\Delta W + W(\tilde{V}+1-W)(W-1)=0,\\ W_x(0)=W_x(1)=0,\\ W'(x)>0,\\ \int_0^1 W(x)\,dx+\tilde{V}=\int_0^1 W_0(x)\,dx+\tilde{V}_0=m.\end{cases}$$









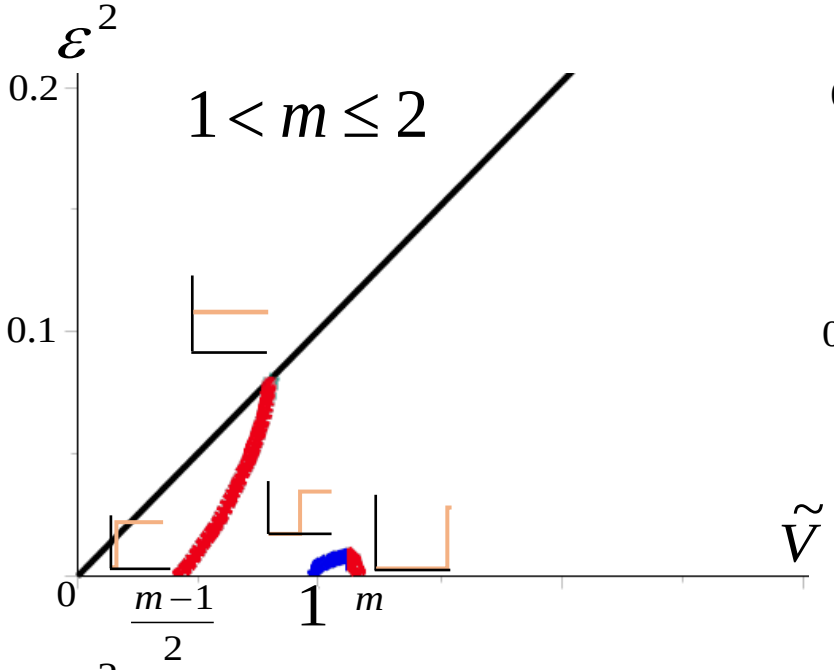
(SLP)の安定性

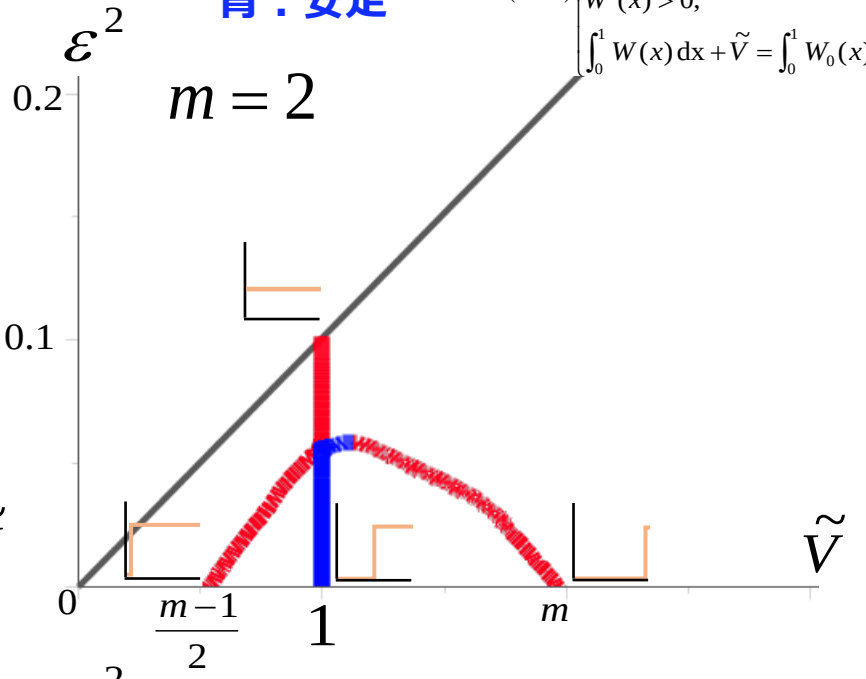
数値計算結果

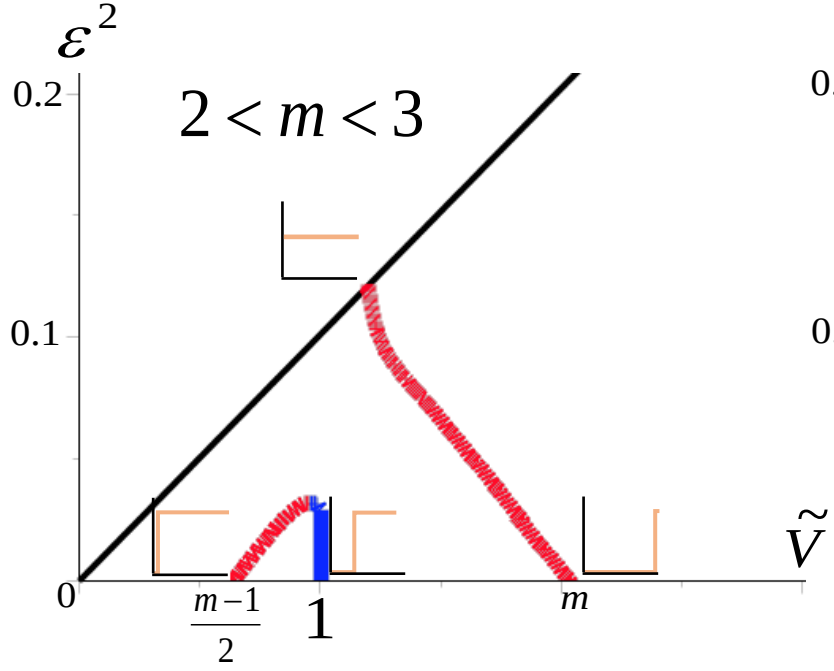
赤：不安定

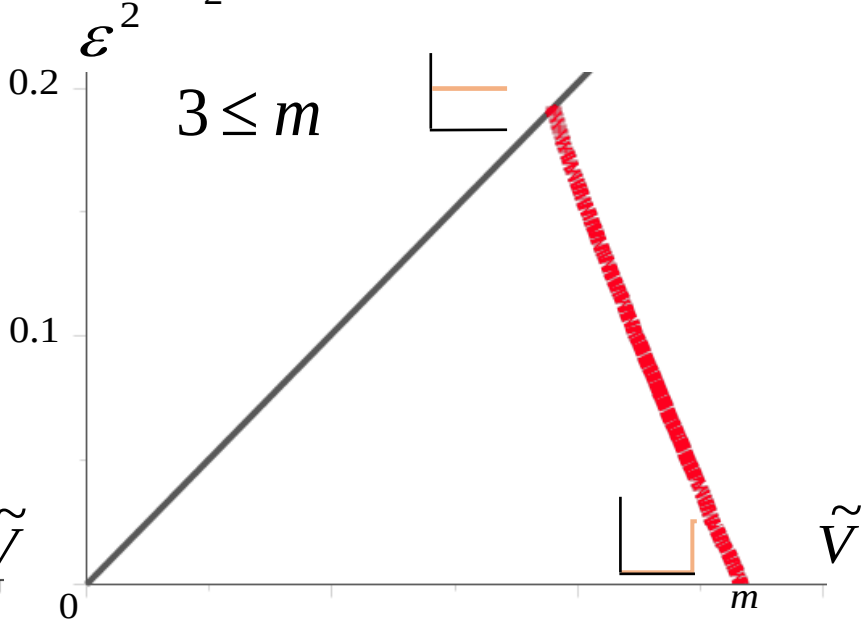
青：安定

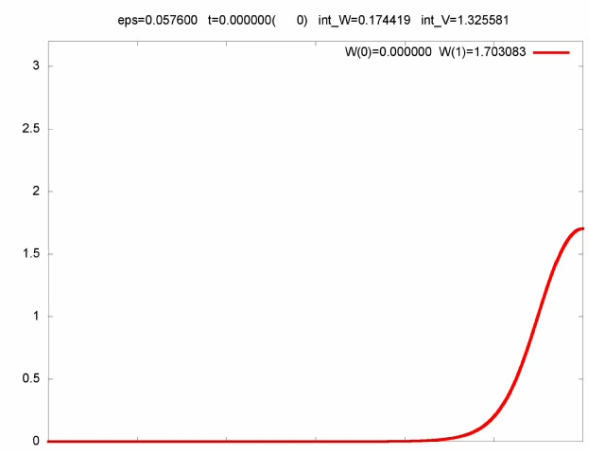
$$(SLP)\begin{cases}\varepsilon^2\Delta W + W(\tilde{V}+1-W)(W-1)=0,\\ W_x(0)=W_x(1)=0,\\ W'(x)>0,\\ \int_0^1 W(x)\,dx+\tilde{V}=\int_0^1 W_0(x)\,dx+\tilde{V}_0=m.\end{cases}$$

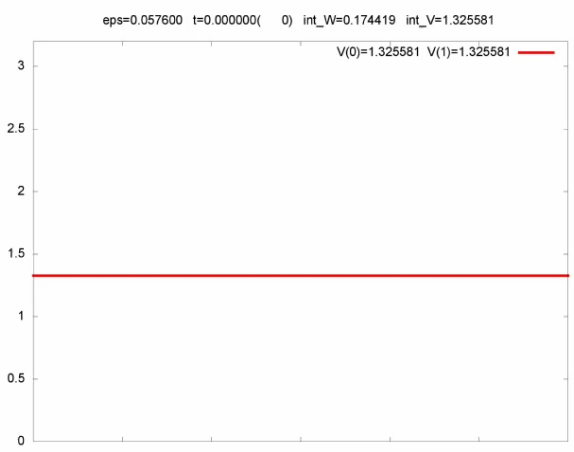




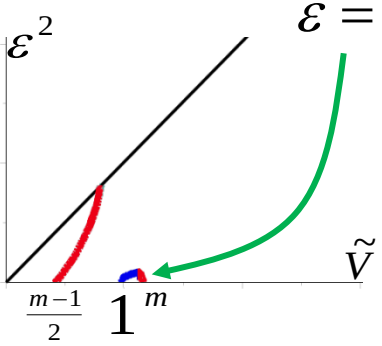


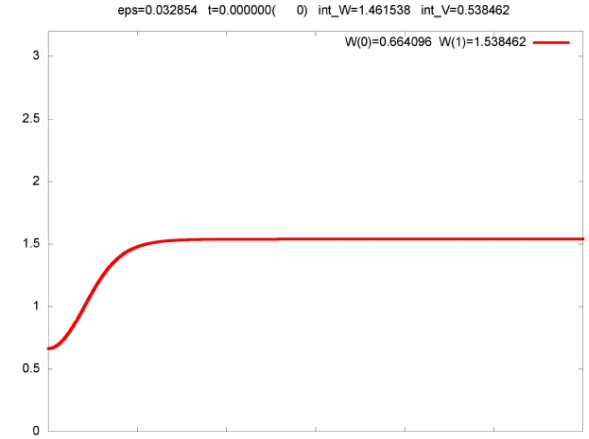


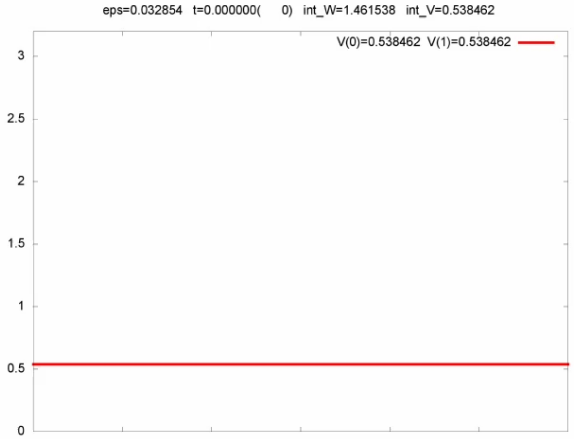




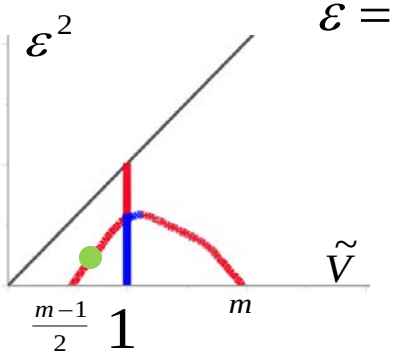
$m = 1.5$ $\tilde{V} = 1.32558$ $\varepsilon = 0.05760$

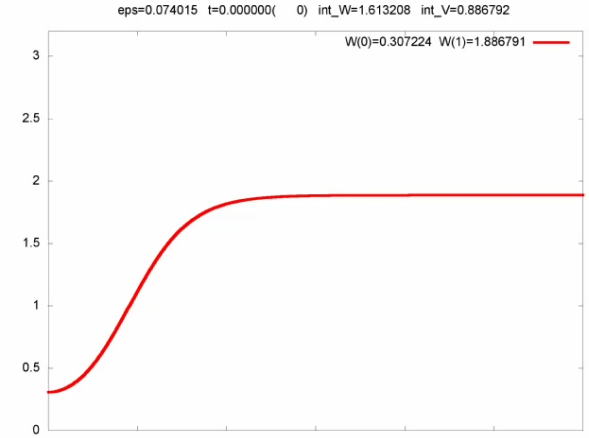


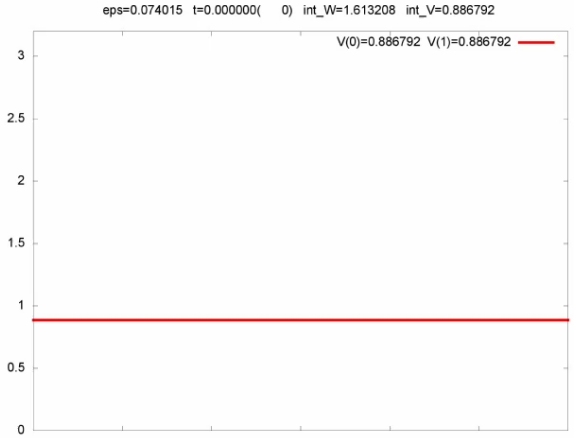




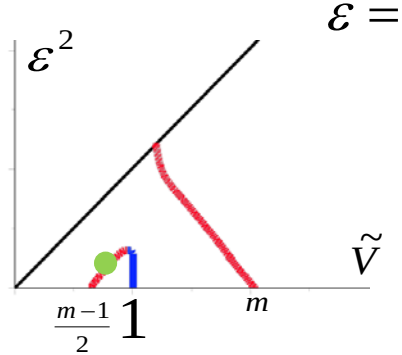
$m = 2$ $\tilde{V} = 0.53846$ $\varepsilon = 0.03285$







$m = 2.5$ $\tilde{V} = 0.88679$ $\varepsilon = 0.07401$

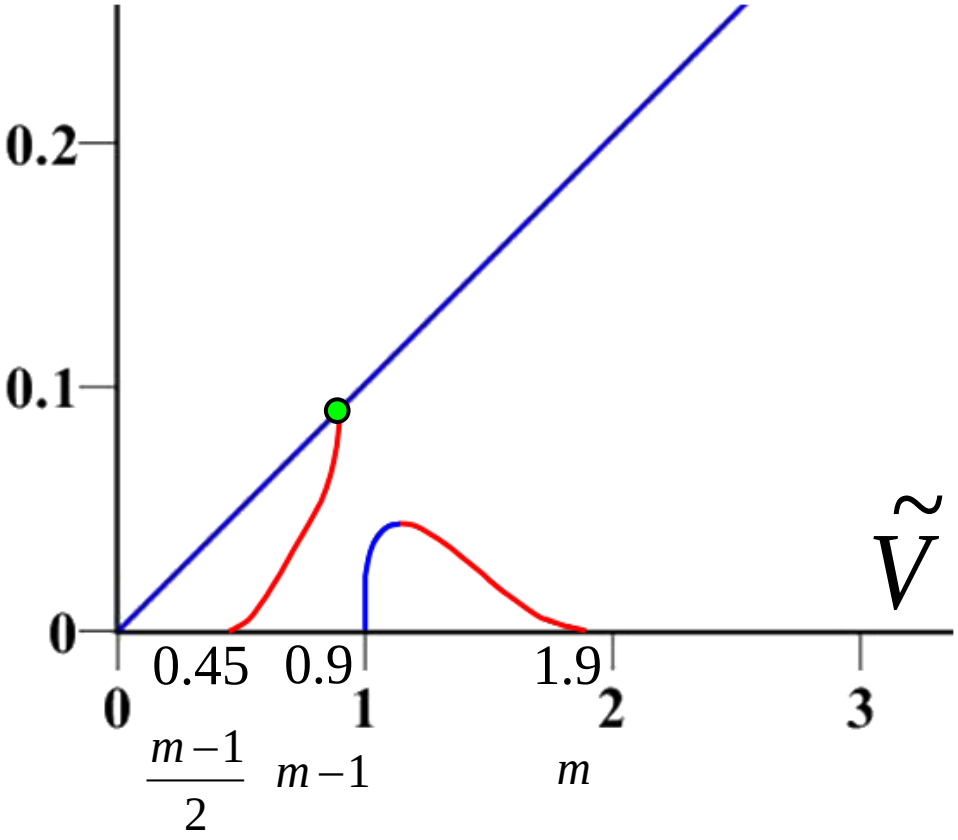


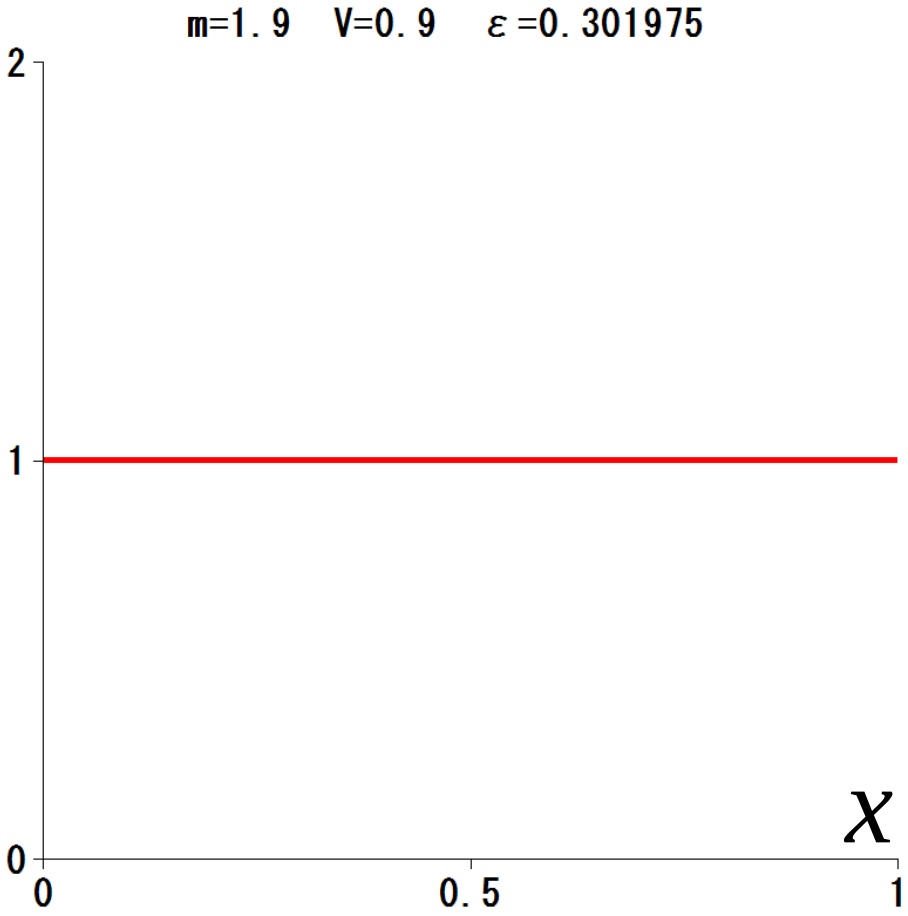
$$(SLP)\begin{cases} \varepsilon^2 W_{xx} + W(W-1)(\tilde{V}+1-W) = 0, \\ W_x(0) = W_x(1) = 0, \\ W'(x) > 0, \\ \int_0^1 W(x)\,\mathrm{dx} + \tilde{V} = m. \end{cases}$$

不安定

$\tilde{V} \to 0.45,\ \varepsilon^2 \to 0$ の形状

$m = 1.9$





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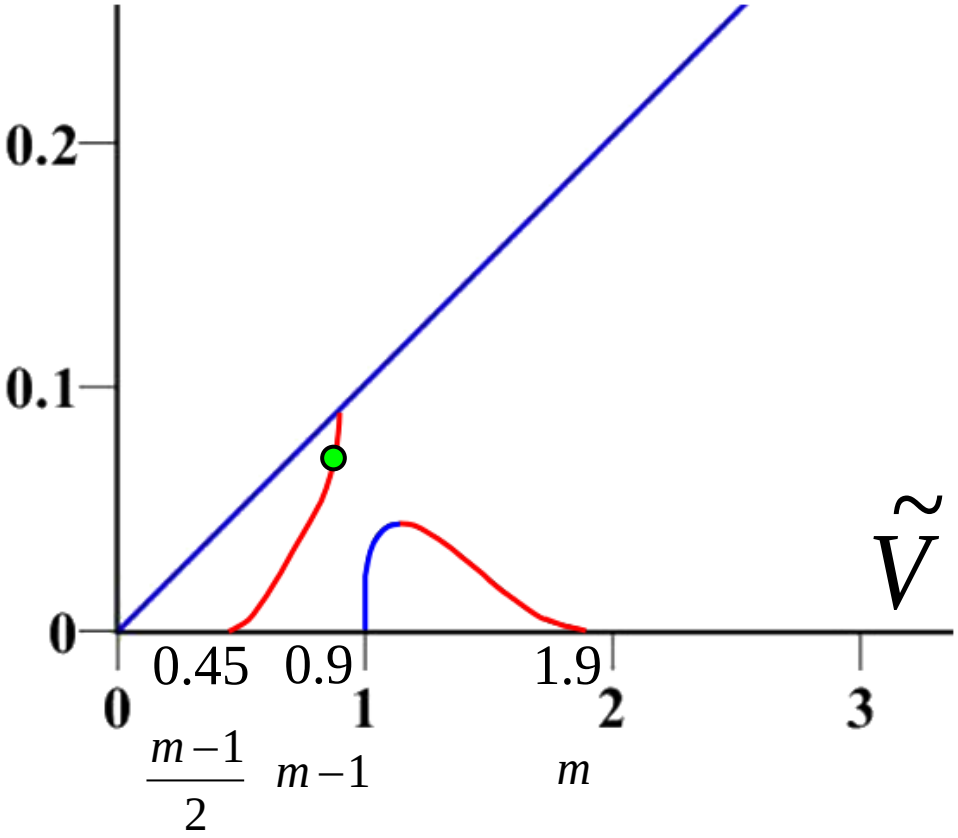

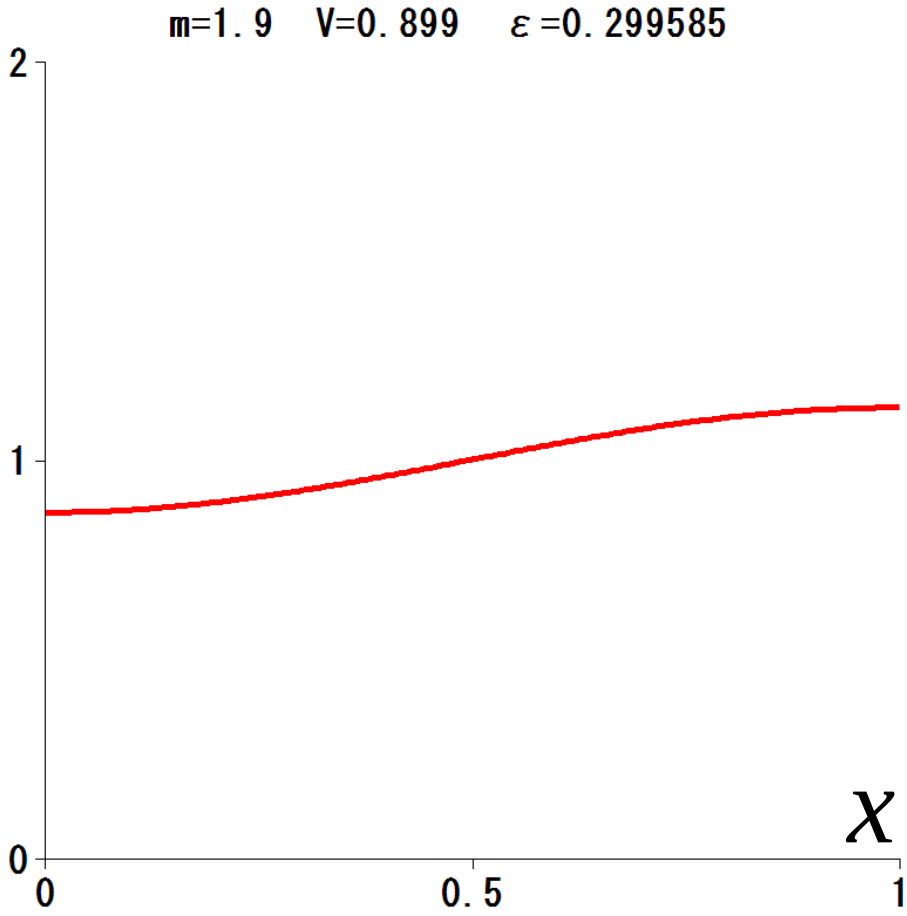

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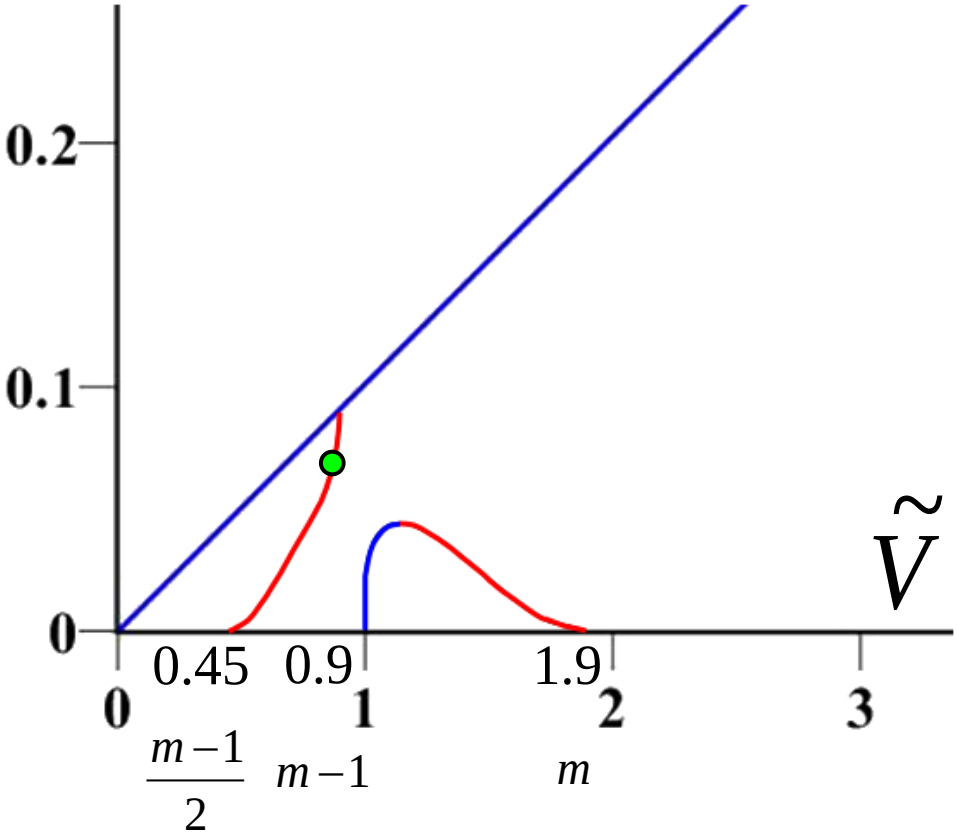

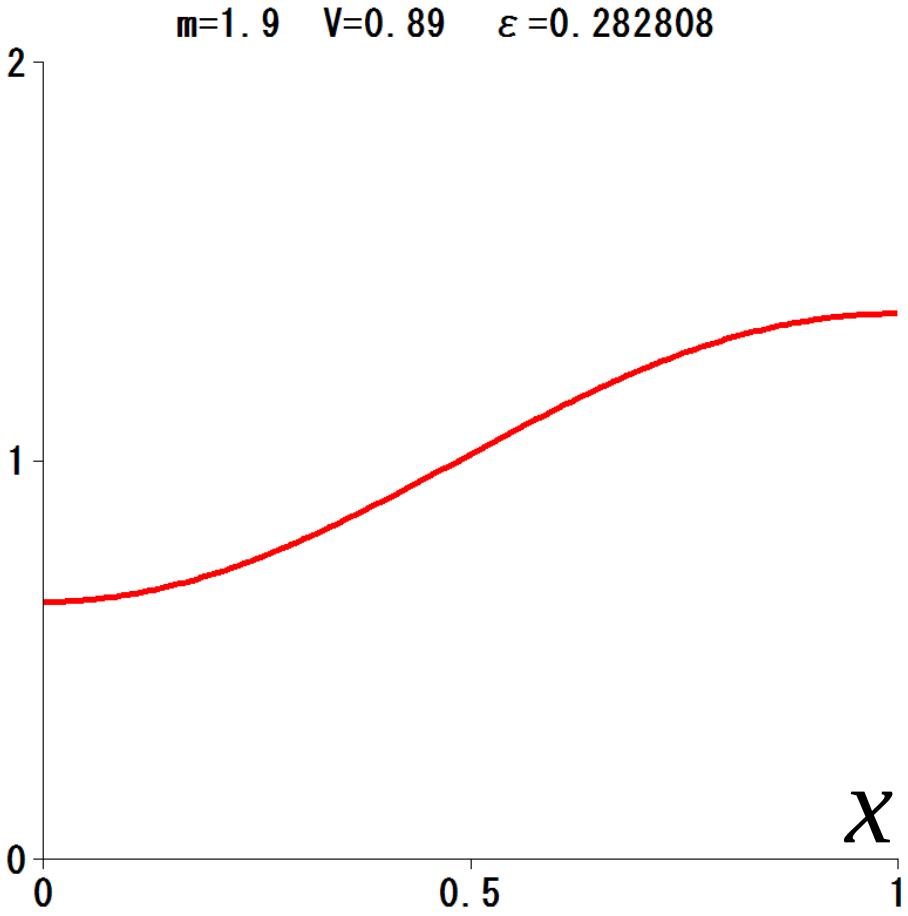

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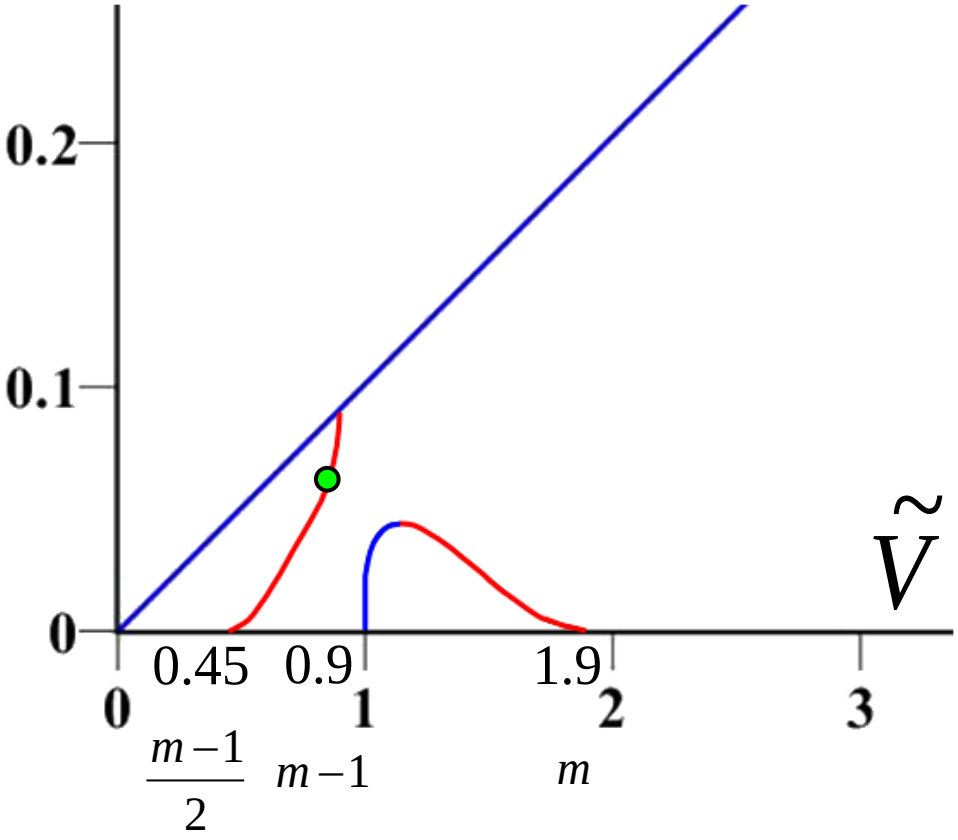
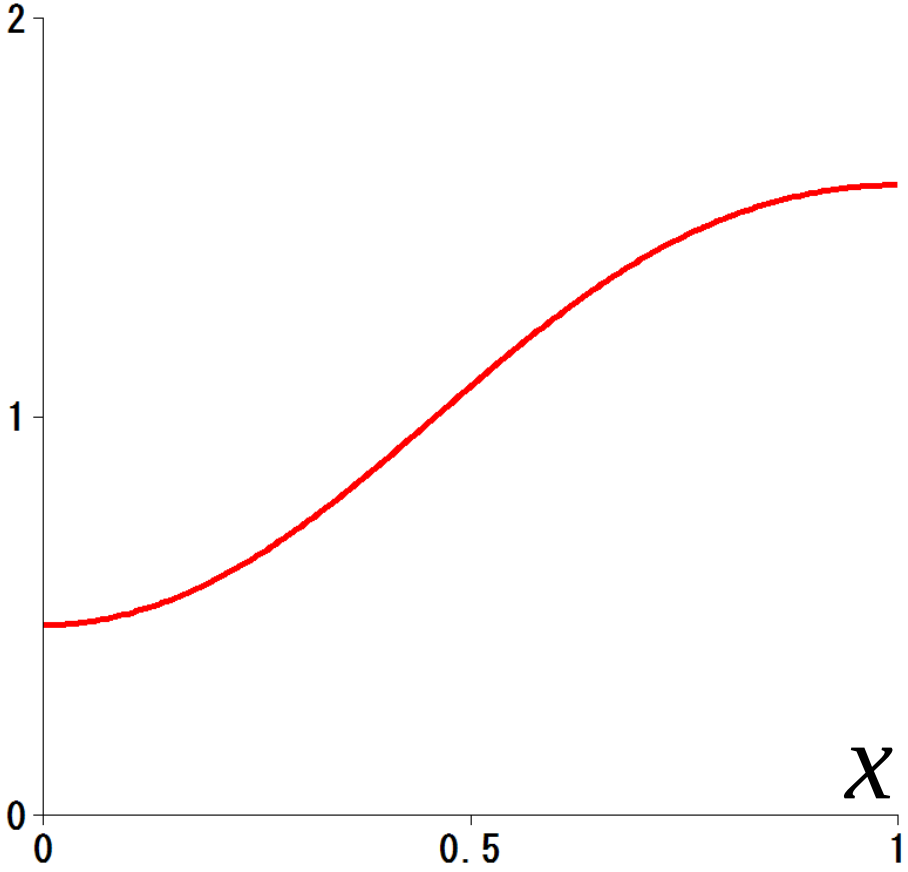
不安定

$\tilde{V} \to 0.45,\ \varepsilon^2 \to 0$ の形状

$m = 1.9$



m=1.9 V=0.85 ε=0.246856



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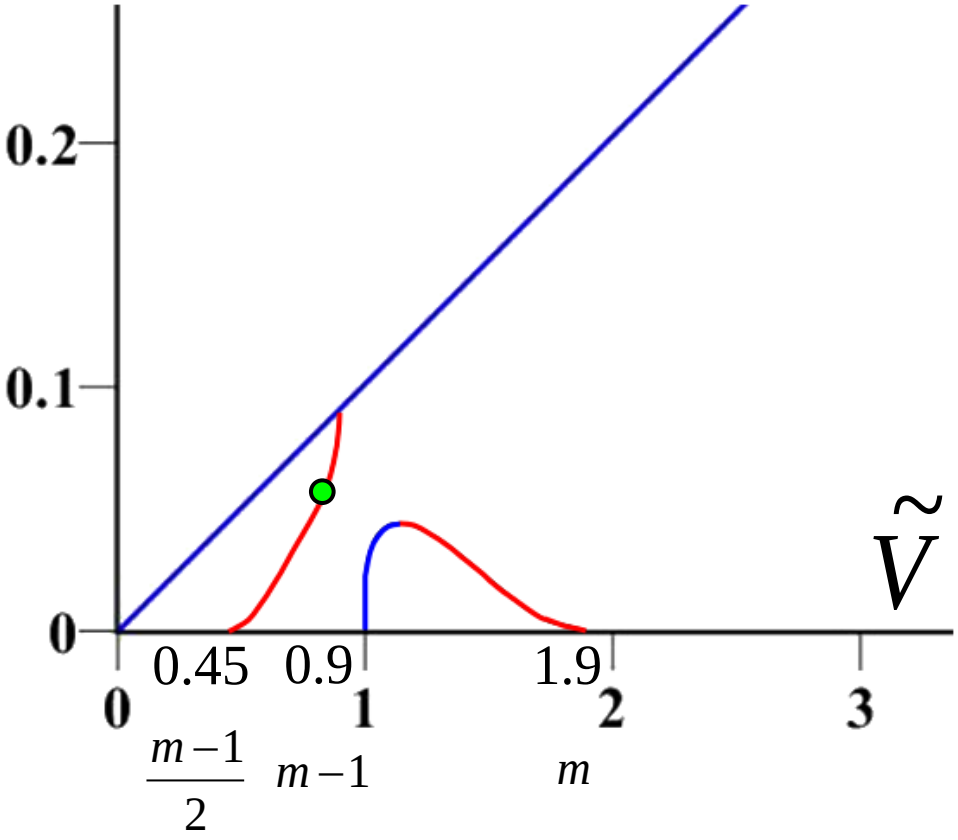
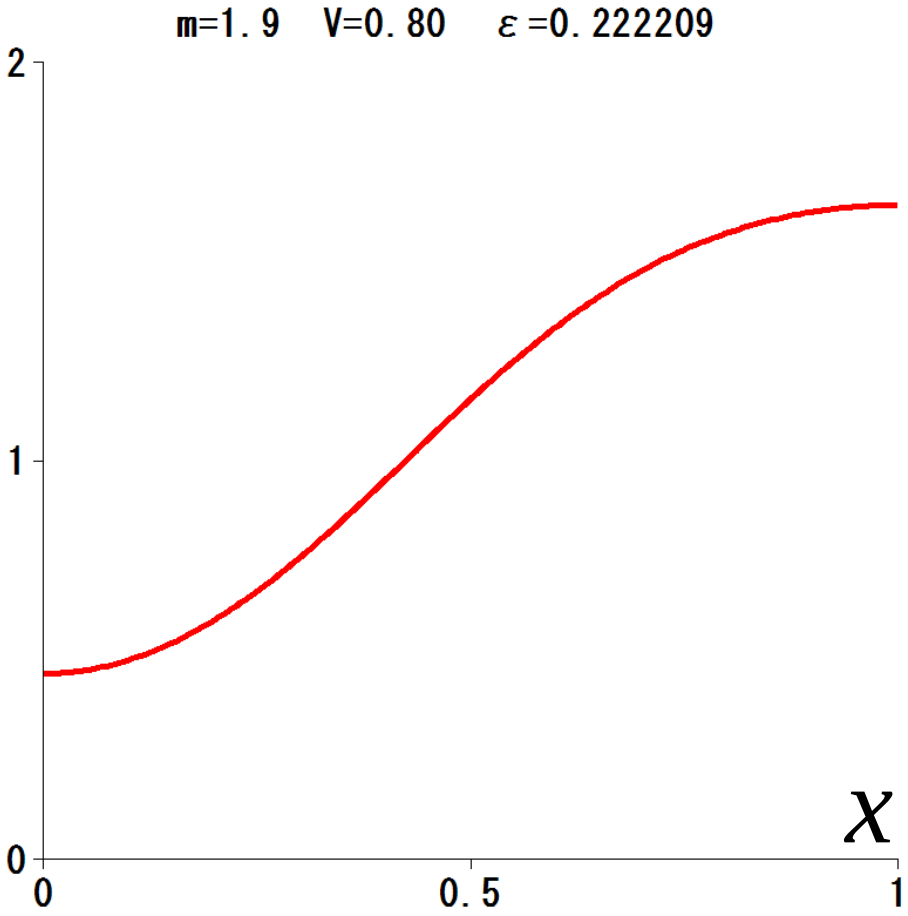
不安定

$\tilde{V} \to 0.45,\ \varepsilon^2 \to 0$ の形状

$m = 1.9$





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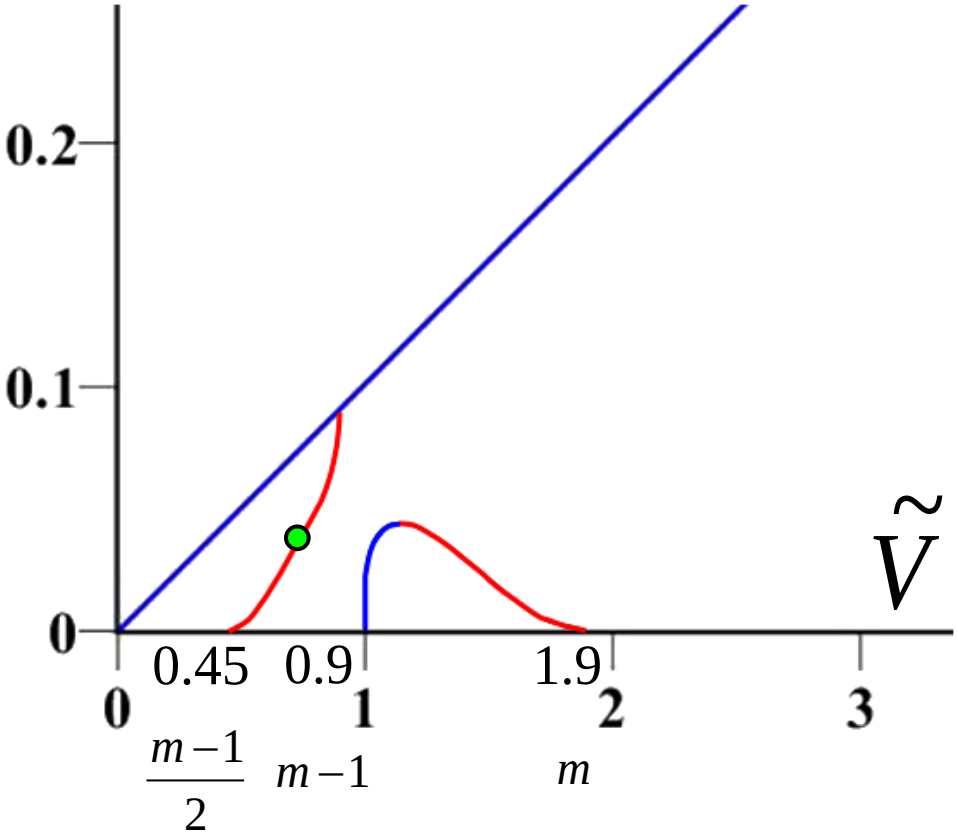
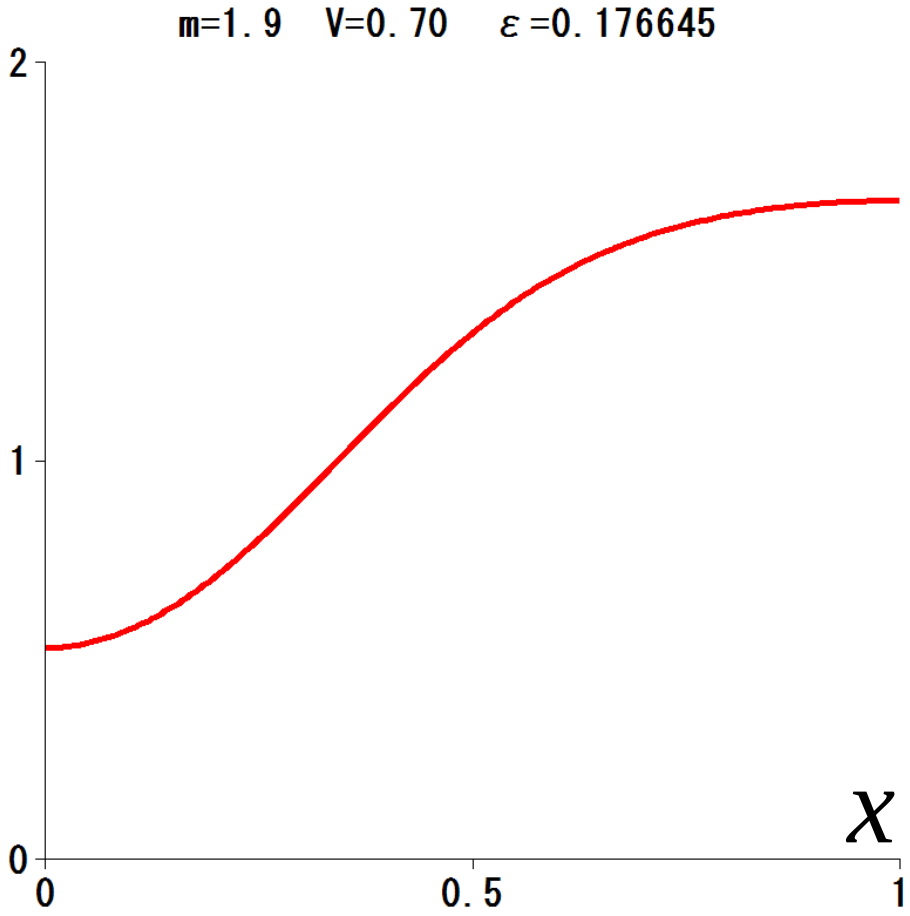
不安定

$\tilde{V} \to 0.45,\ \varepsilon^2 \to 0$ の形状

$m = 1.9$





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不安定

$\tilde{V} \to 0.45,\ \varepsilon^2 \to 0$ の形状

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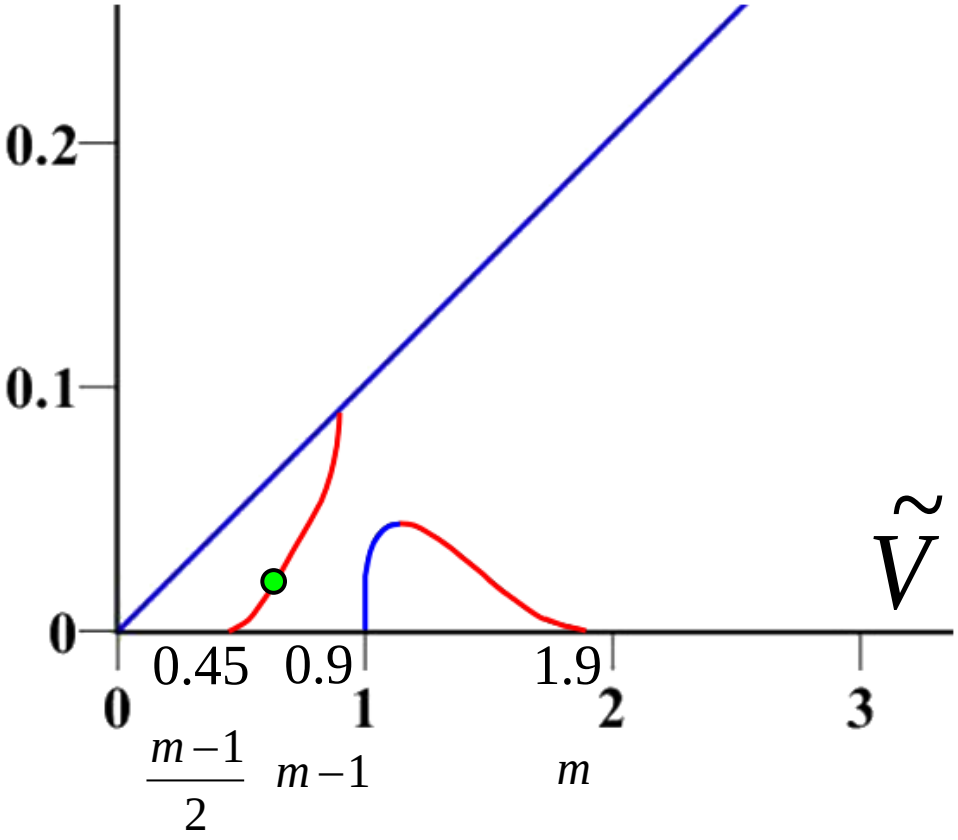

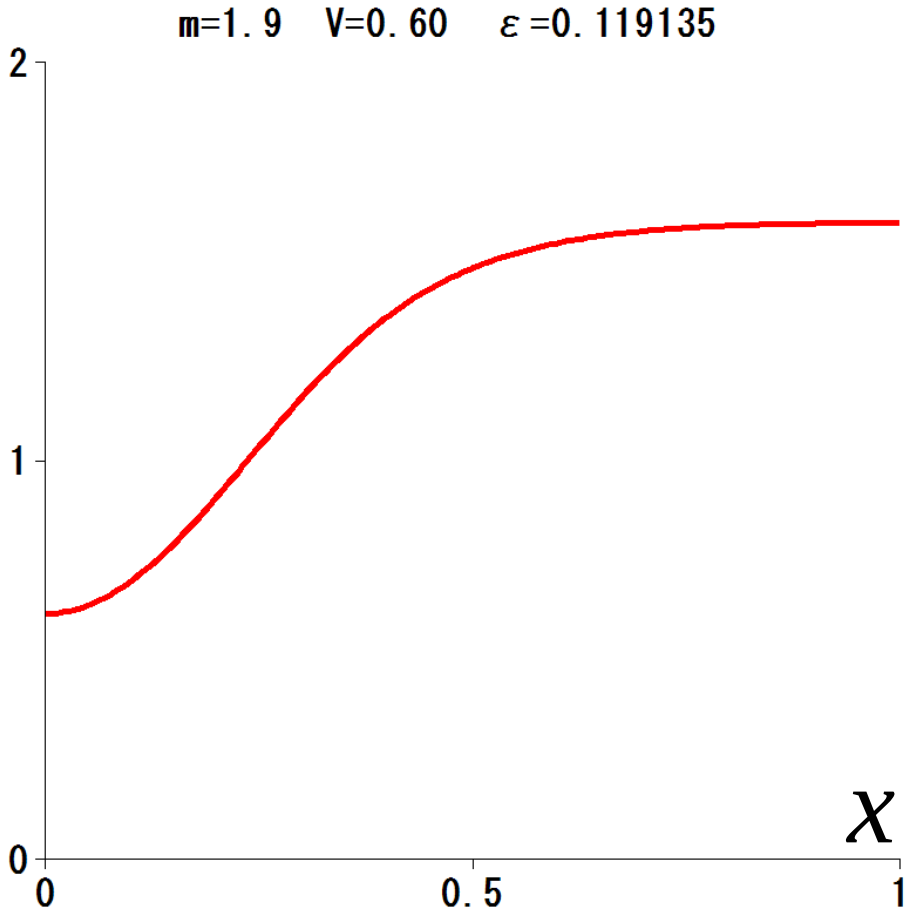

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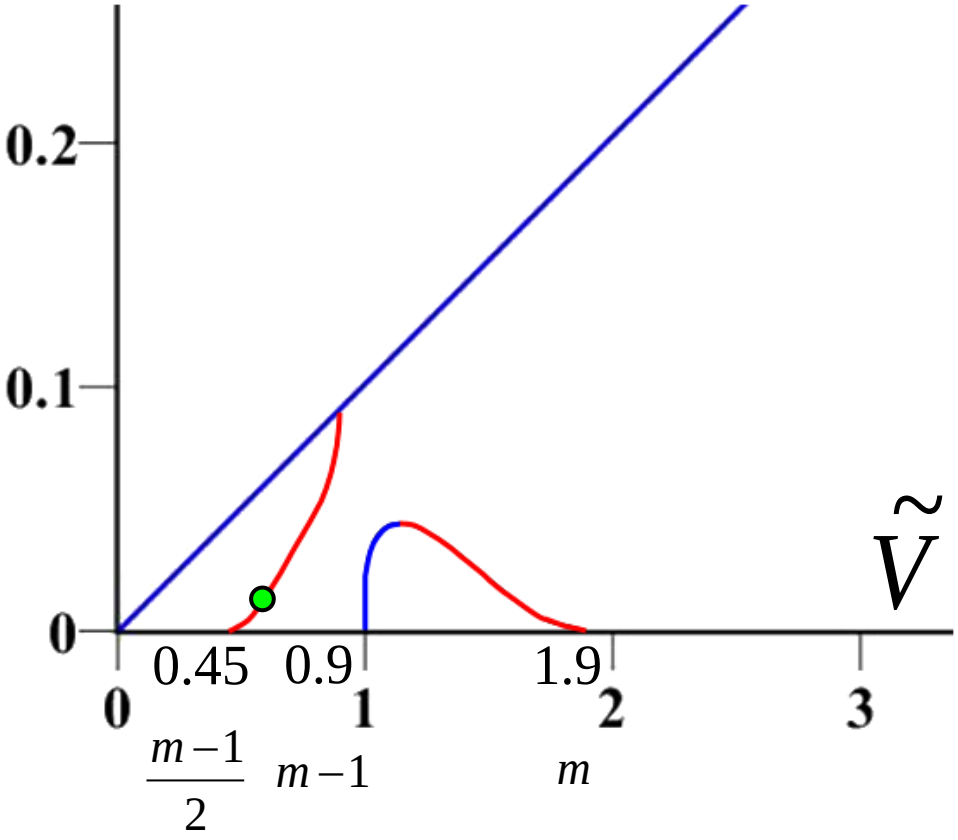
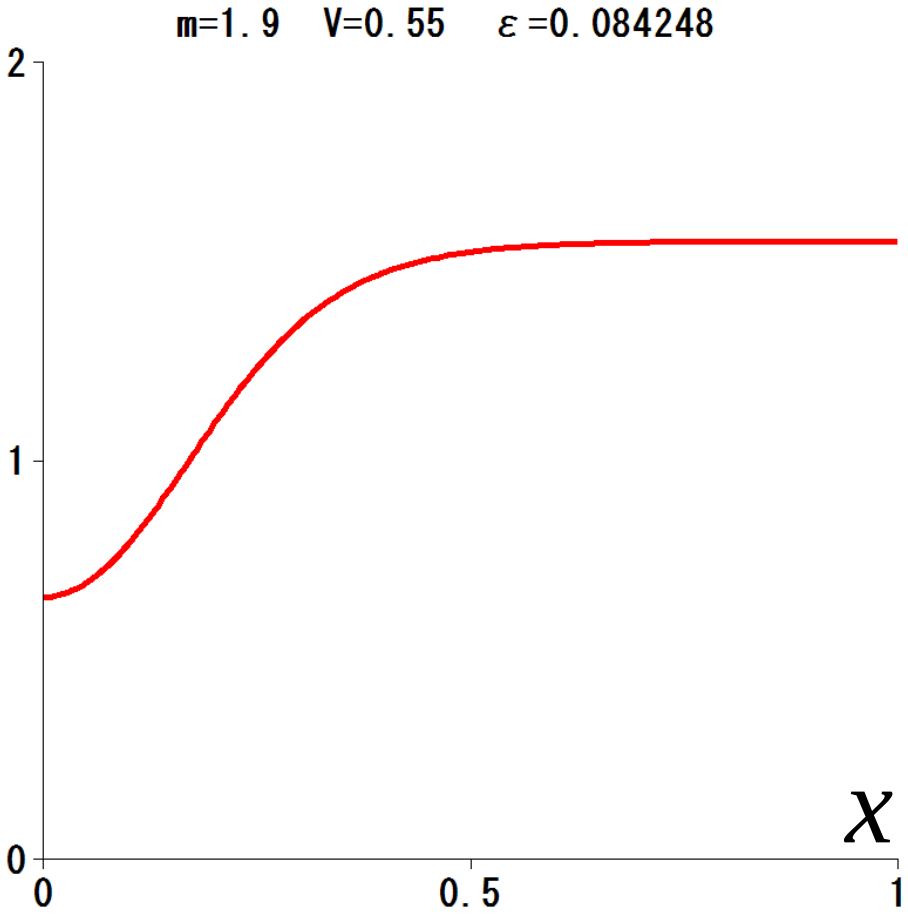
不安定

$\tilde{V} \to 0.45,\ \varepsilon^2 \to 0$ の形状

$m = 1.9$



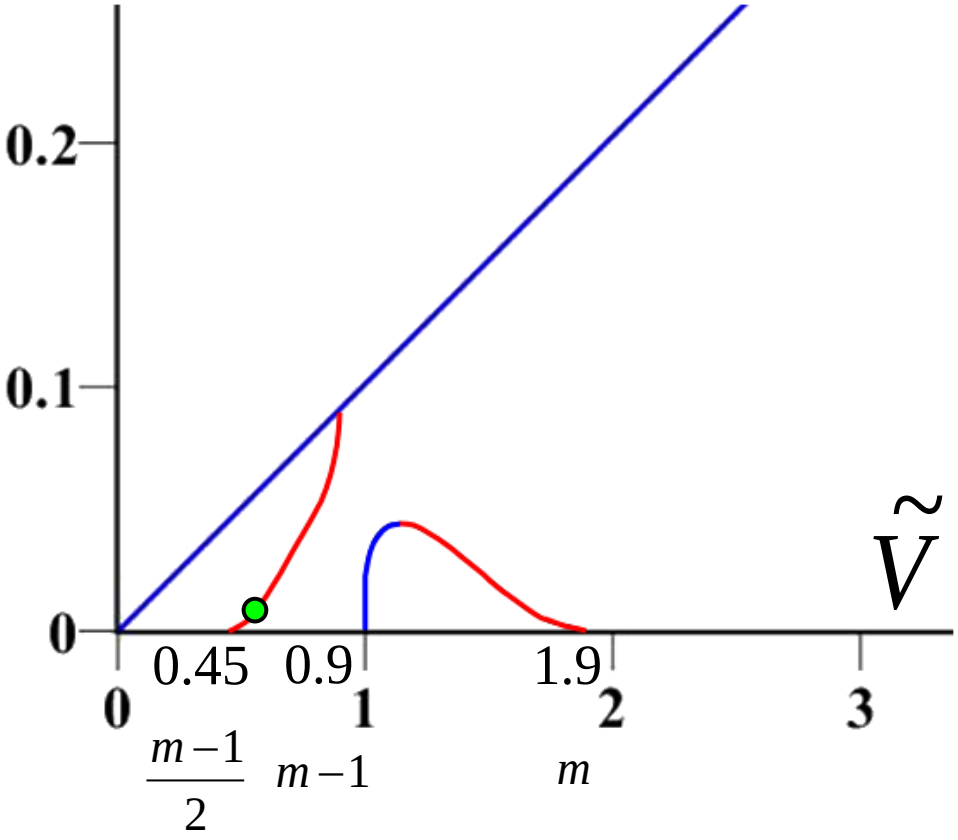


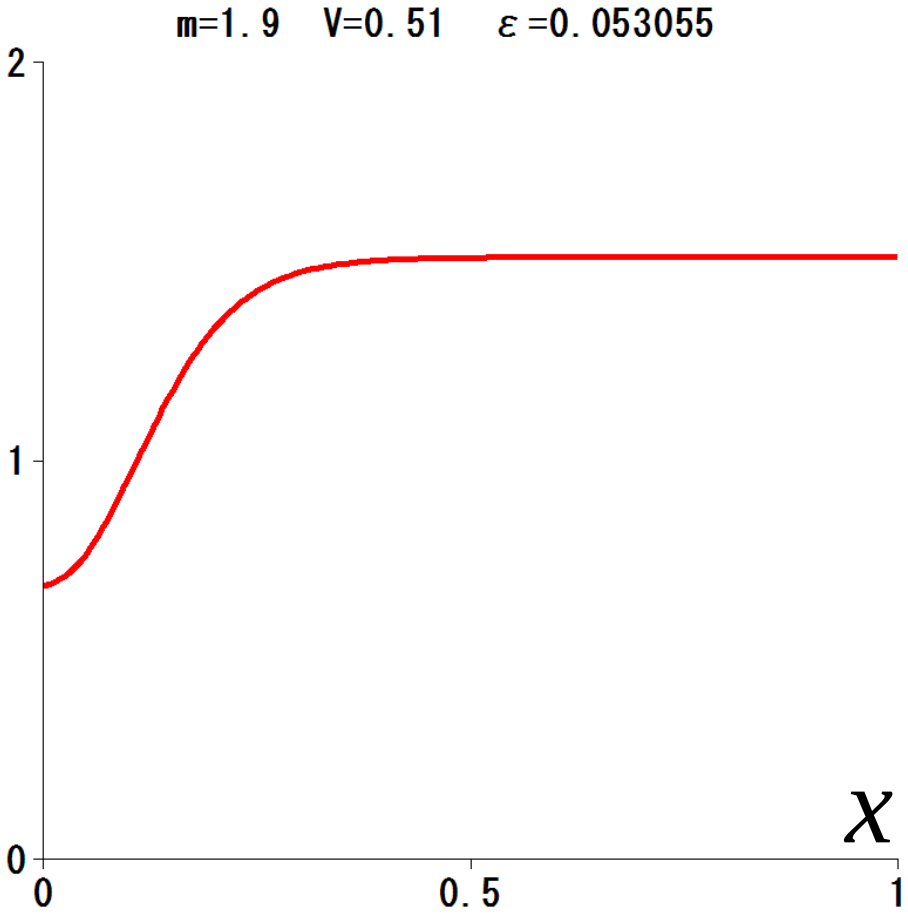
$$(SLP)\begin{cases} \varepsilon^2 W_{xx} + W(W-1)(\tilde{V}+1-W) = 0, \\ W_x(0) = W_x(1) = 0, \\ W'(x) > 0, \\ \int_0^1 W(x)\,dx + \tilde{V} = m. \end{cases}$$

不安定

$\tilde{V} \to 0.45,\ \varepsilon^2 \to 0$ の形状

$m = 1.9$





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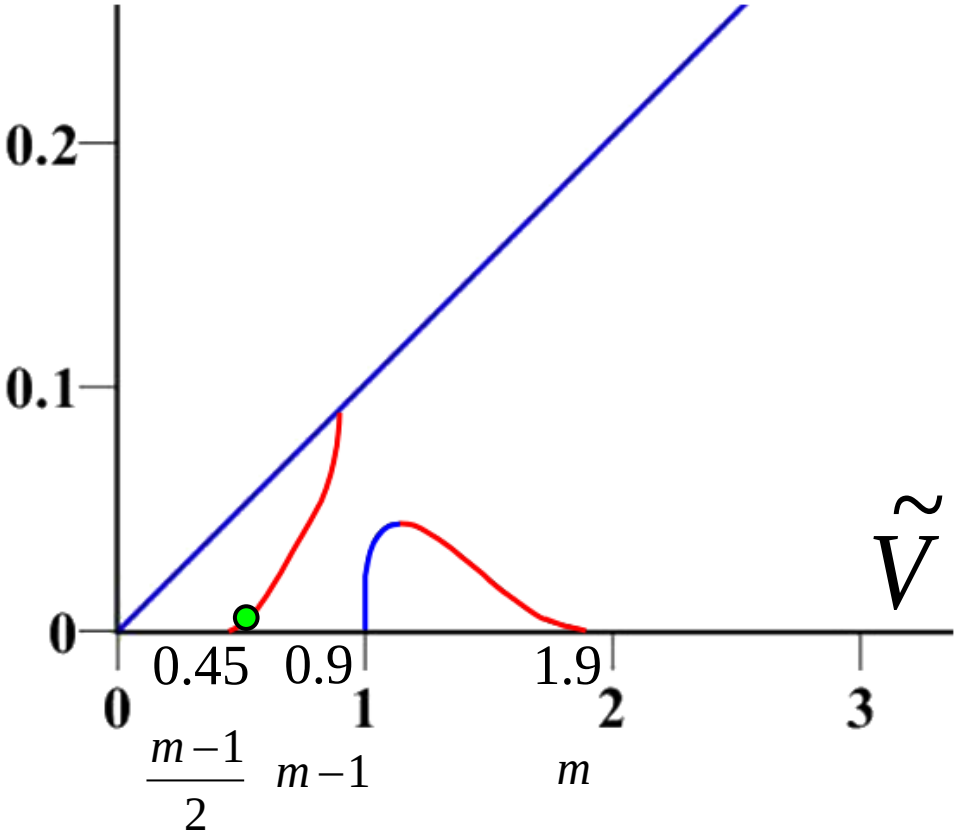
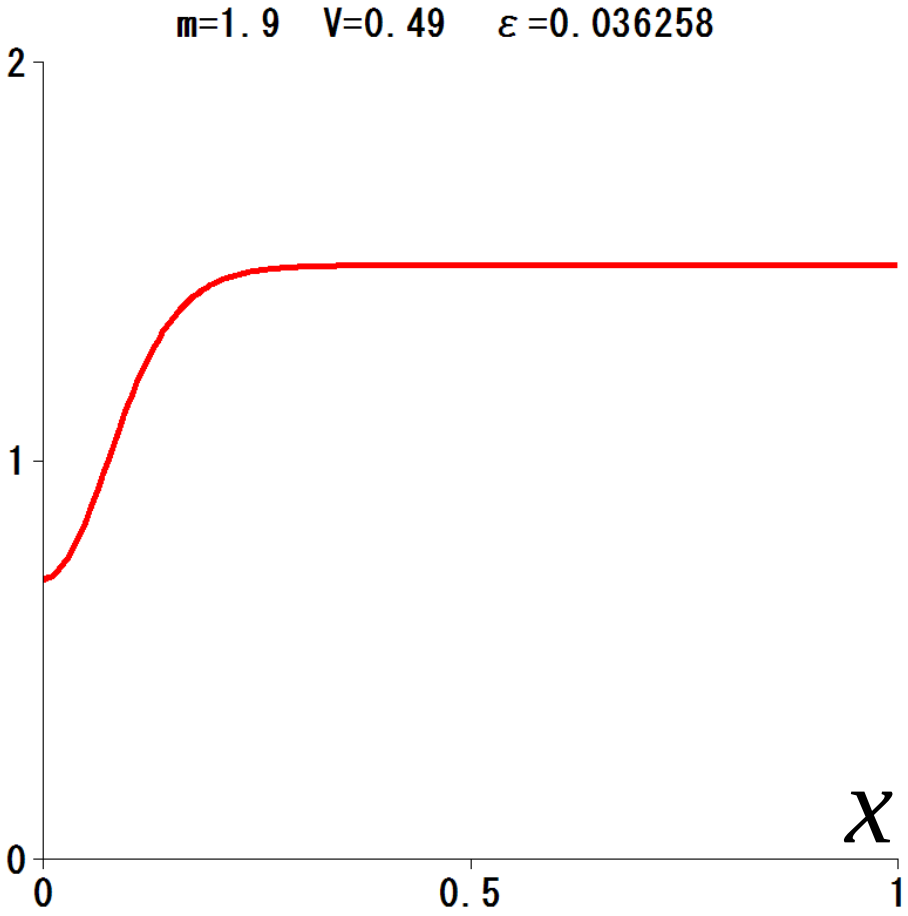
不安定

$\tilde{V} \to 0.45,\ \varepsilon^2 \to 0$ の形状

$m = 1.9$





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$\tilde{V} \to 0.45,\ \varepsilon^2 \to 0$ の形状

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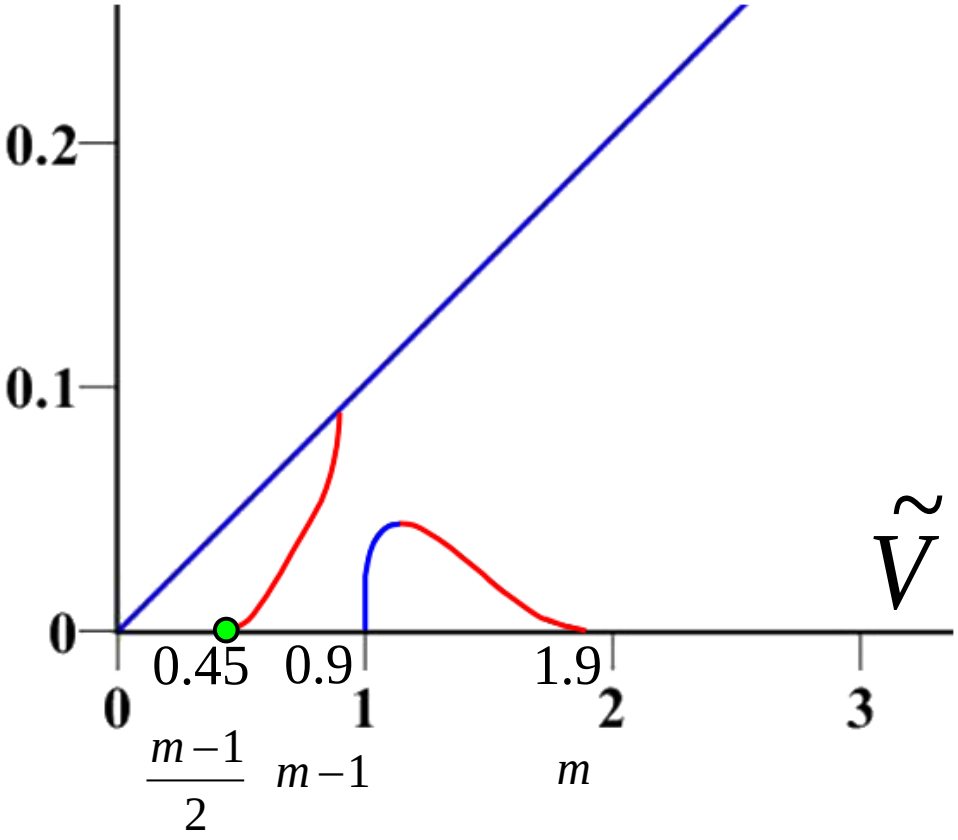
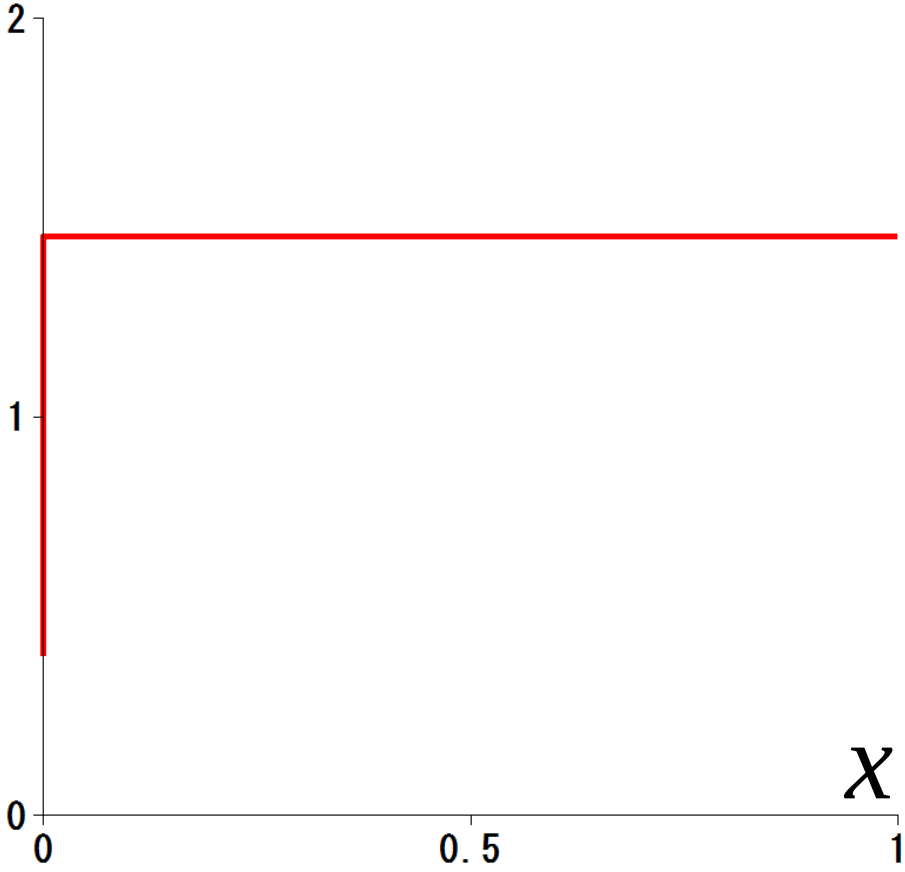


m=1.9 $\tilde{V} \to 0.45$ $\varepsilon \to 0$



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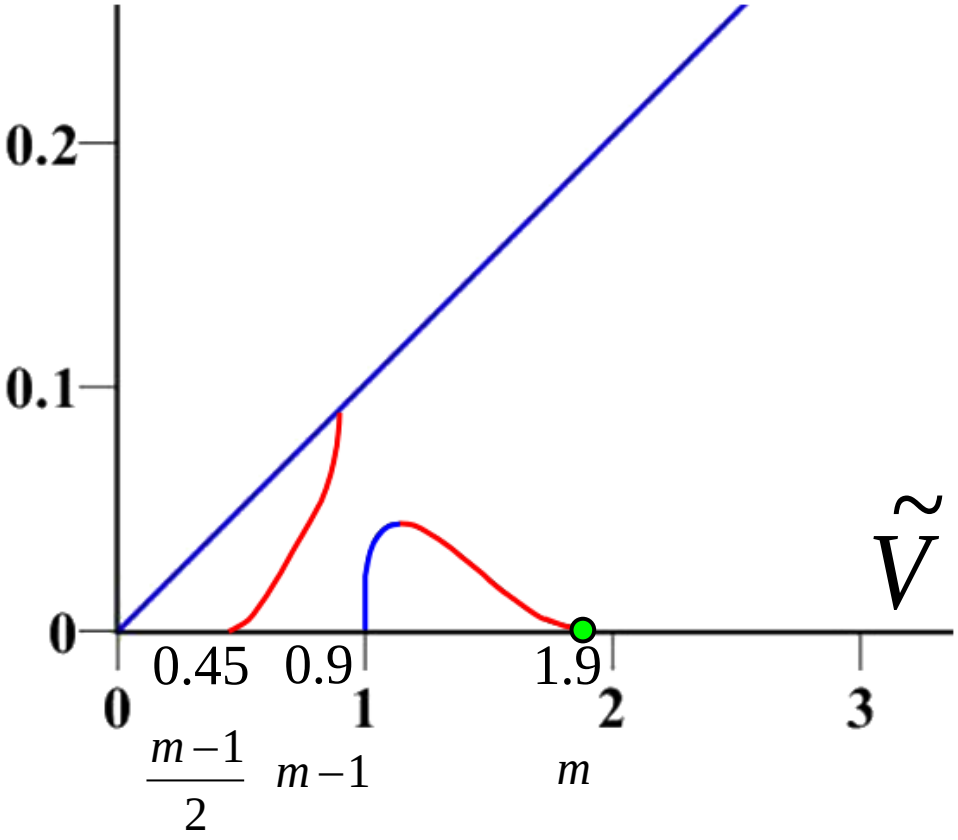
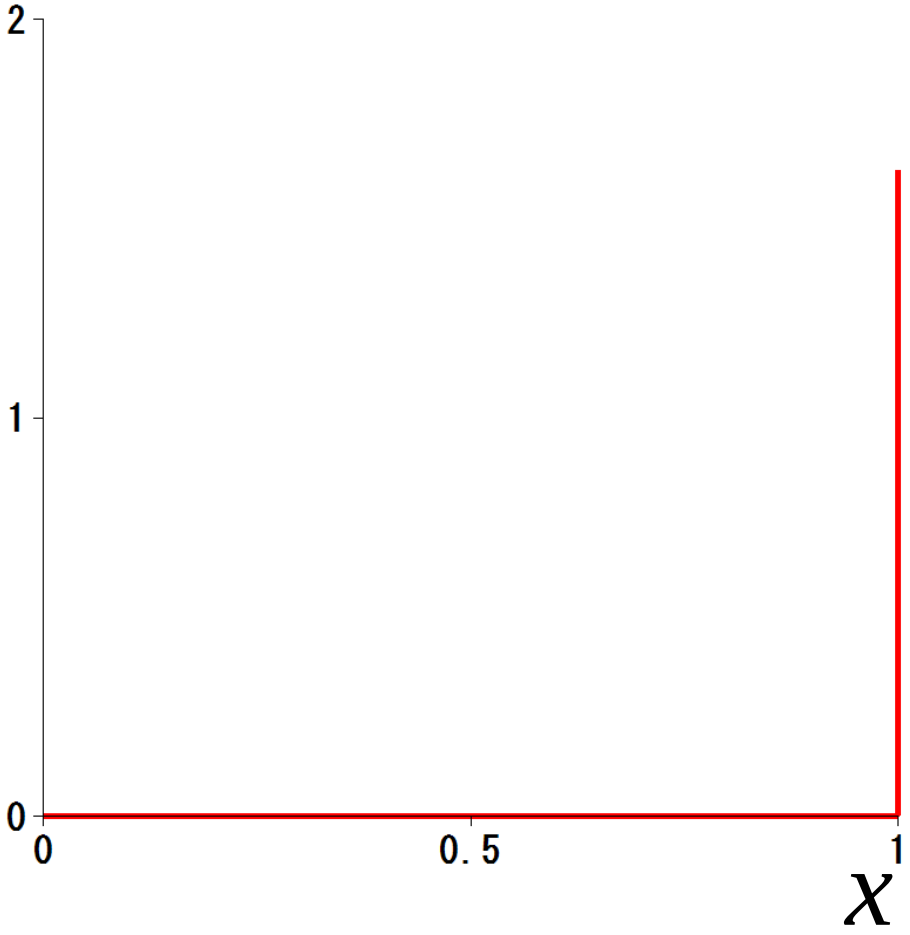
不安定

$\tilde{V} \to 1.9, \varepsilon^2 \to 0$ の形状

$m = 1.9$





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不安定

$\tilde{V} \to 1.9, \varepsilon^2 \to 0$ の形状

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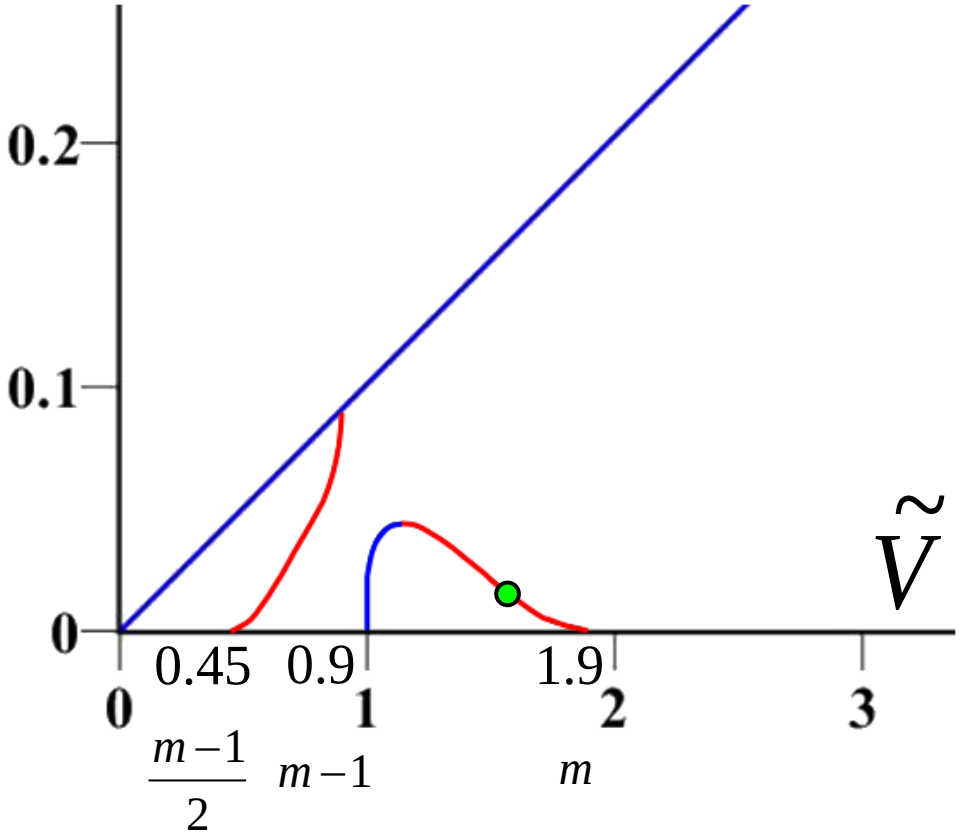

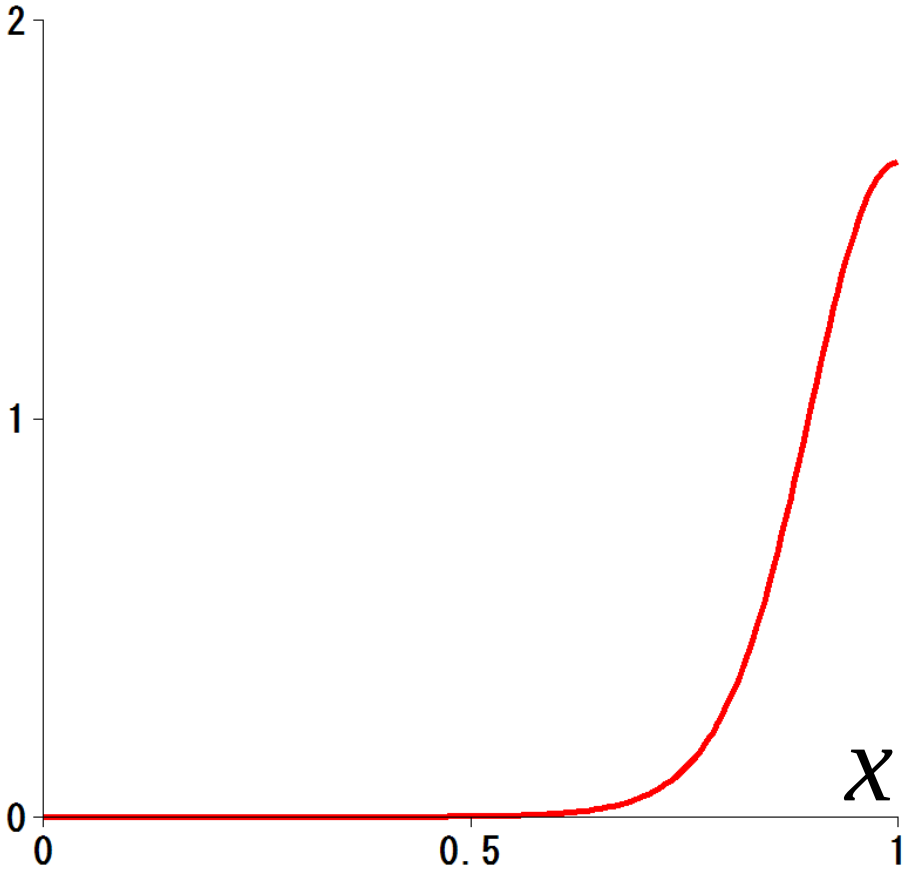

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不安定

$\tilde{V} \to 1.9, \varepsilon^2 \to 0$ の形状

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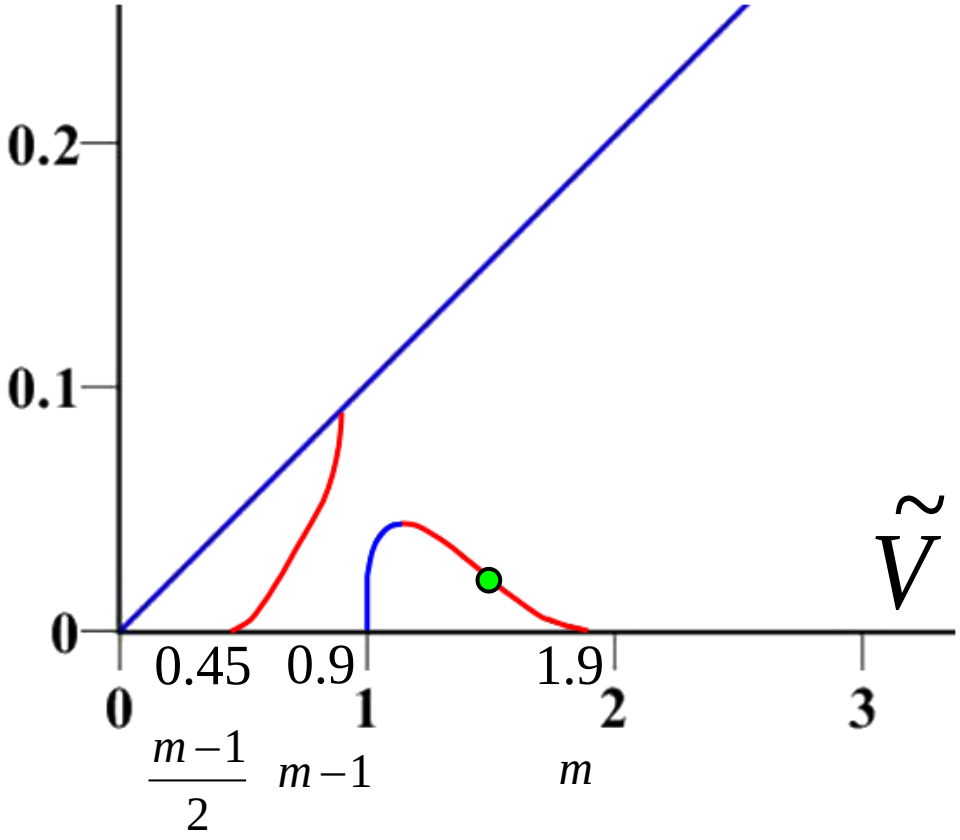

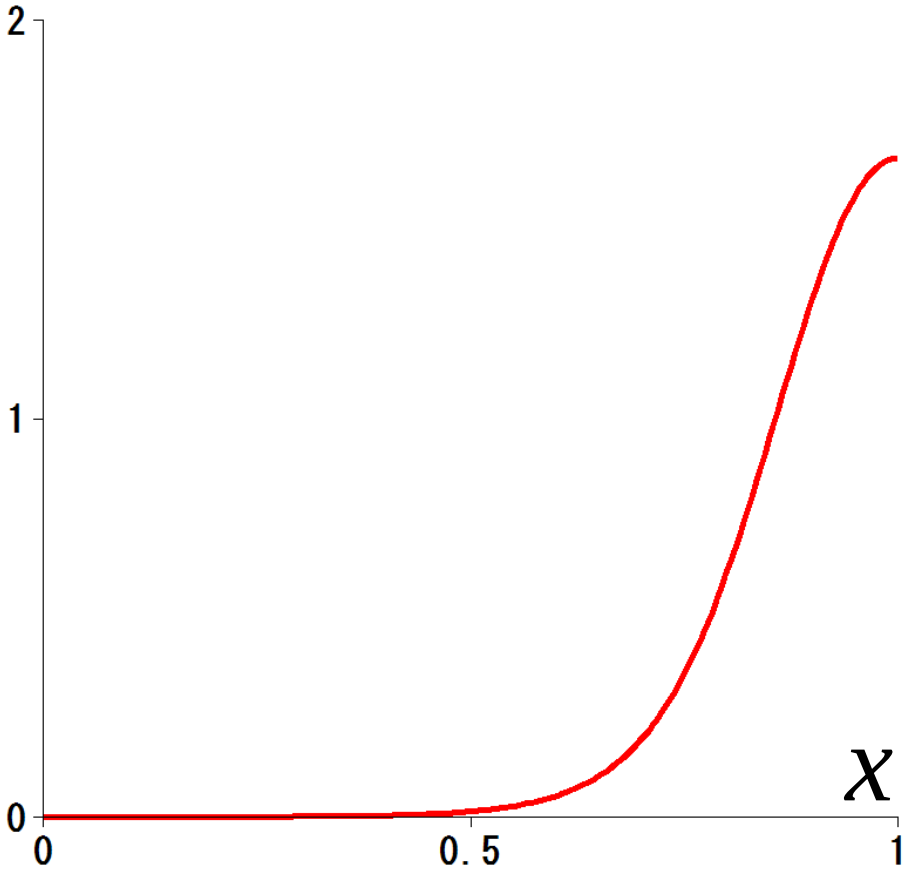

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$\tilde{V} \to 1.9, \varepsilon^2 \to 0$ の形状

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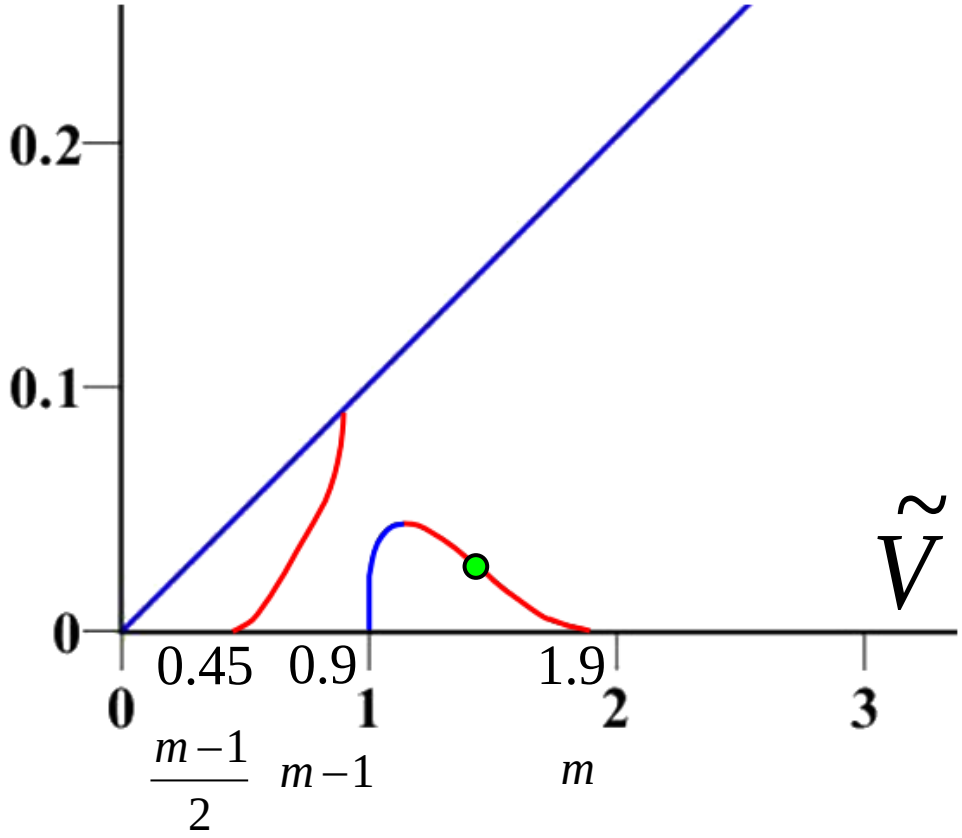

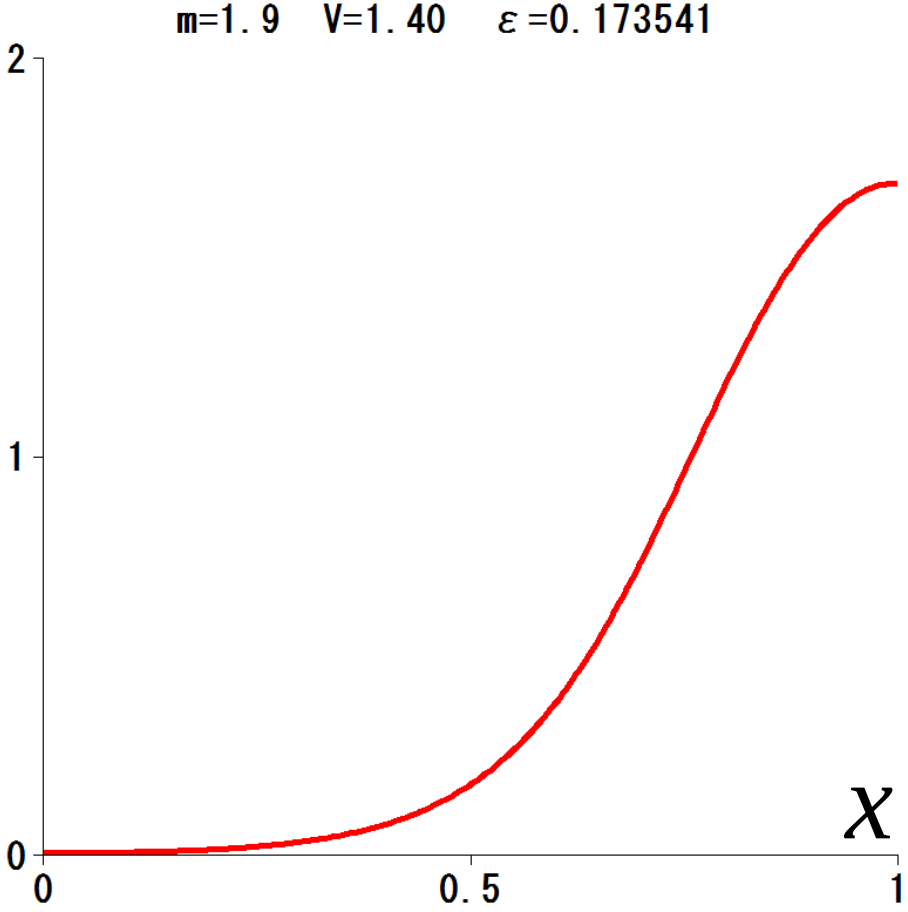

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不安定

$\tilde{V} \to 1.9, \varepsilon^2 \to 0$ の形状

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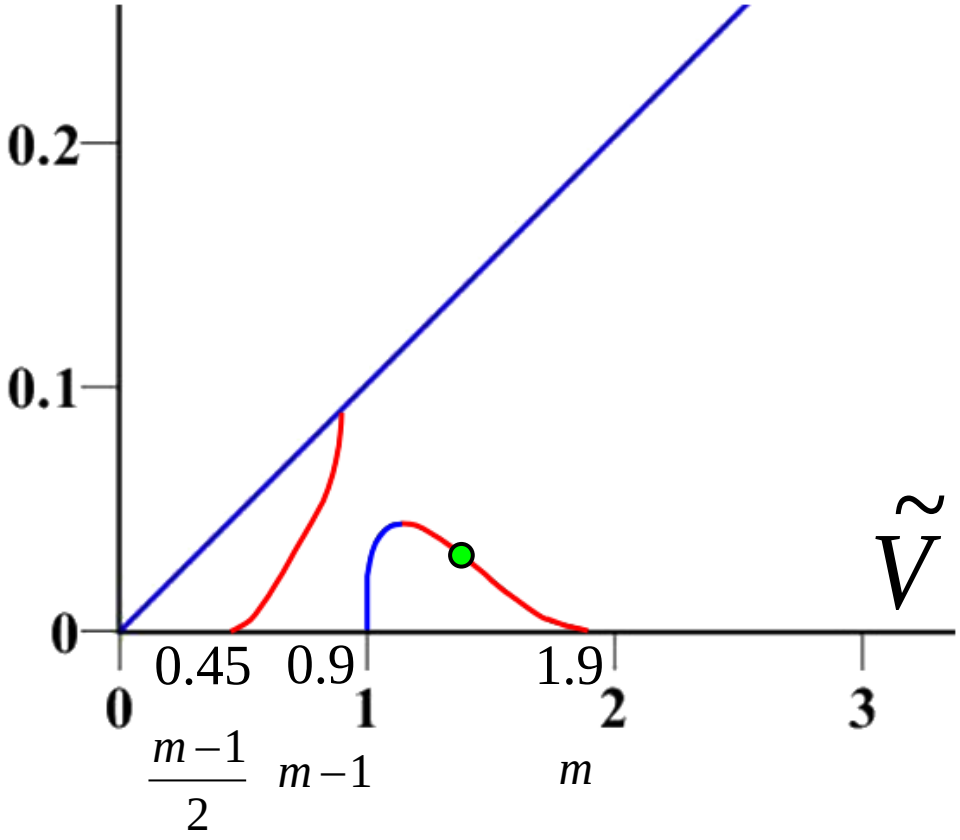

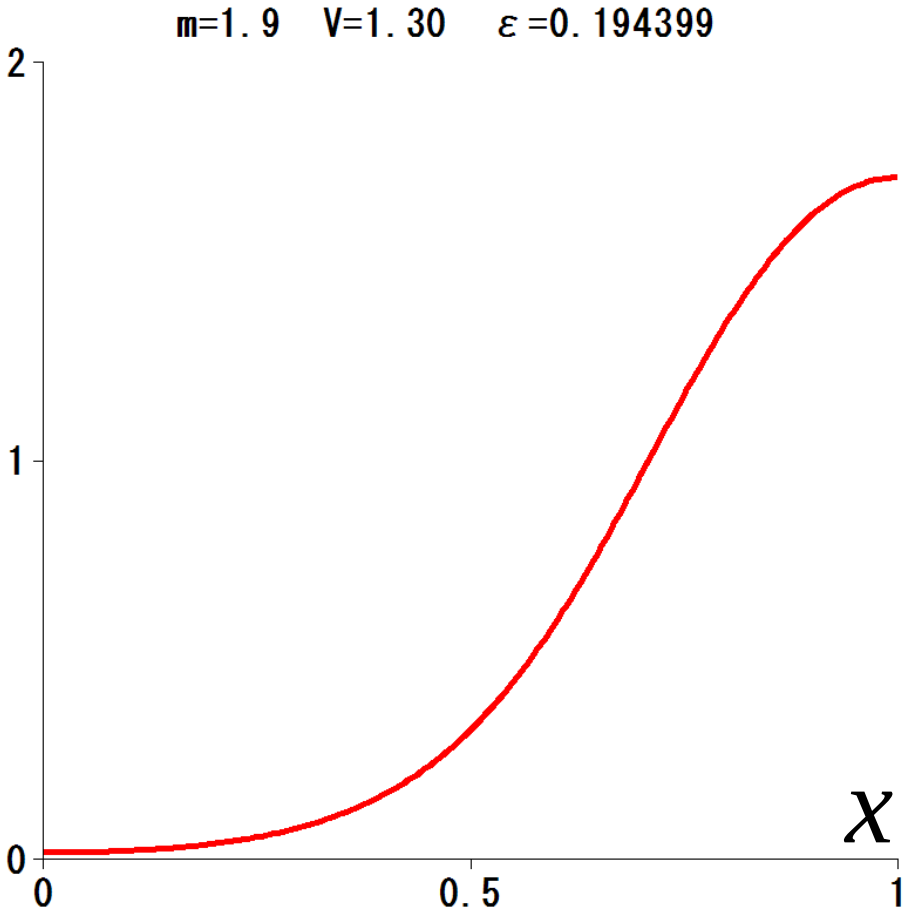

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不安定

$\tilde{V} \to 1.9, \varepsilon^2 \to 0$ の形状

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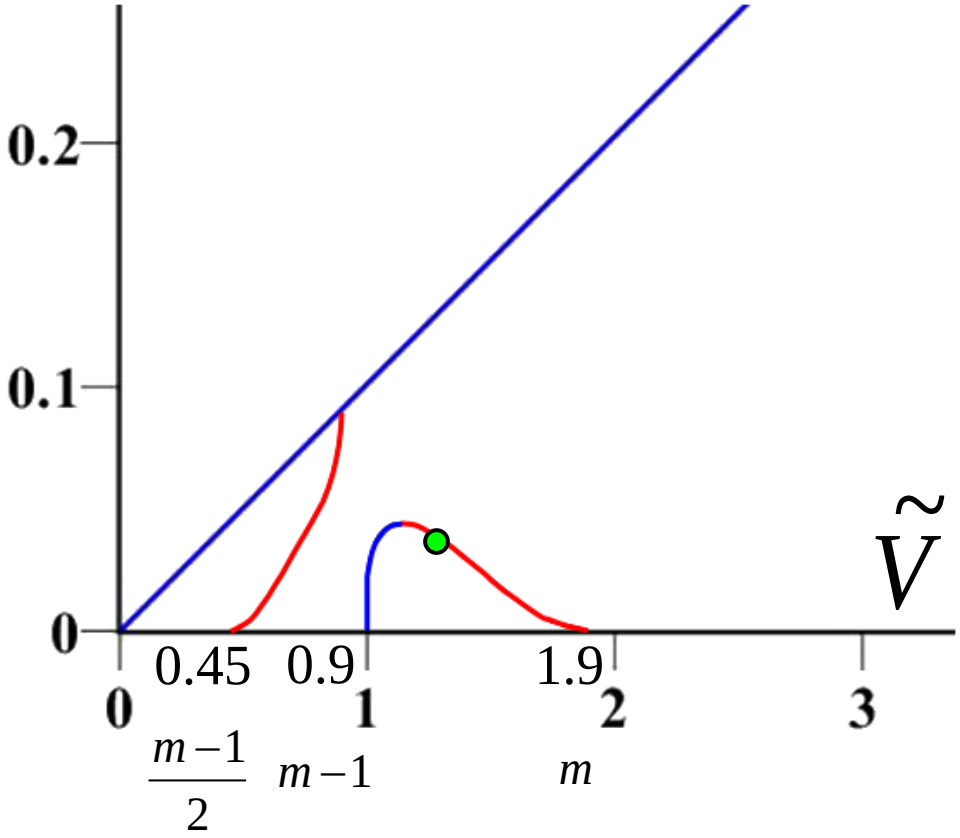
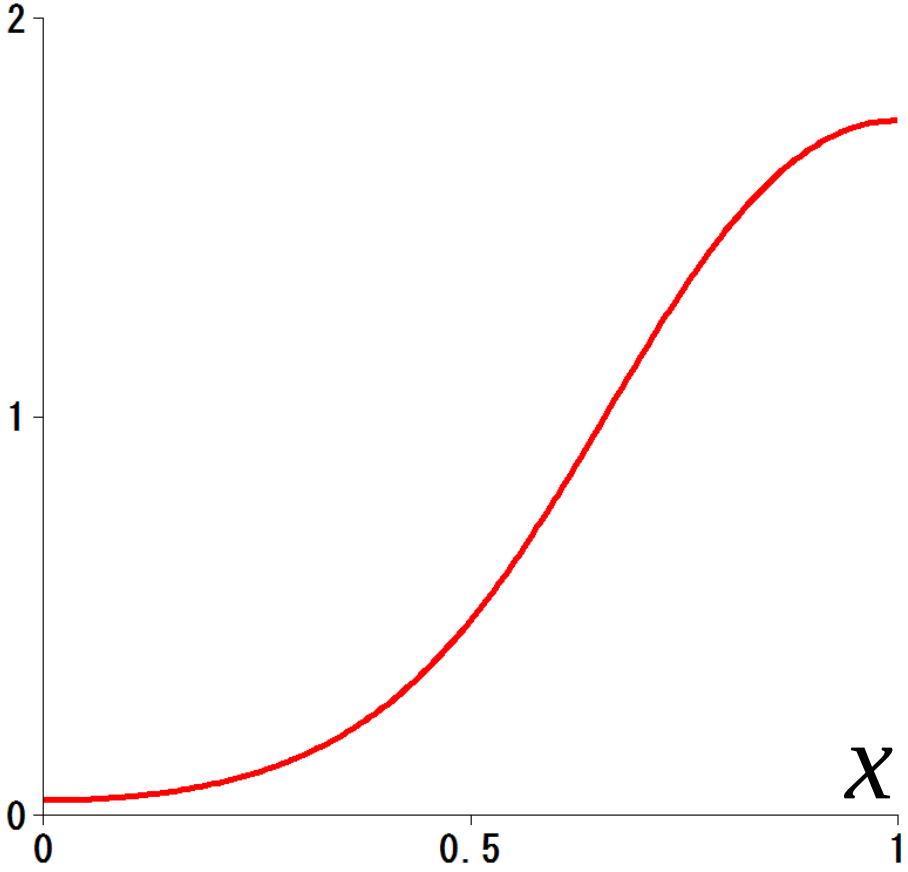


m=1.9 V=1.20 ε=0.207529



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安定

$\tilde{V} \to 1.9, \varepsilon^2 \to 0$ の形状

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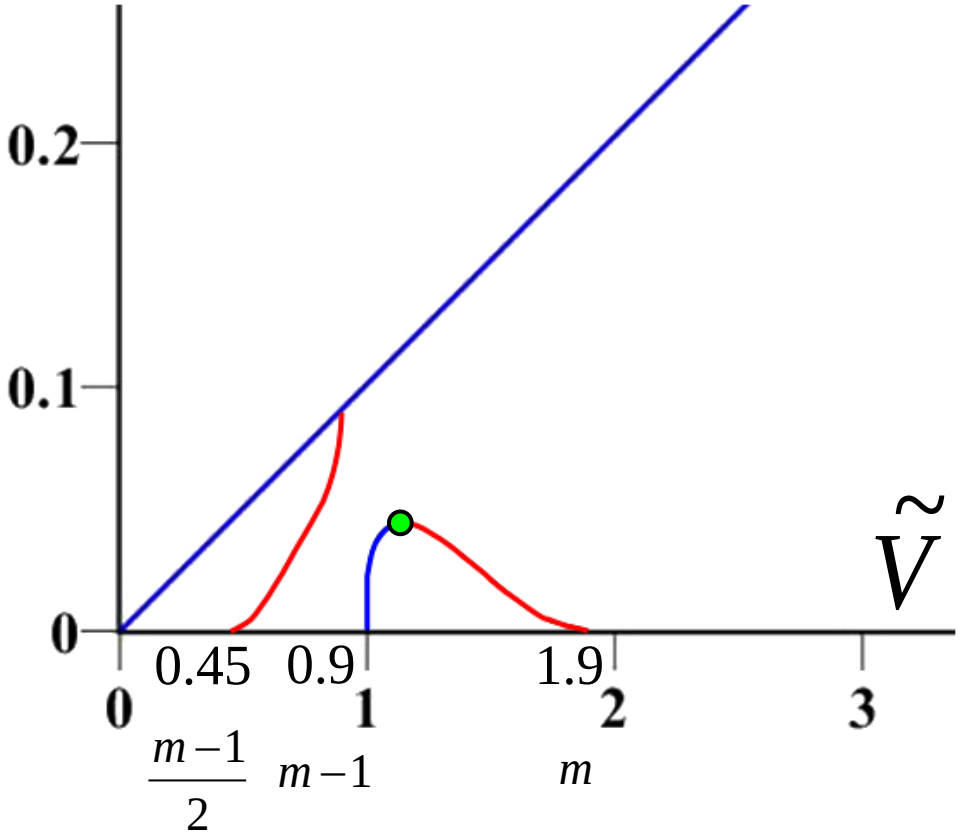

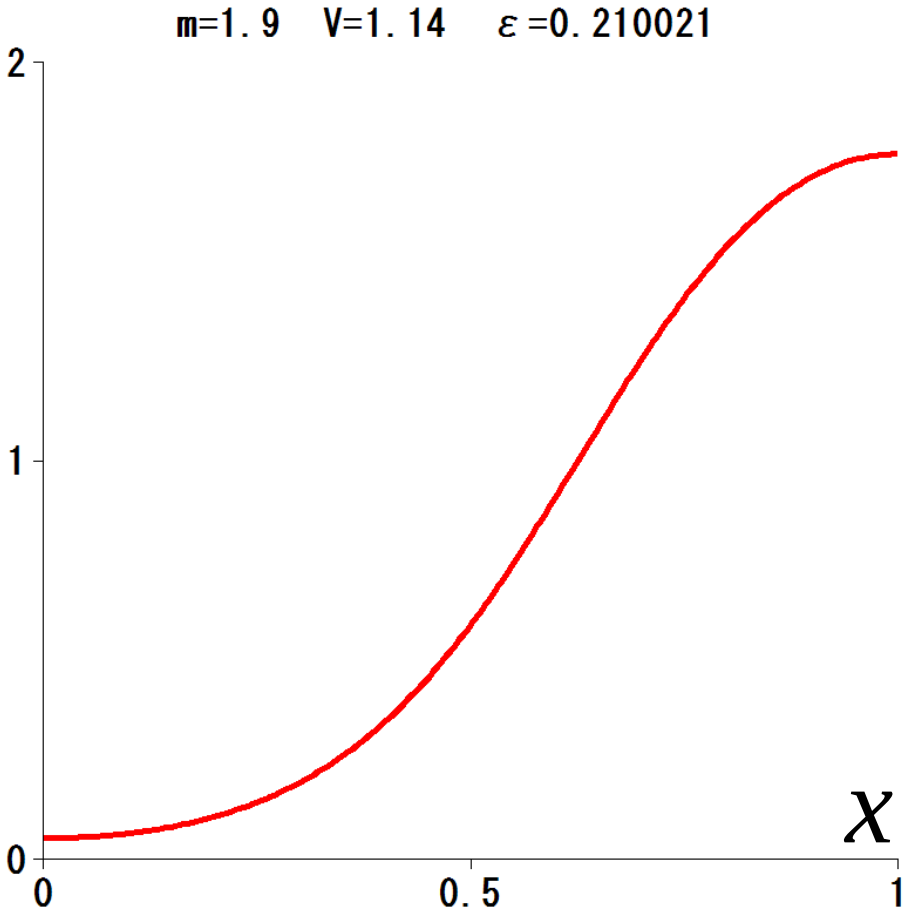

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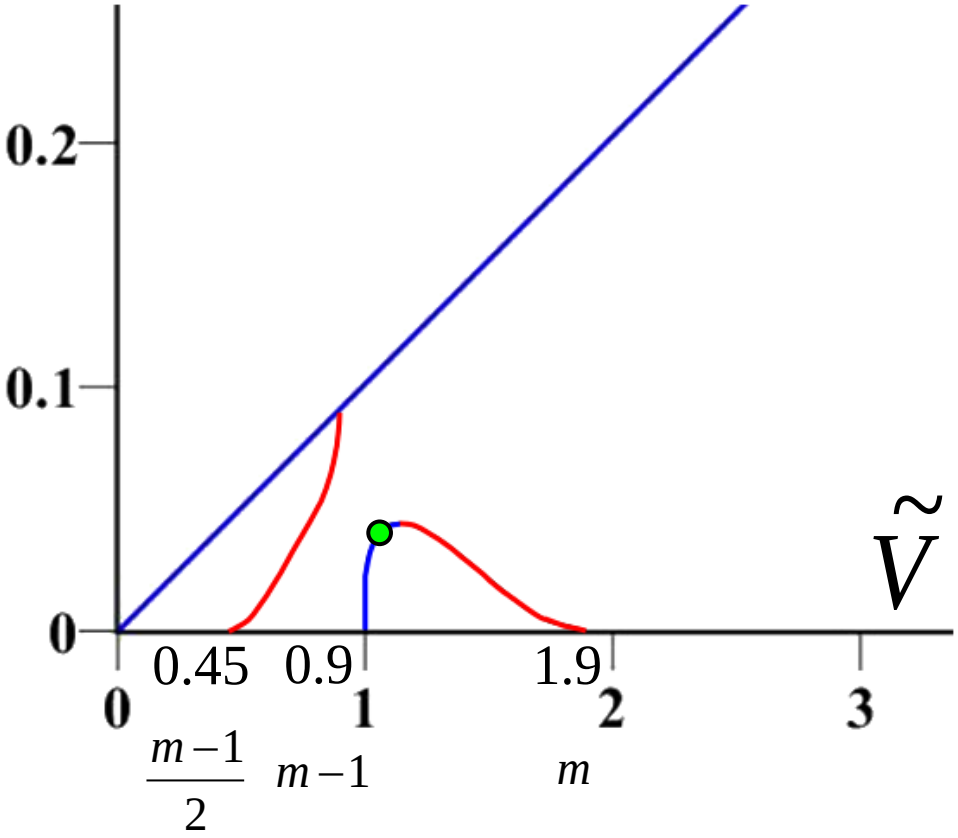
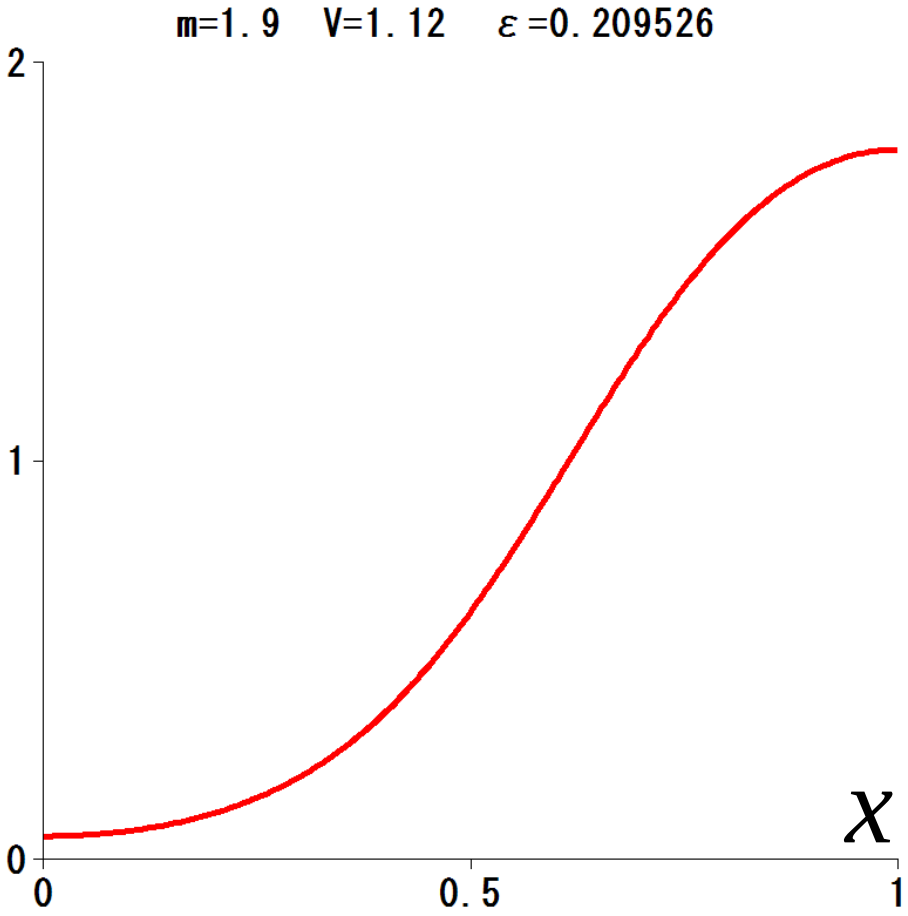
安定

$\tilde{V} \to 1,\ \varepsilon^2 \to 0$ の形状

$m = 1.9$





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安定

$\tilde{V} \to 1,\ \varepsilon^2 \to 0$ の形状

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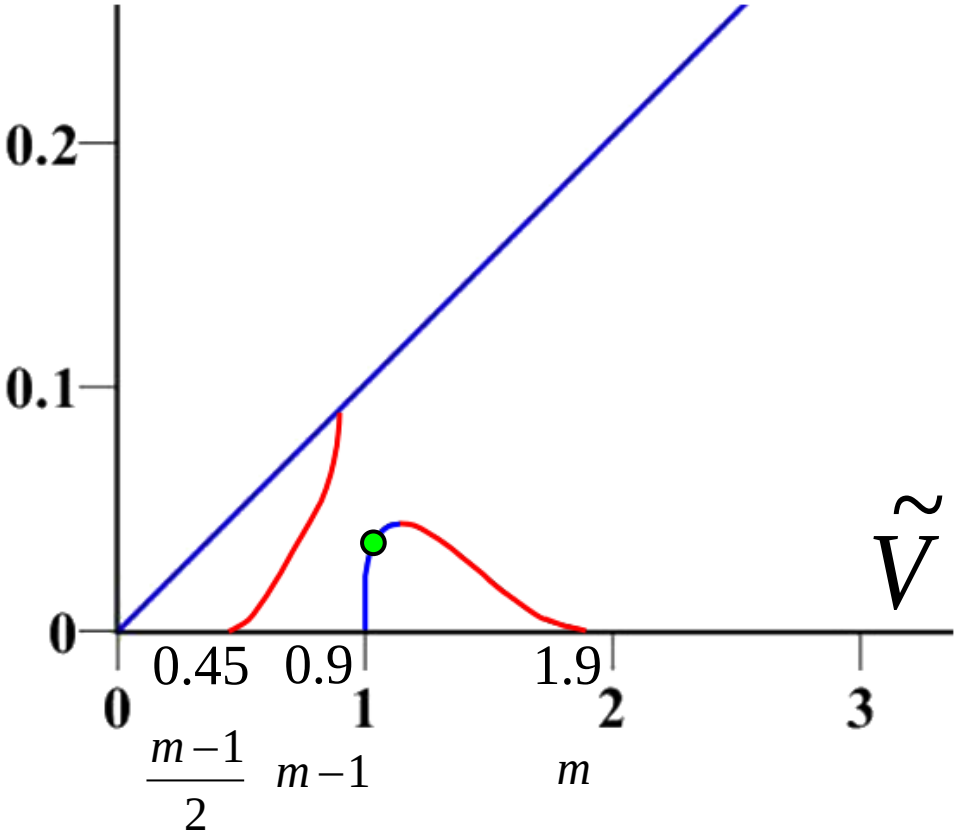

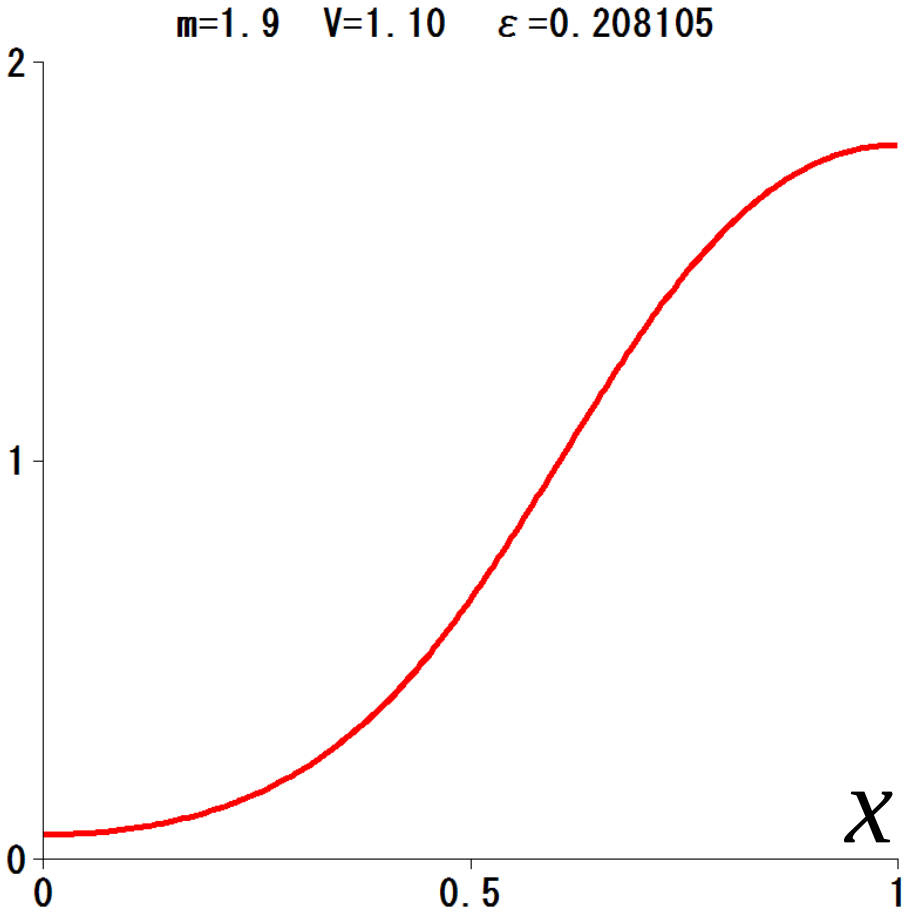

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$\tilde{V} \to 1,\ \varepsilon^2 \to 0$ の形状

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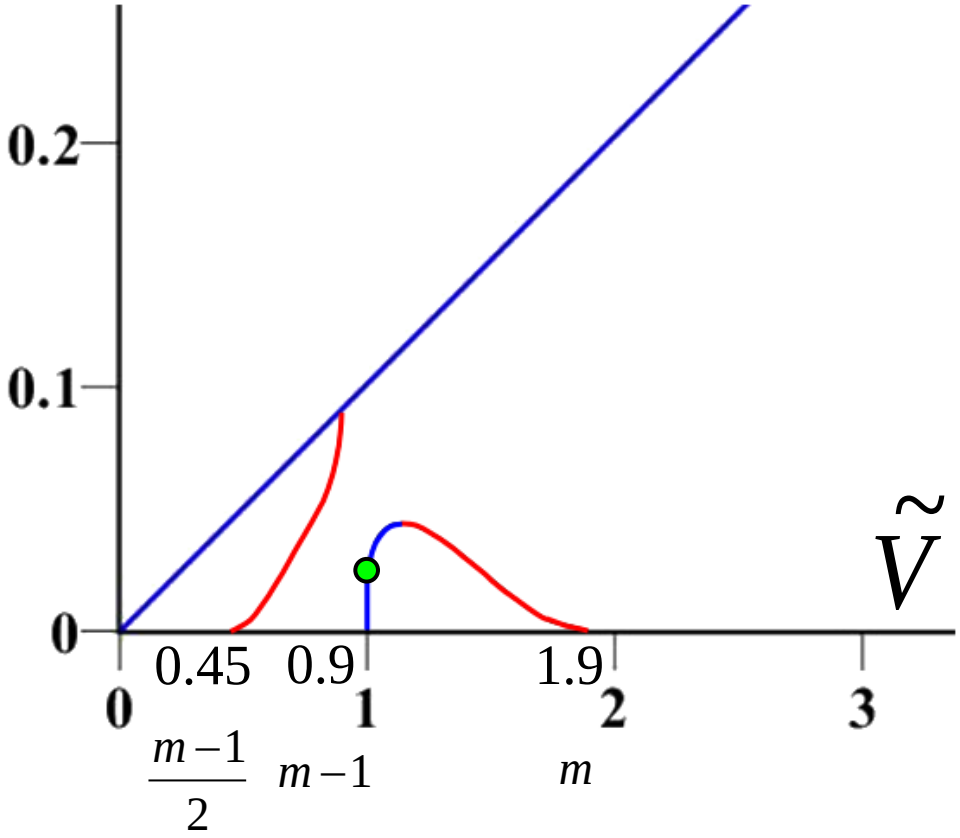

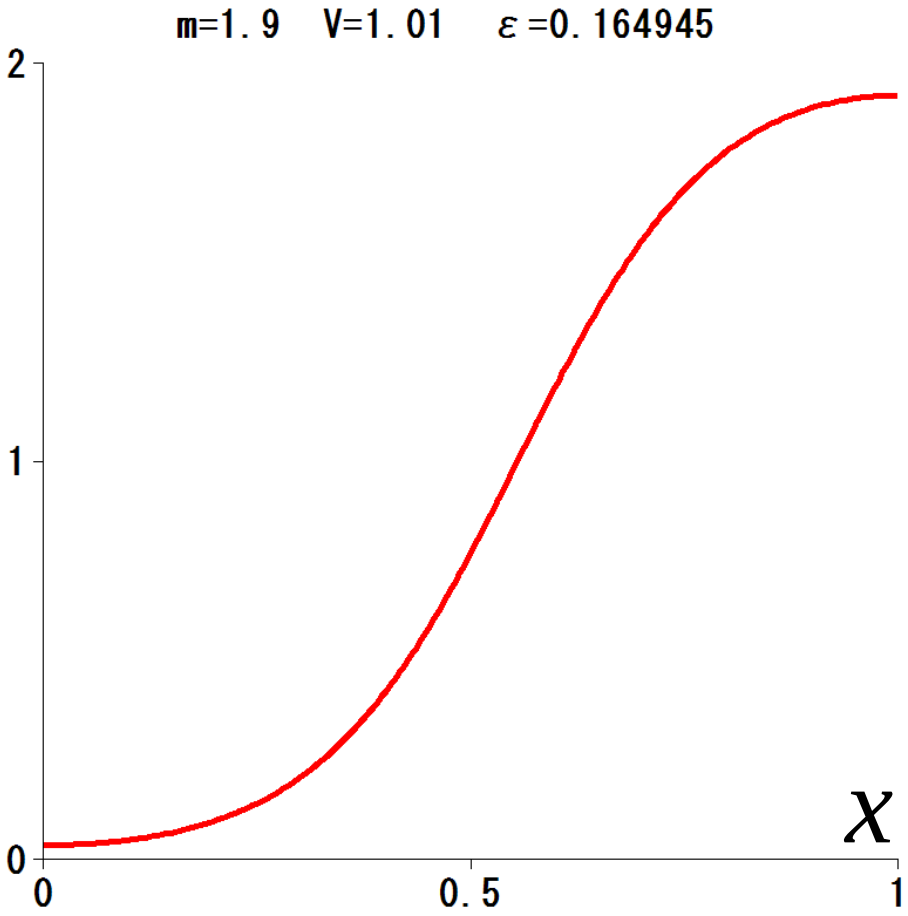

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安定

$\tilde{V} \to 1,\ \varepsilon^2 \to 0$ の形状

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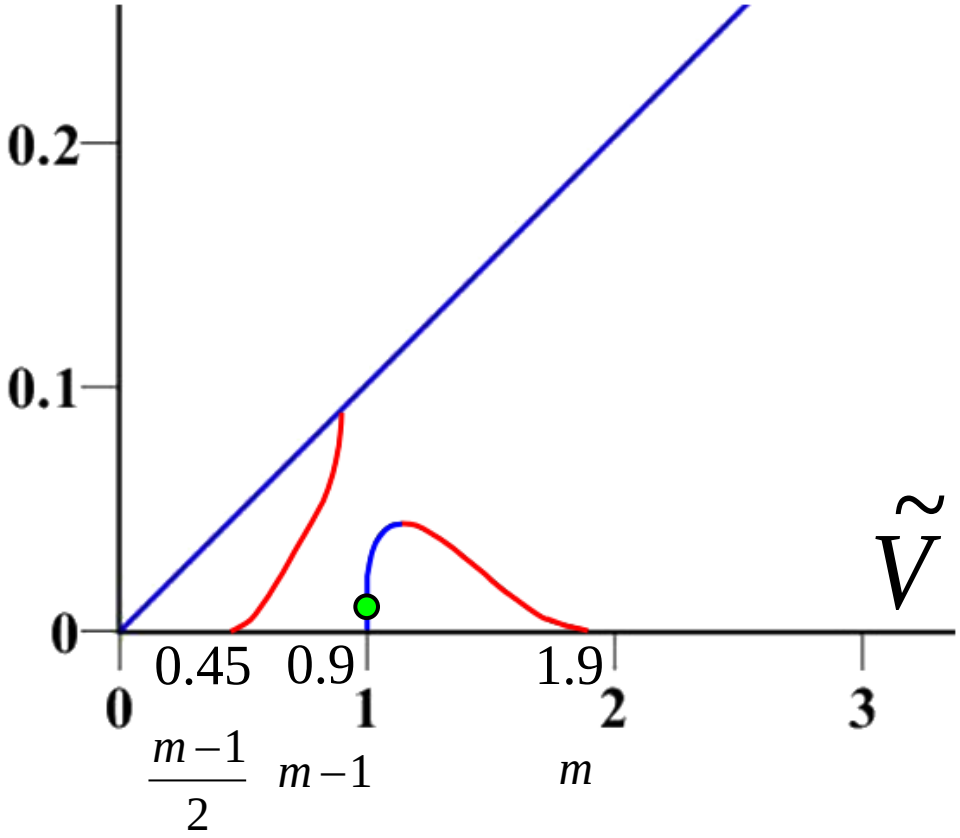

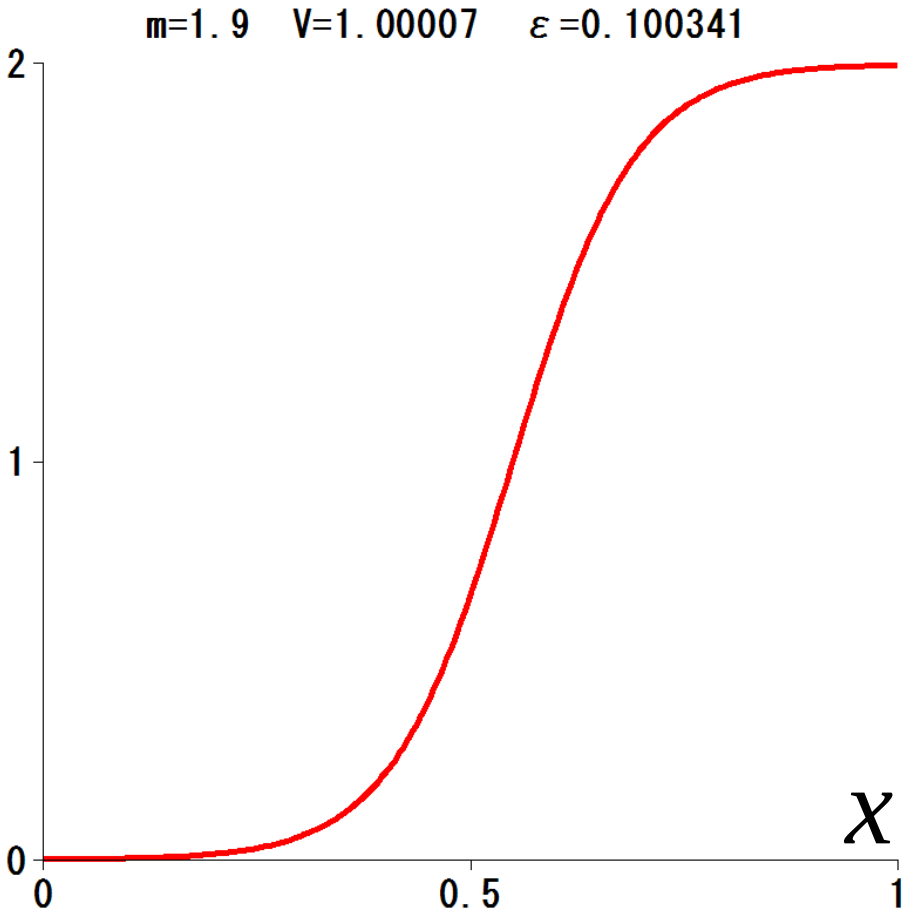

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$\tilde{V} \to 1,\ \varepsilon^2 \to 0$ の形状

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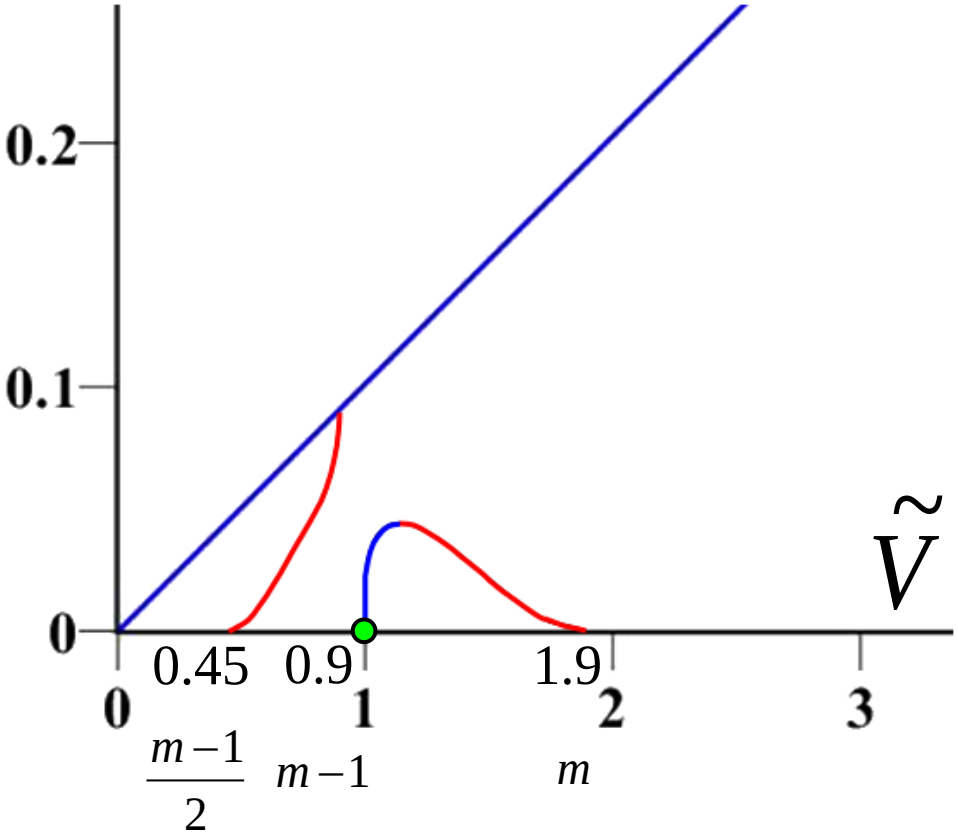

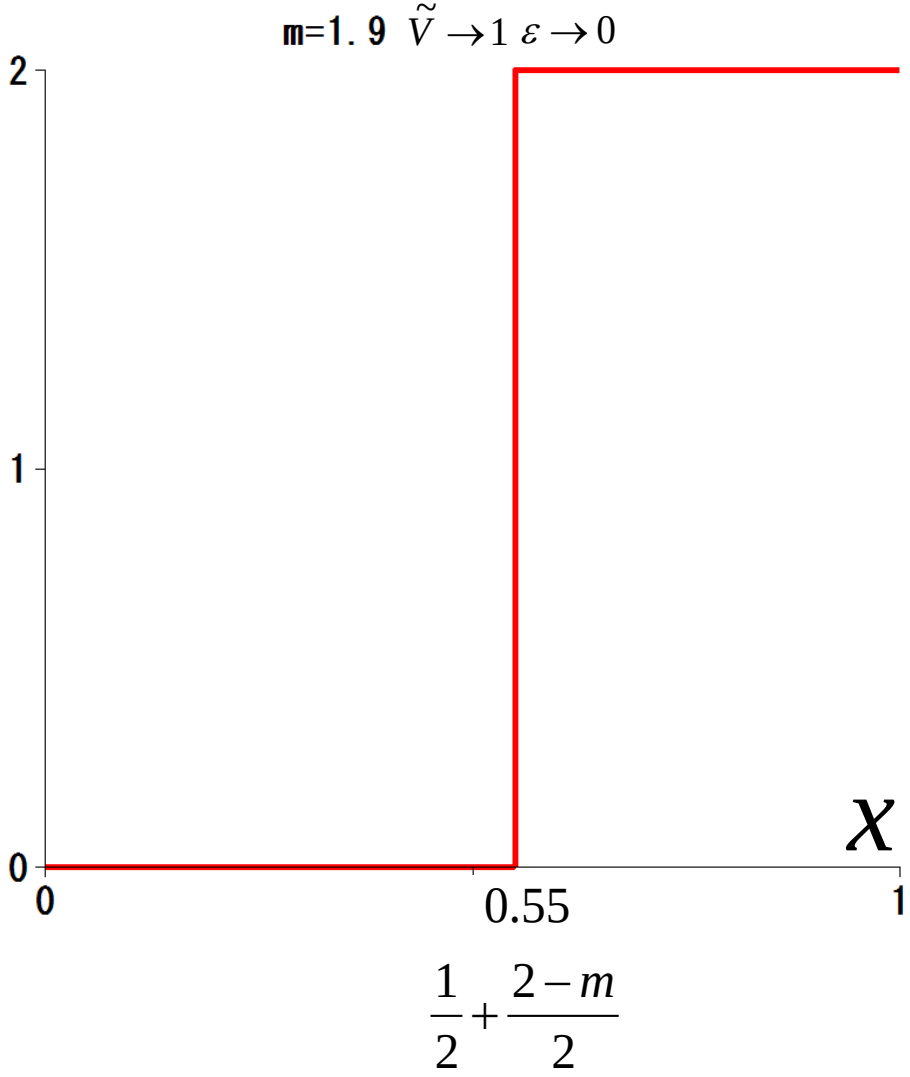

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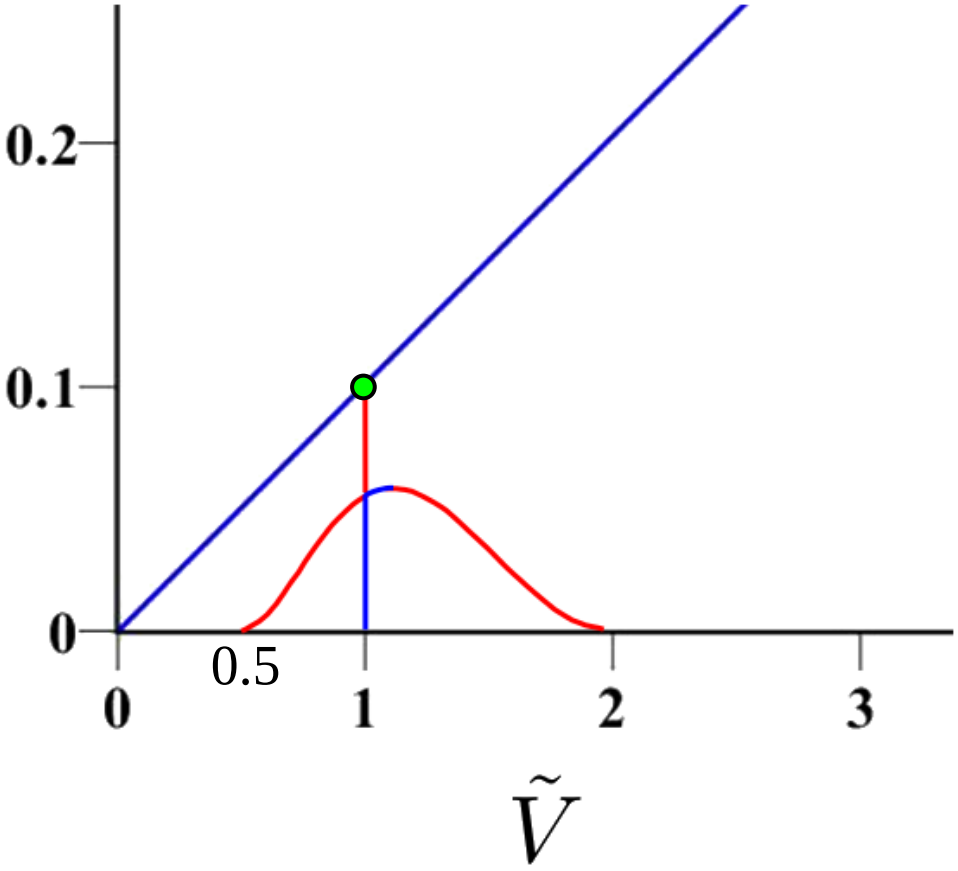
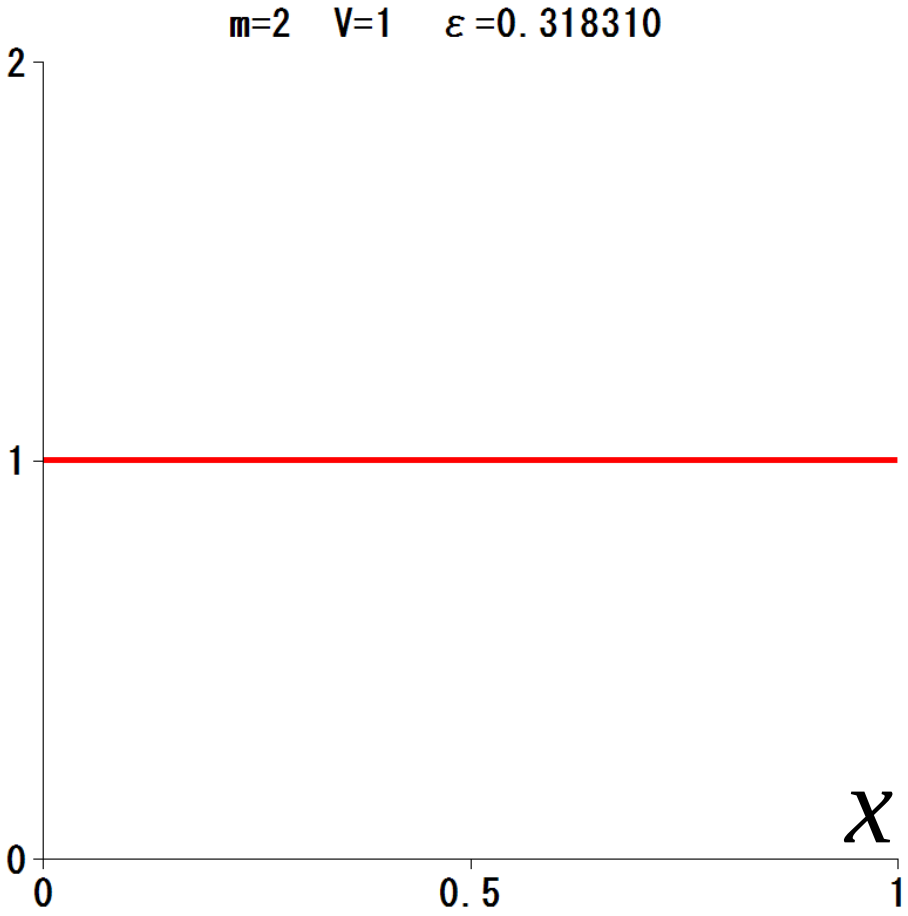
不安定

$\tilde{V} = 1, \quad \varepsilon^2 \to 0$ の形状

$m = 2$





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不安定

$\tilde{V} = 1, \quad \varepsilon^2 \to 0$ の形状

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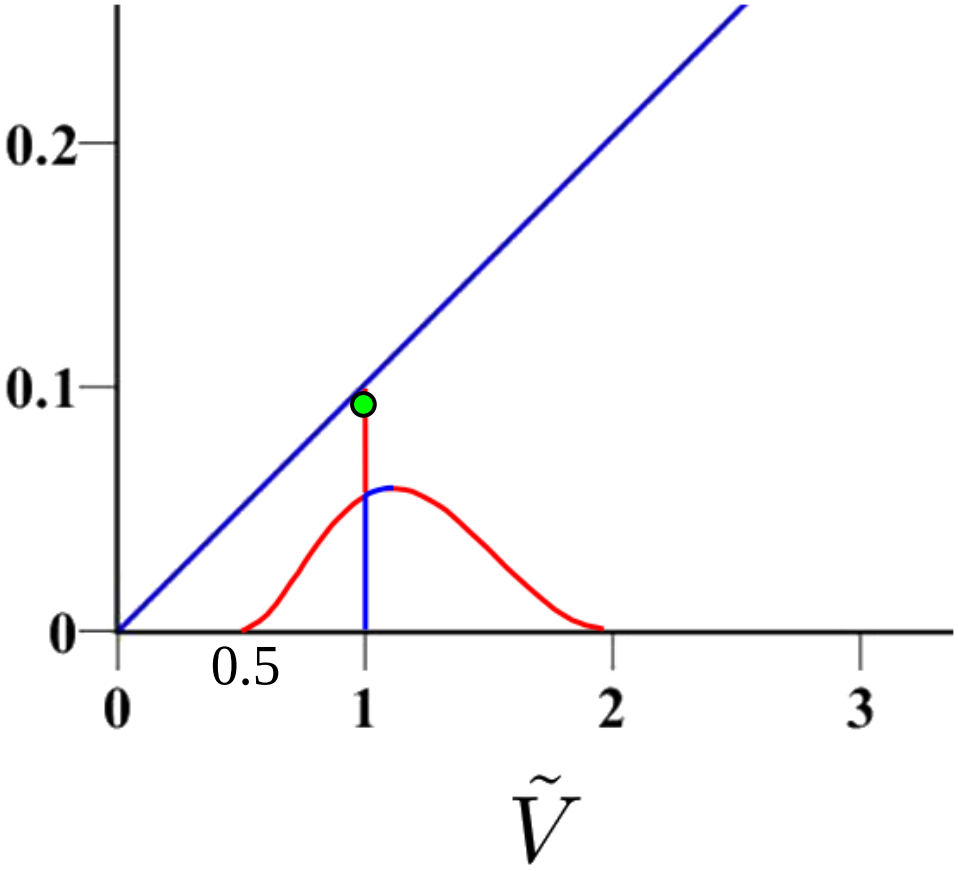

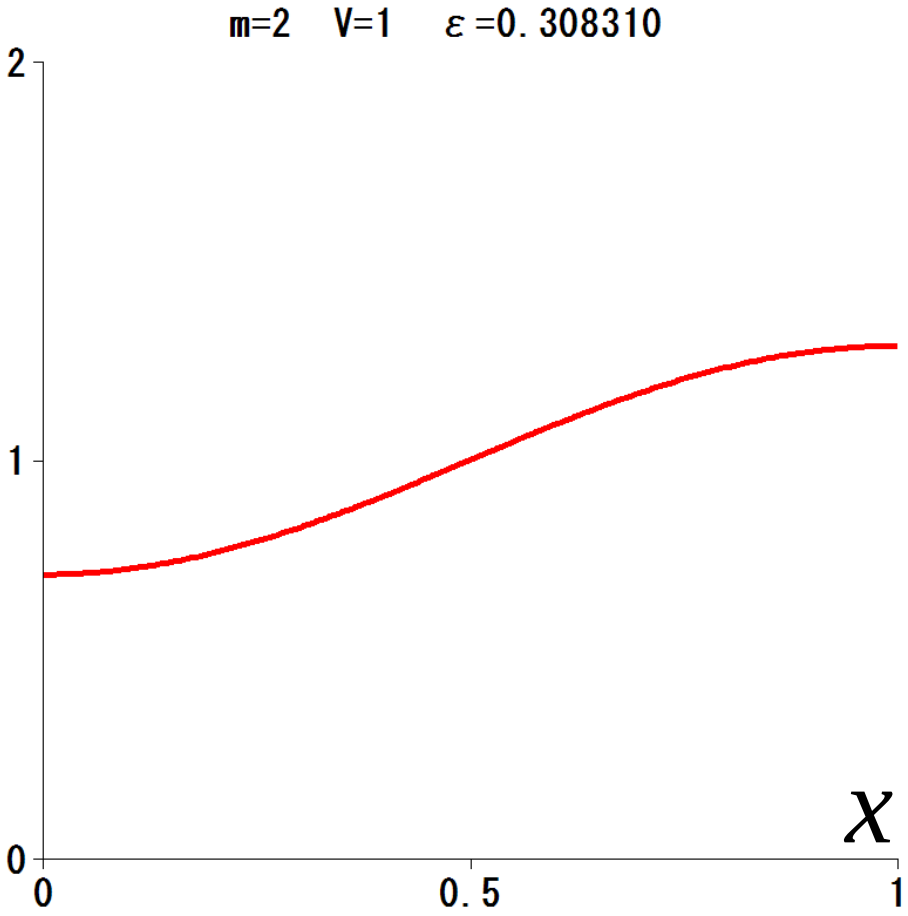

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不安定

$\tilde{V} = 1, \quad \varepsilon^2 \to 0$ の形状

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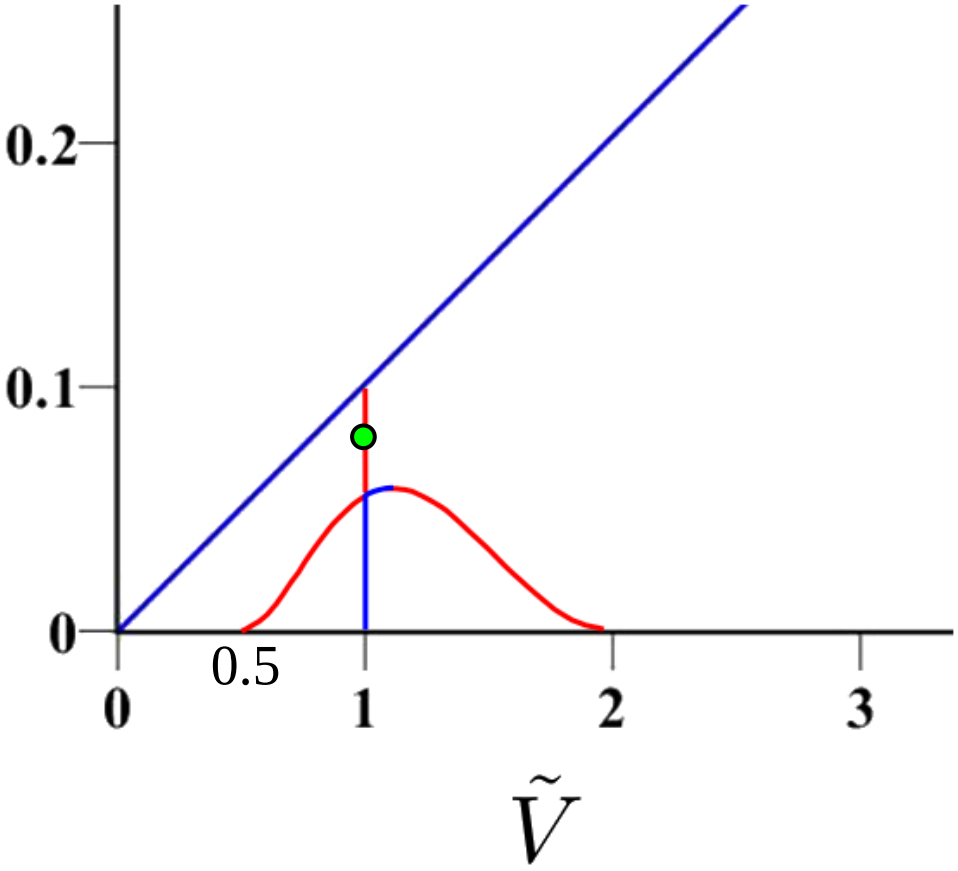

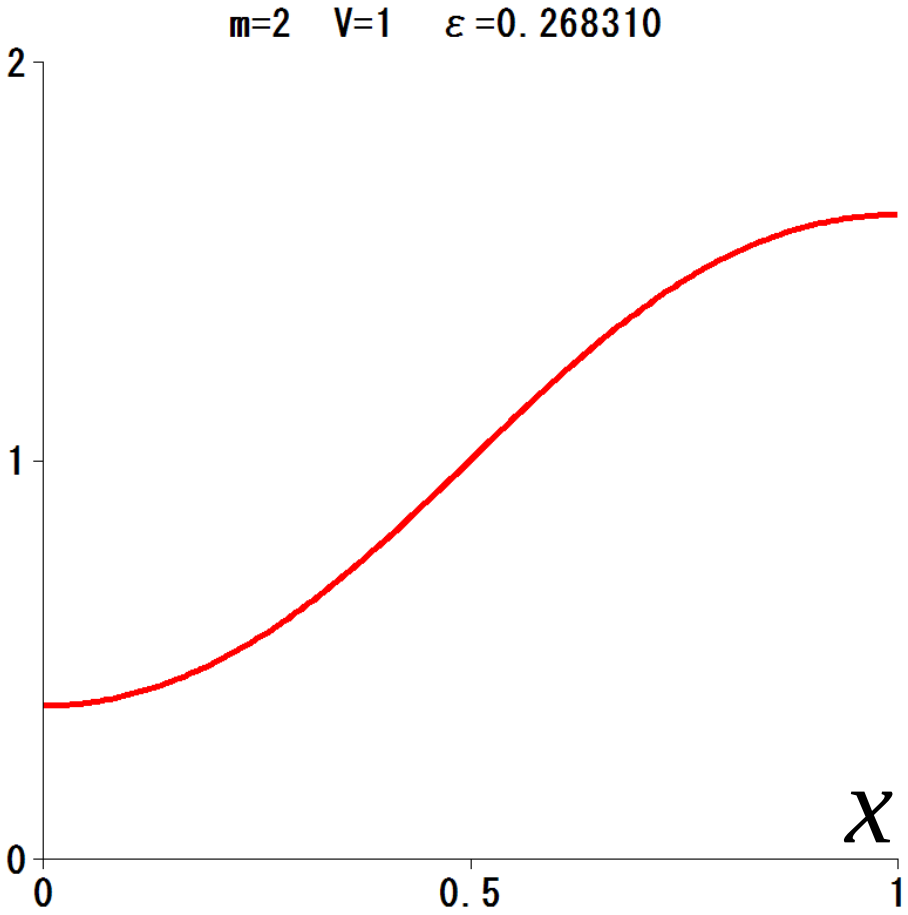

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不安定

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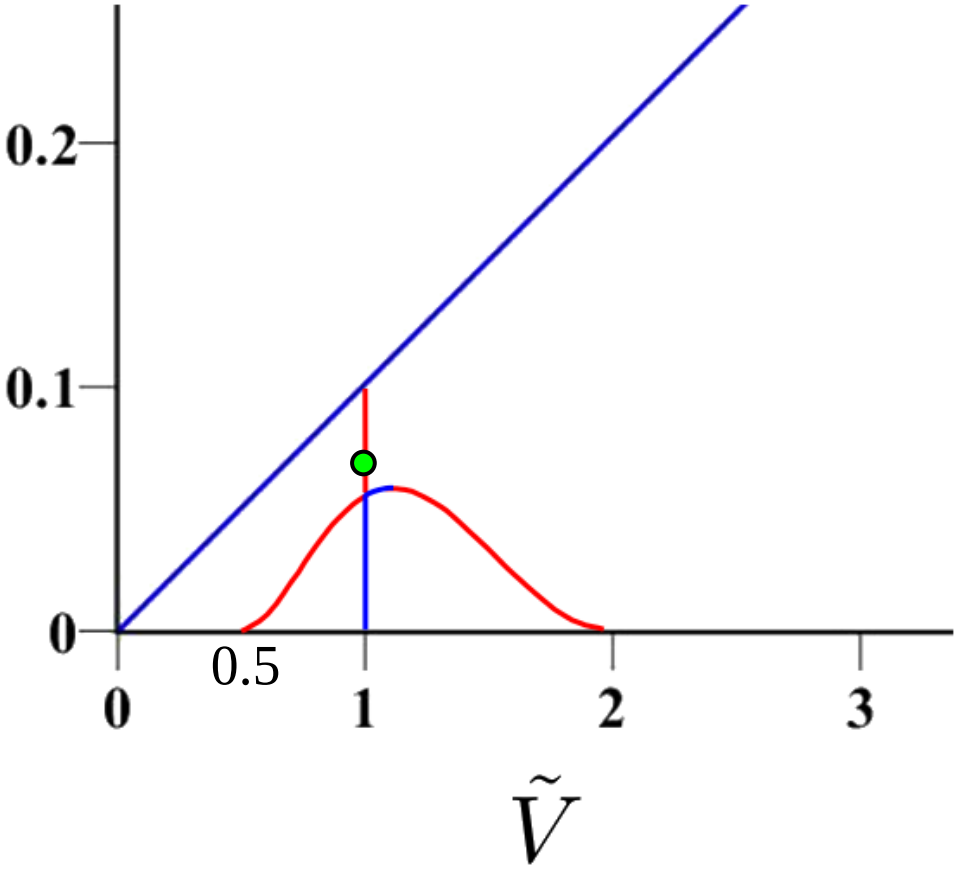

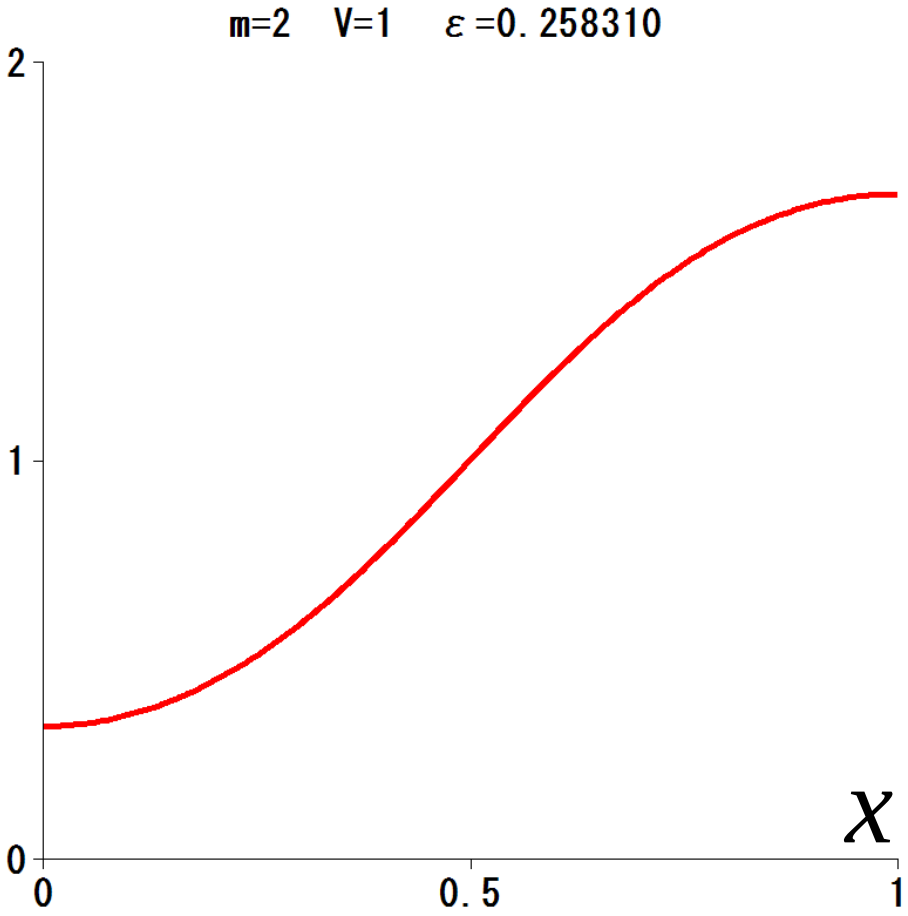

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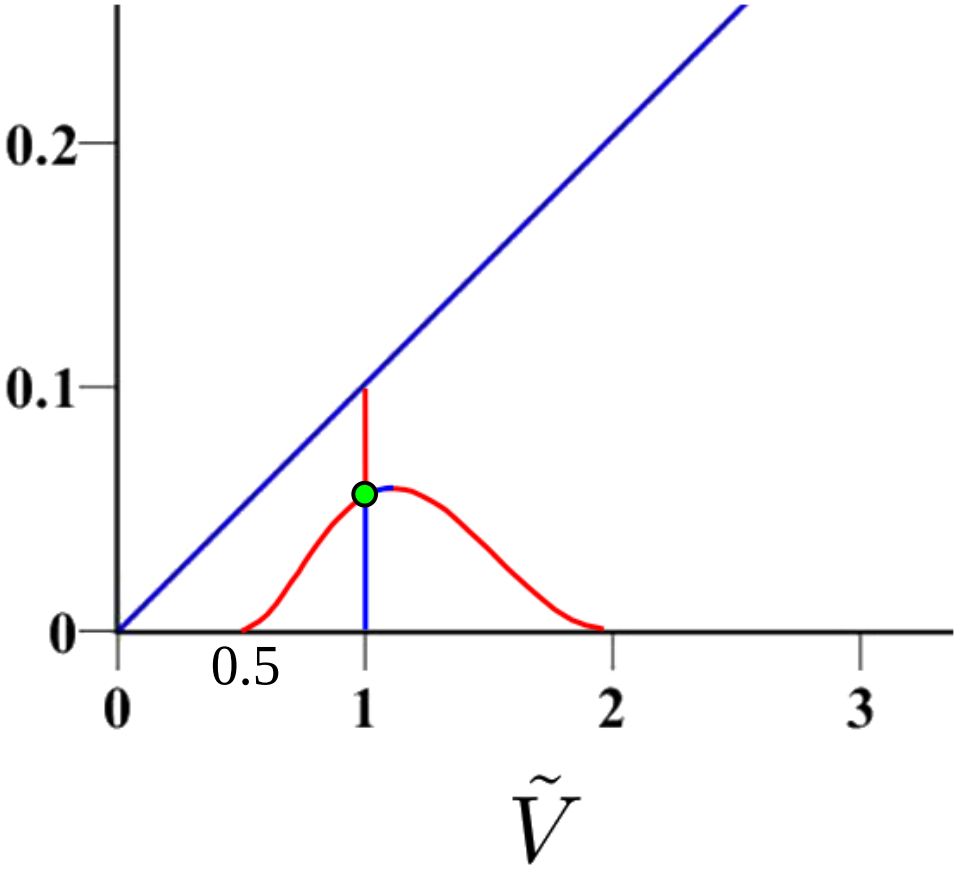
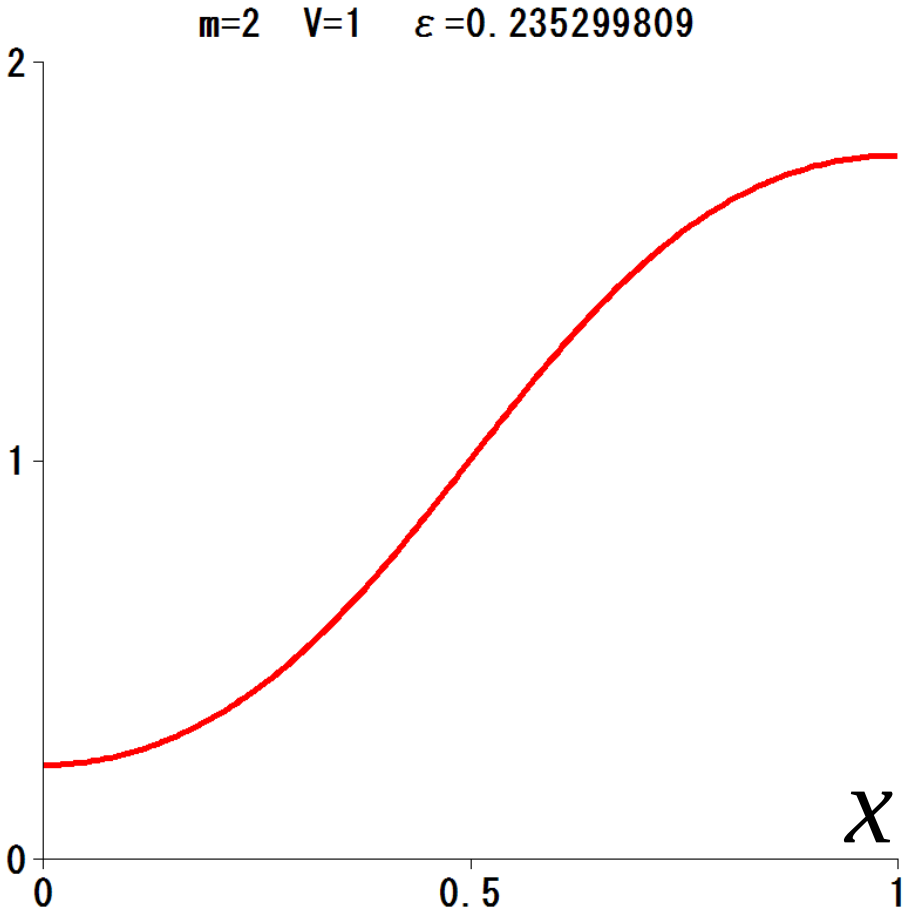
$\tilde{V} = 1, \quad \varepsilon^2 \to 0$ の形状

$m = 2$





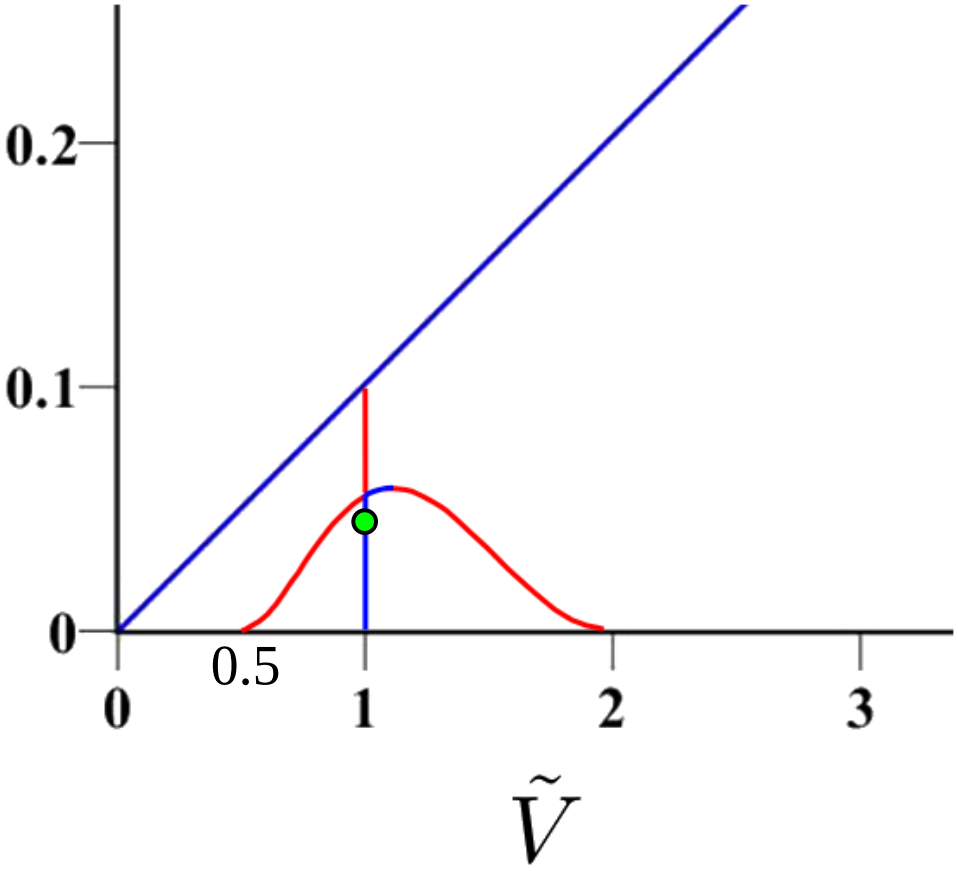
2 次分岐点

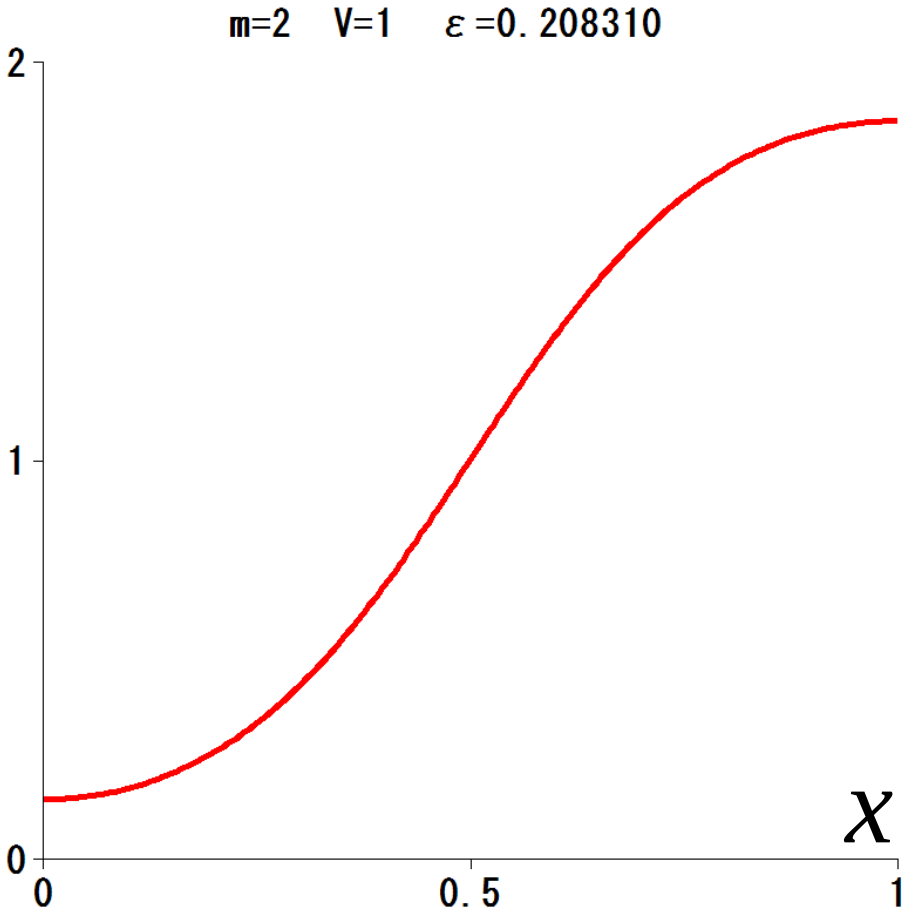
$$(SLP)\begin{cases} \varepsilon^2 W_{xx} + W(W-1)(\tilde{V}+1-W) = 0, \\ W_x(0) = W_x(1) = 0, \\ W'(x) > 0, \\ \int_0^1 W(x)\,\mathrm{dx} + \tilde{V} = m. \end{cases}$$

安定

$\tilde{V} = 1, \quad \varepsilon^2 \to 0$ の形状

$m = 2$





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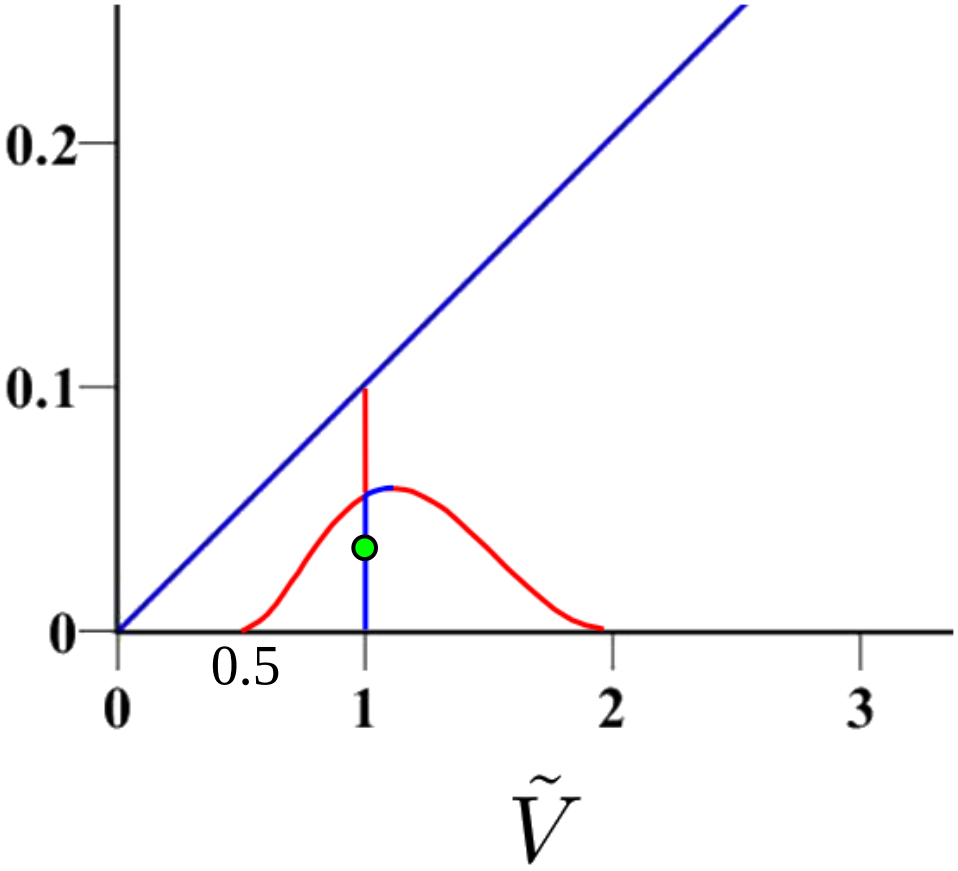

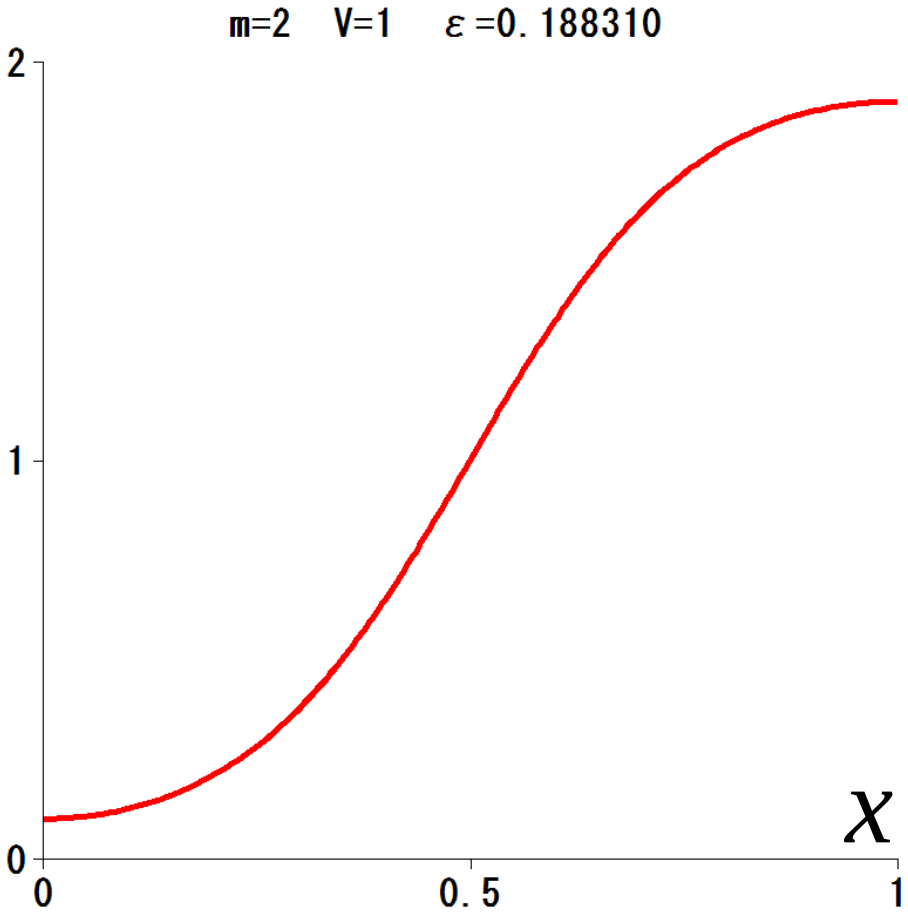

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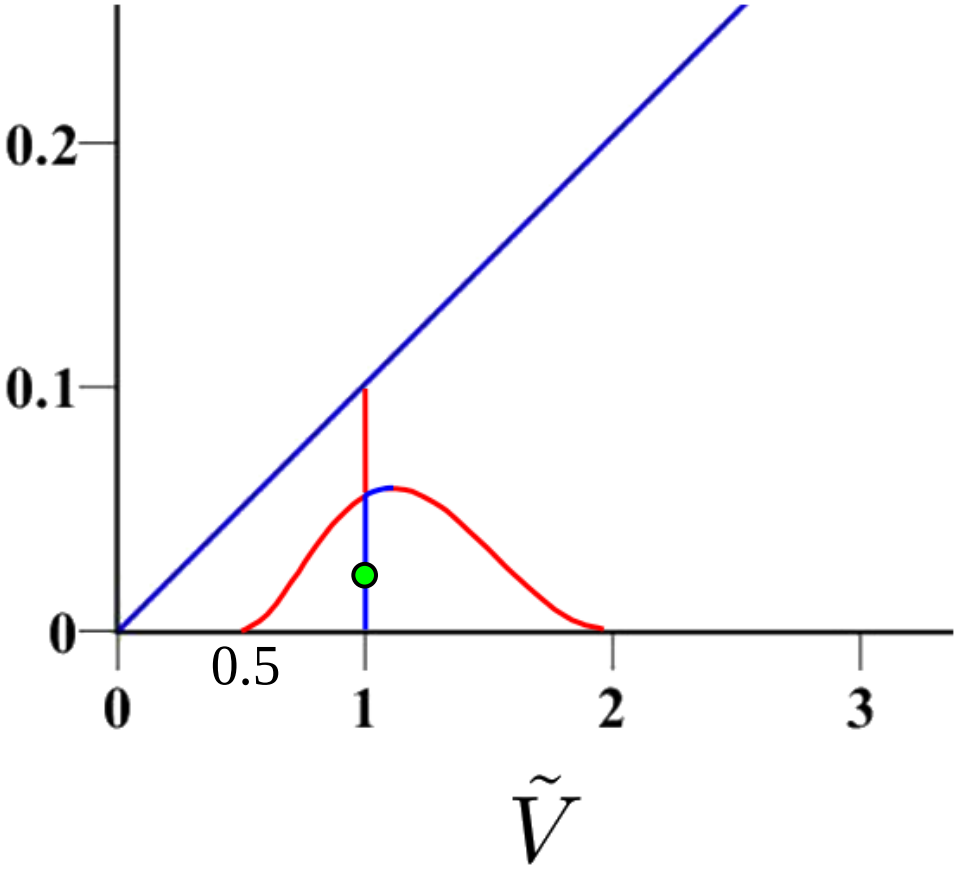

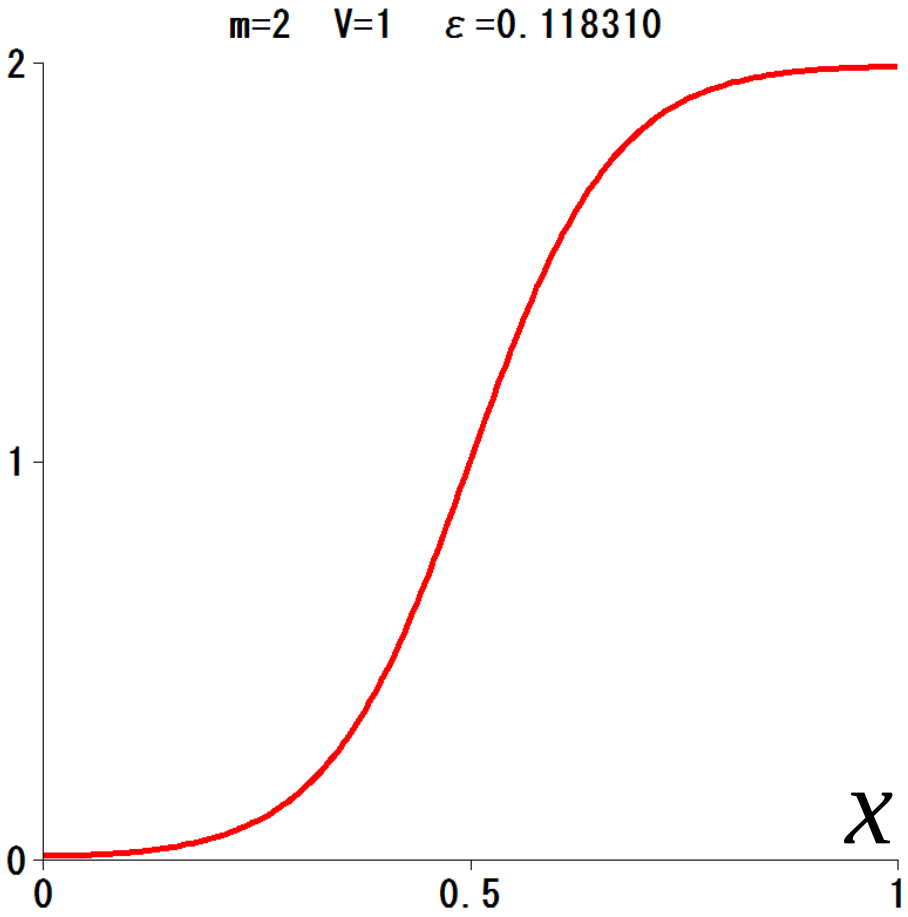

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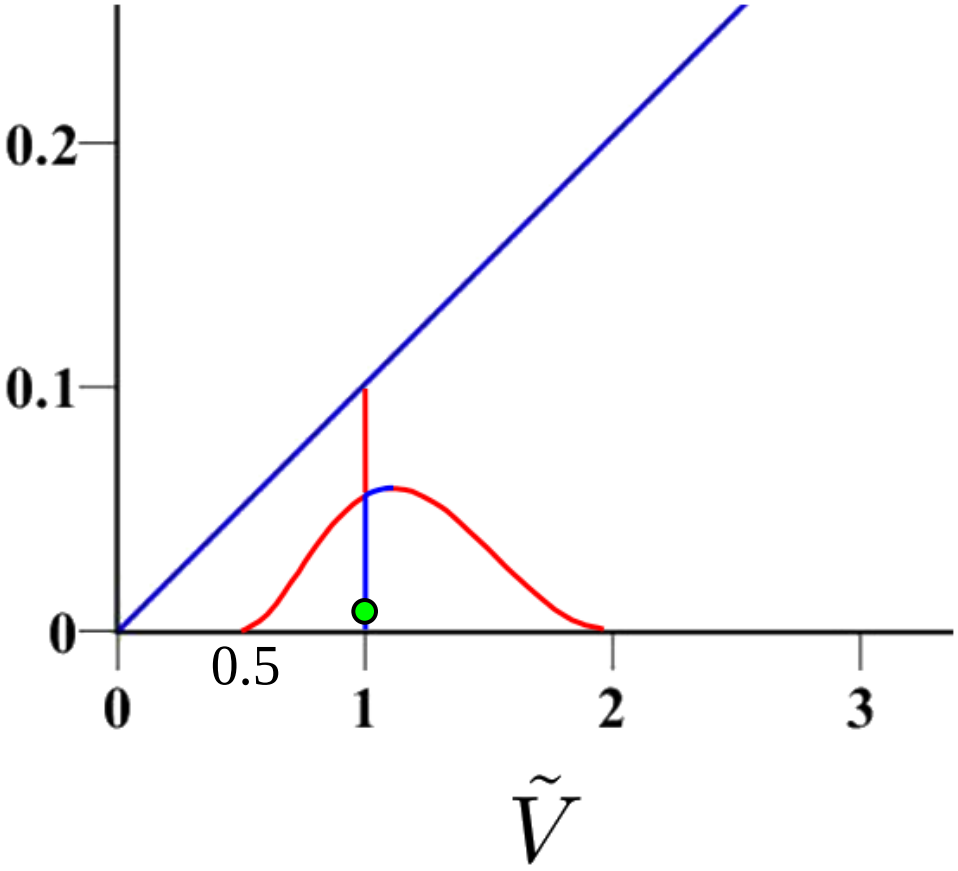

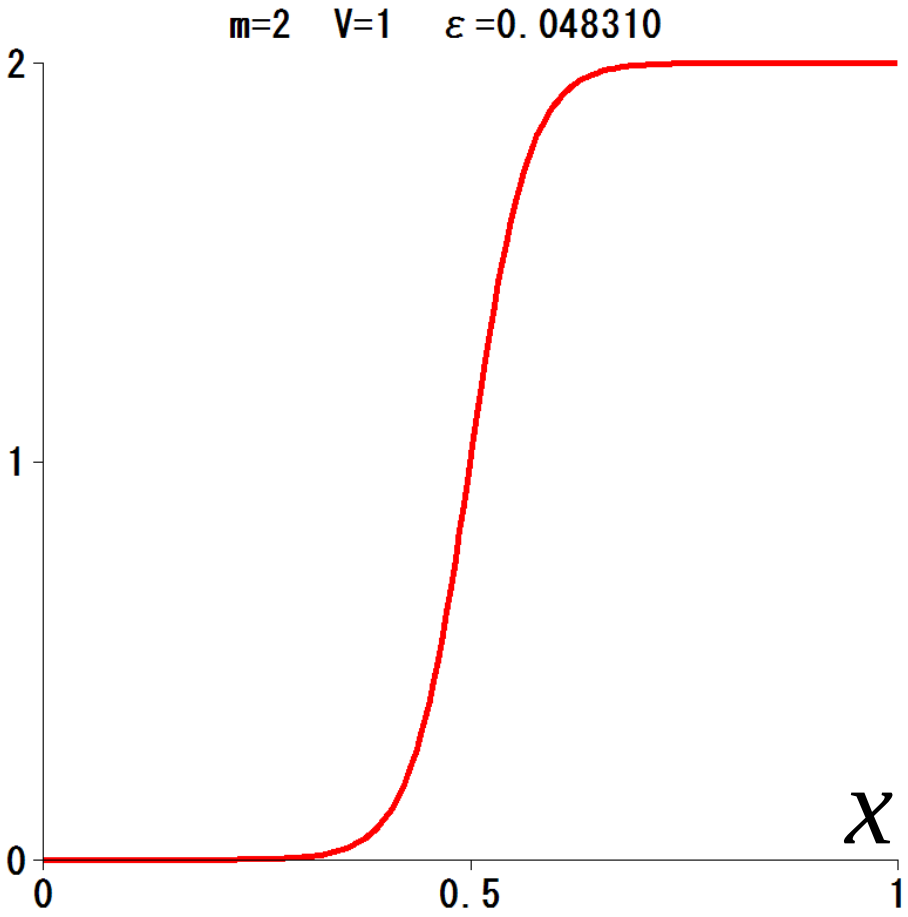

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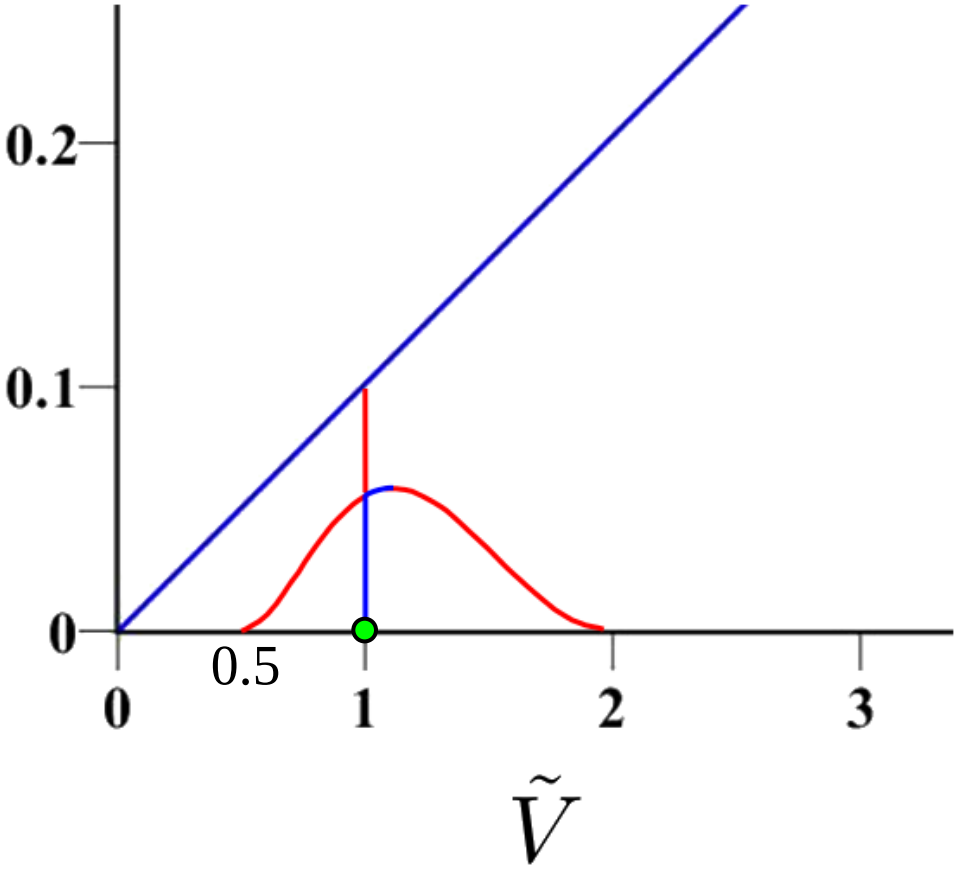

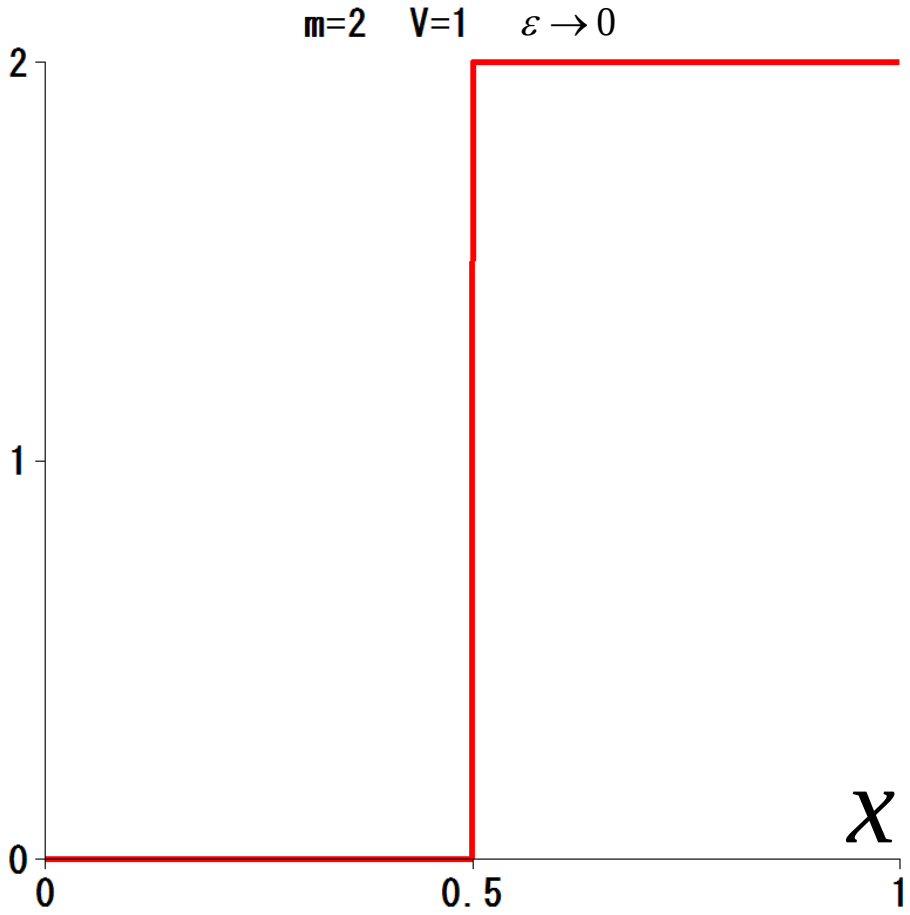

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不安定

$\tilde{V} \to 2,\ \varepsilon^2 \to 0$ の形状

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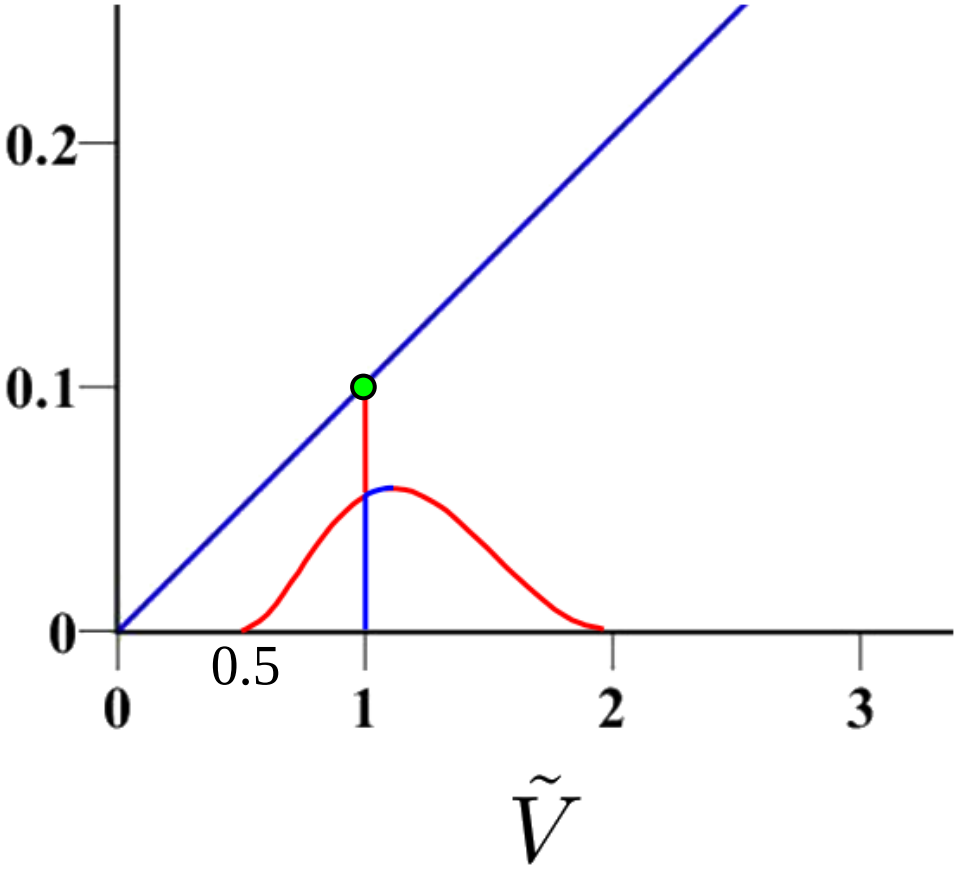

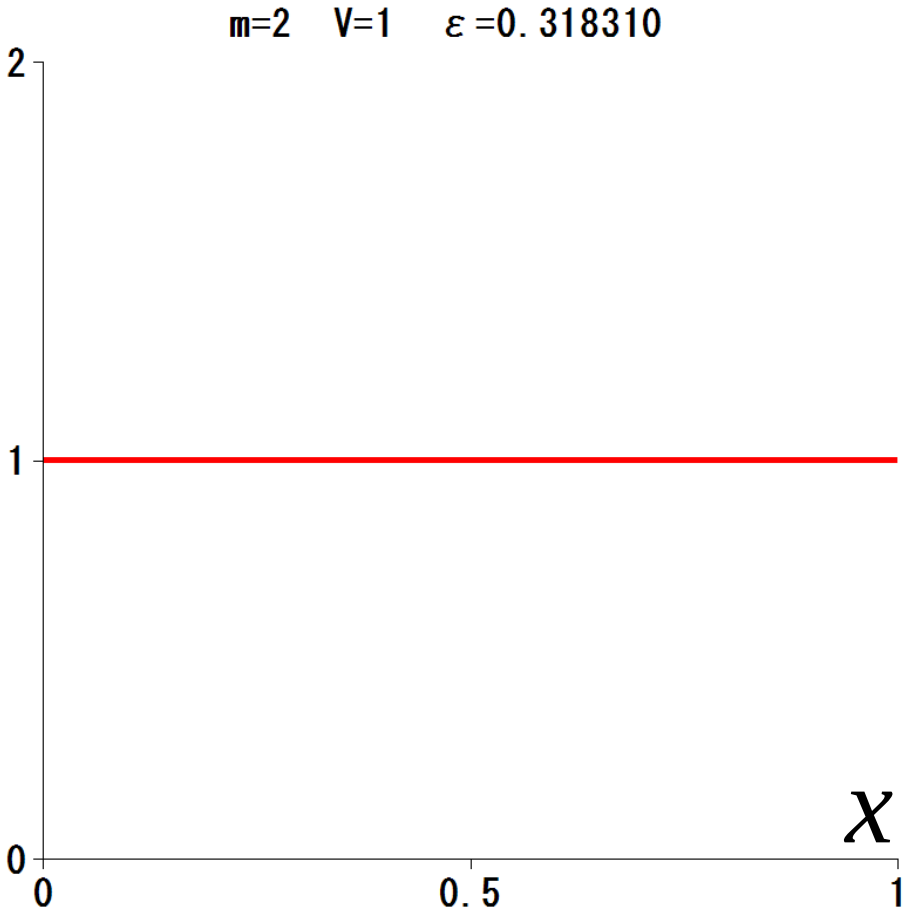

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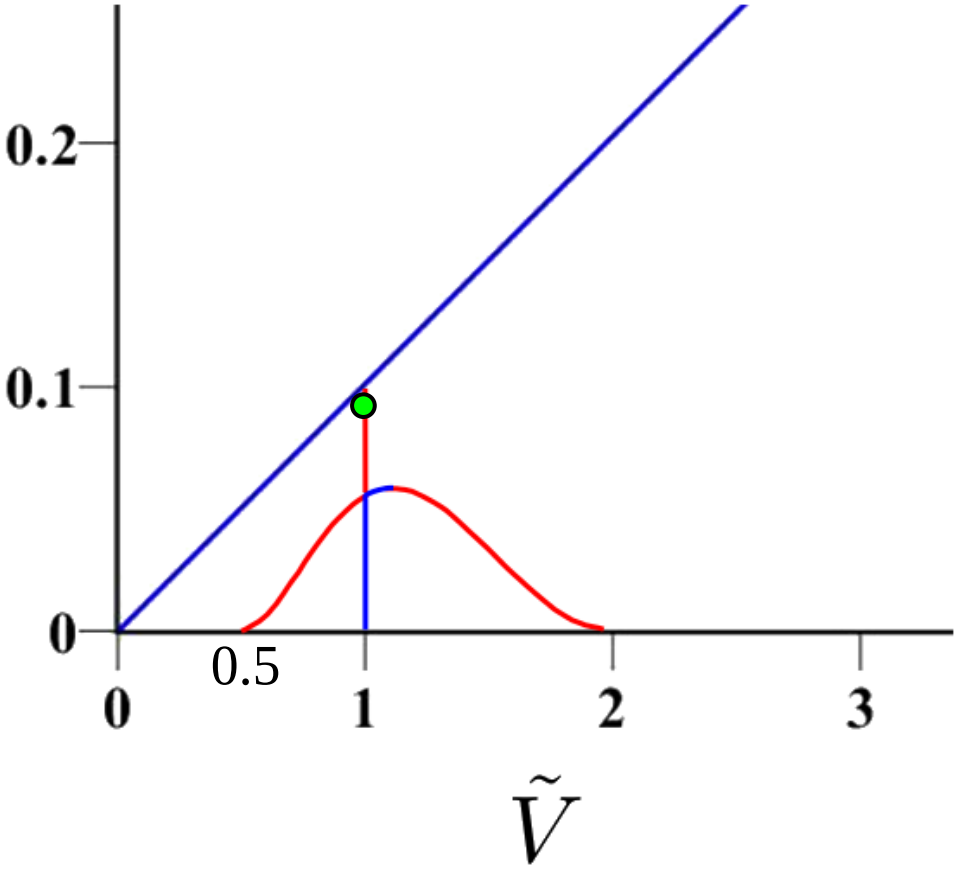

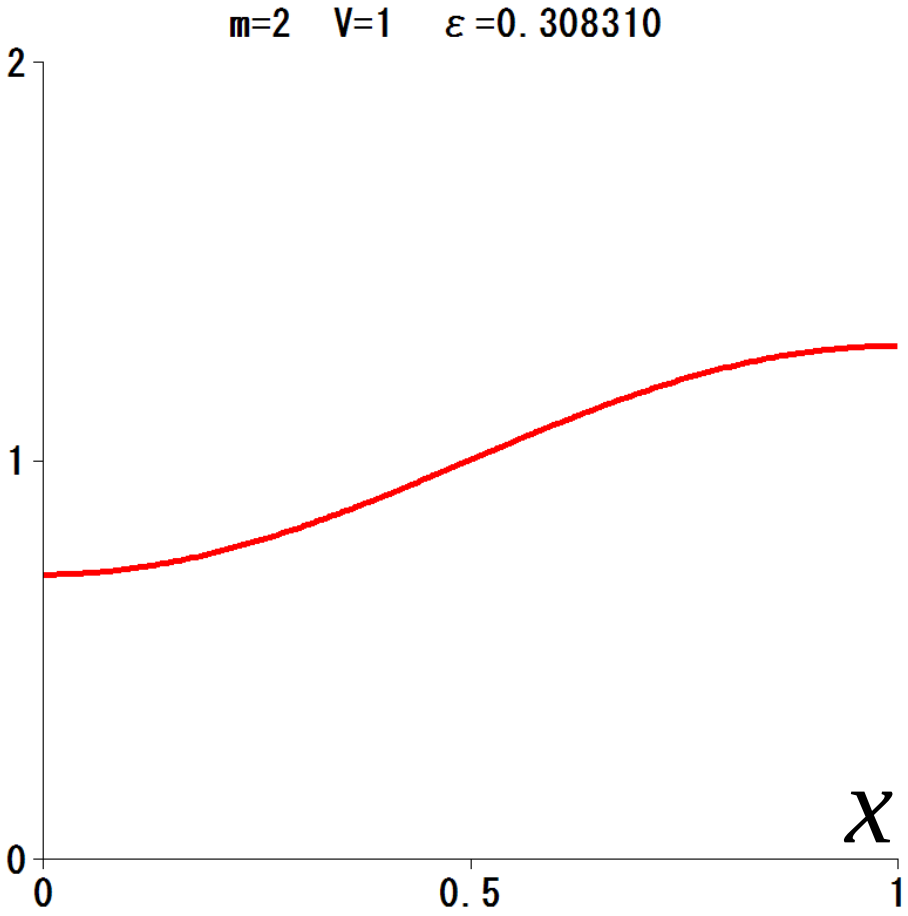

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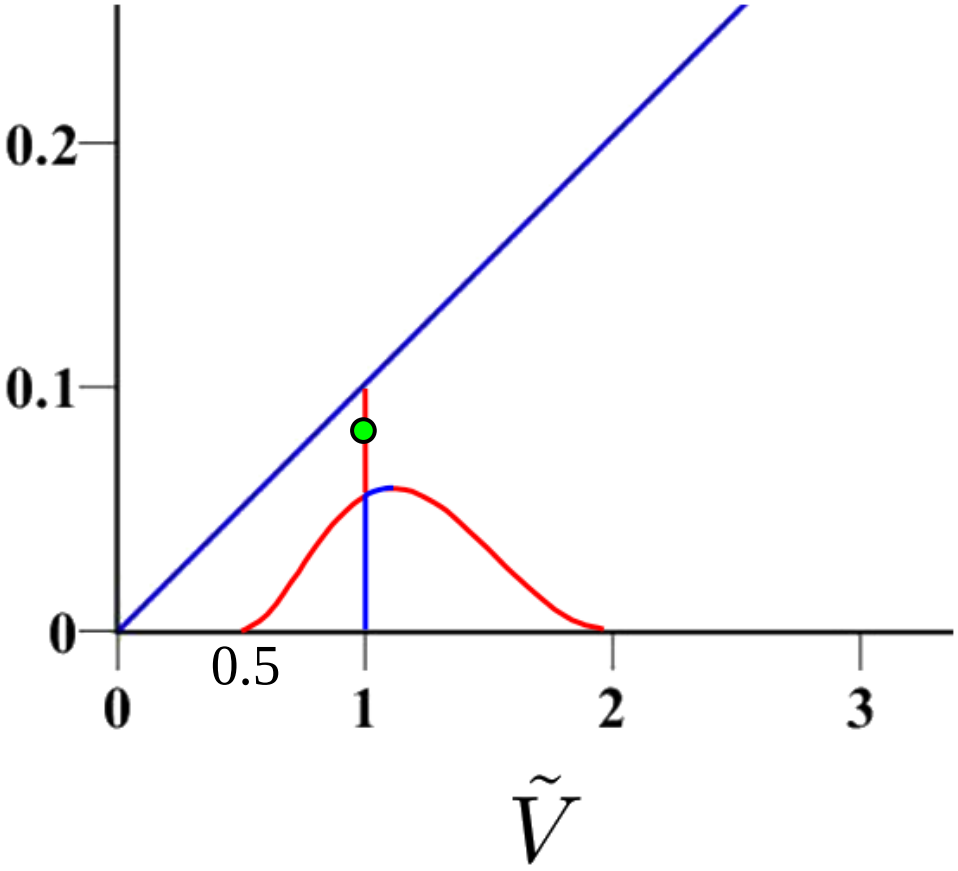

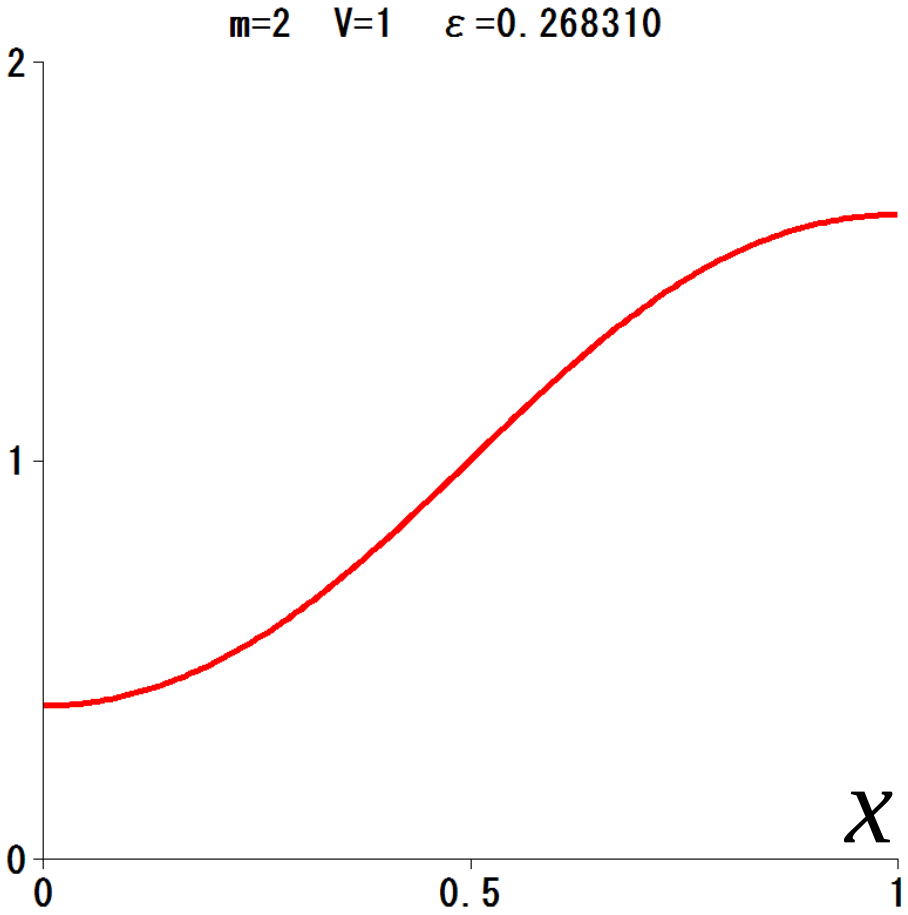

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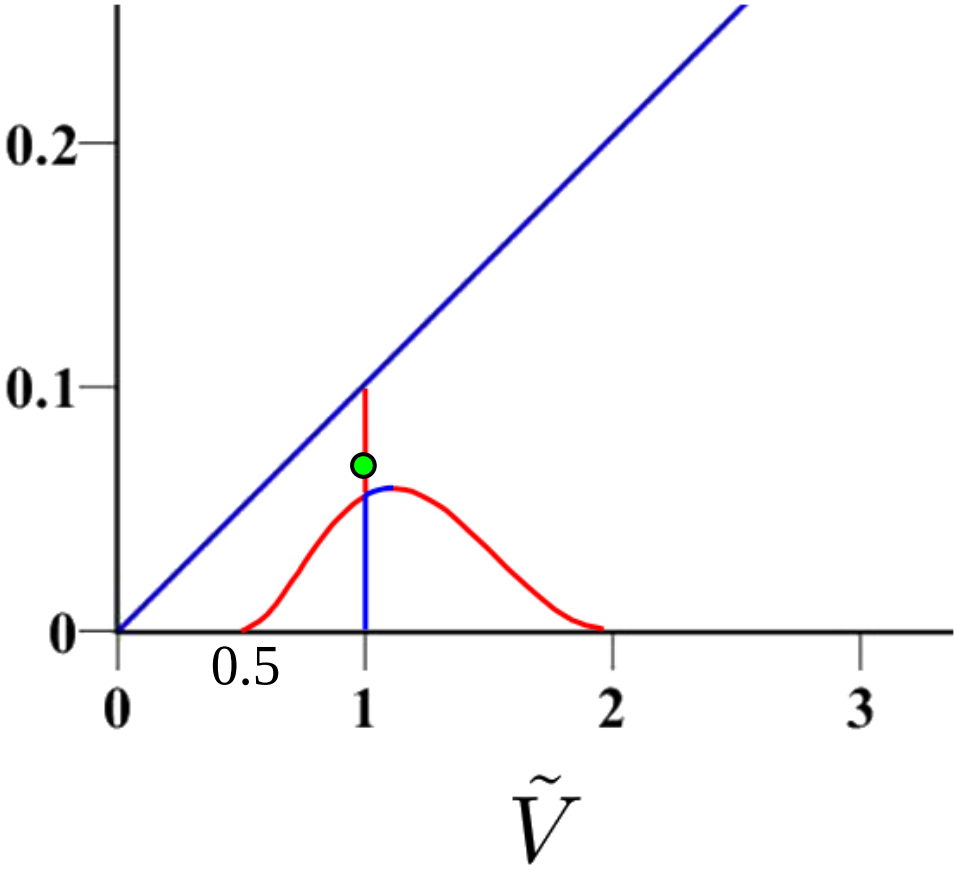

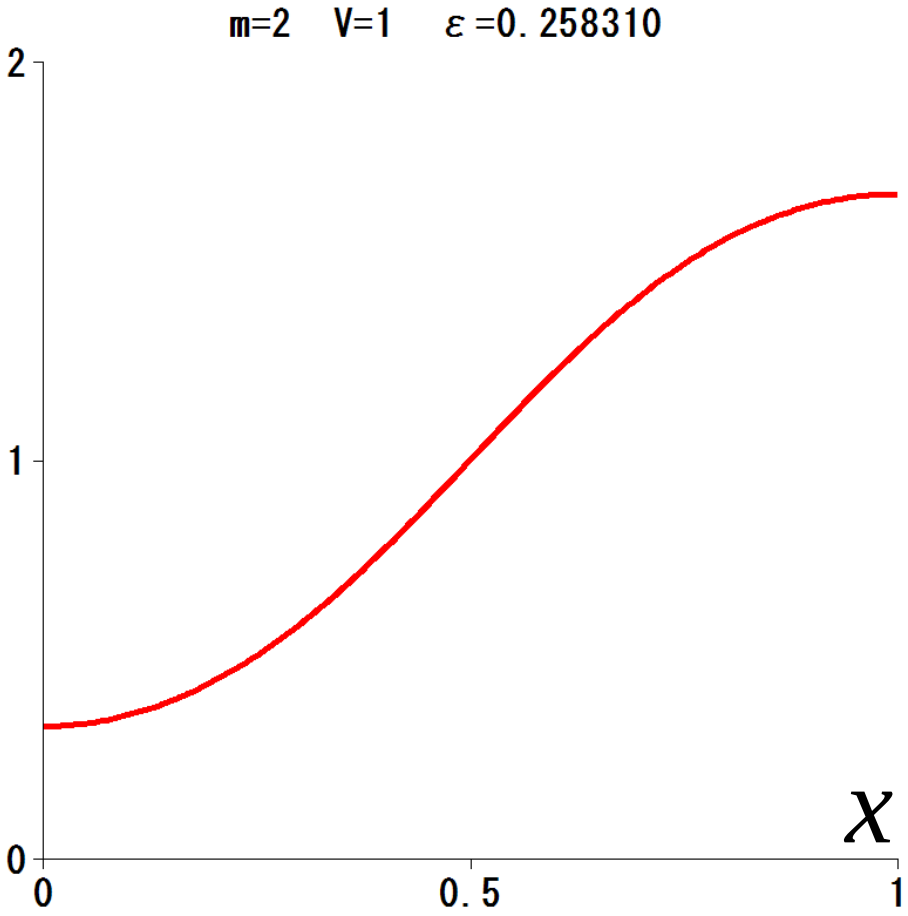

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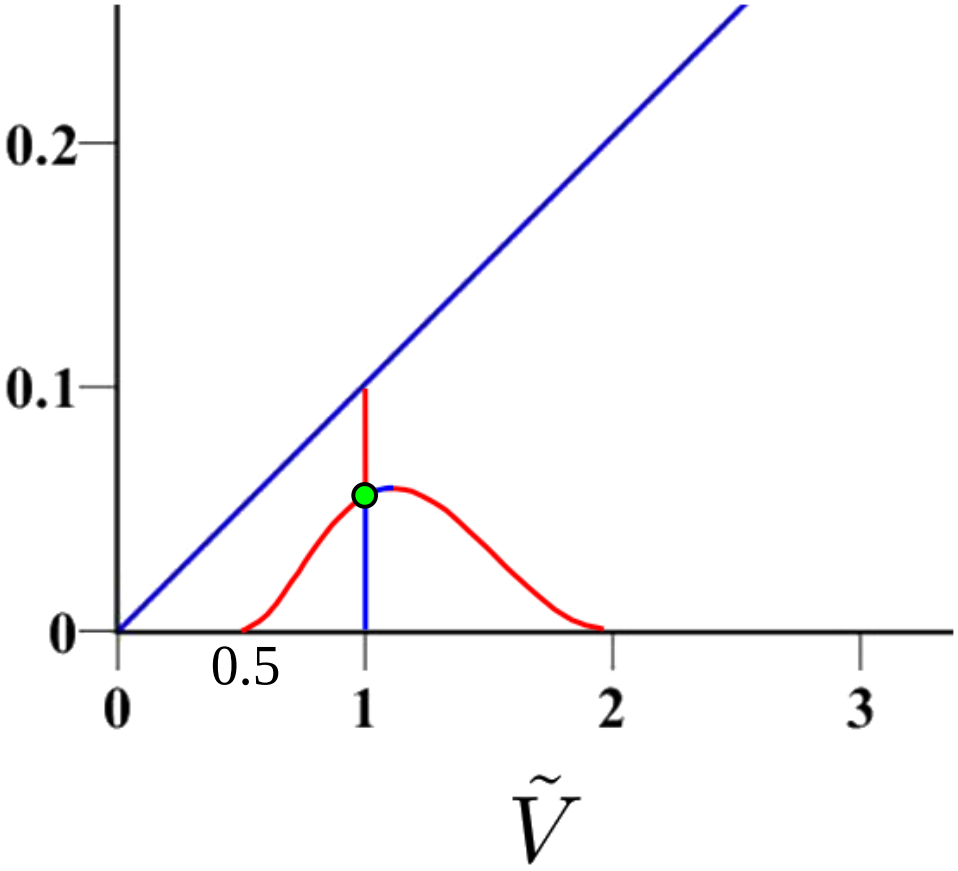

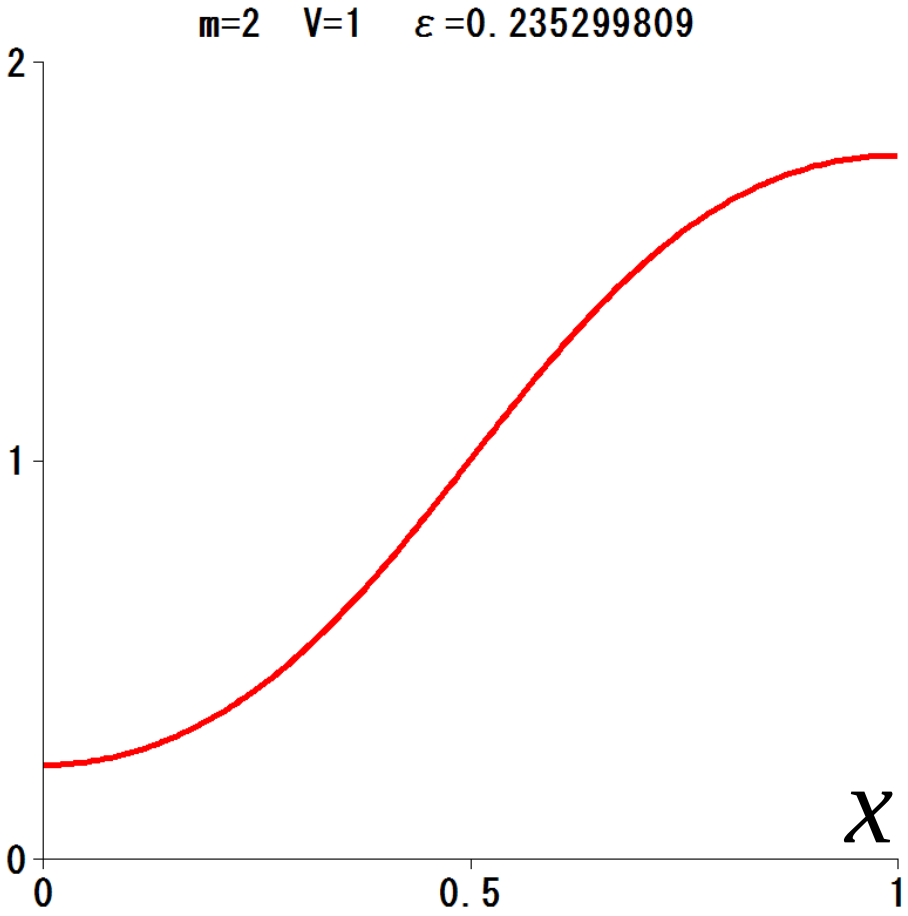

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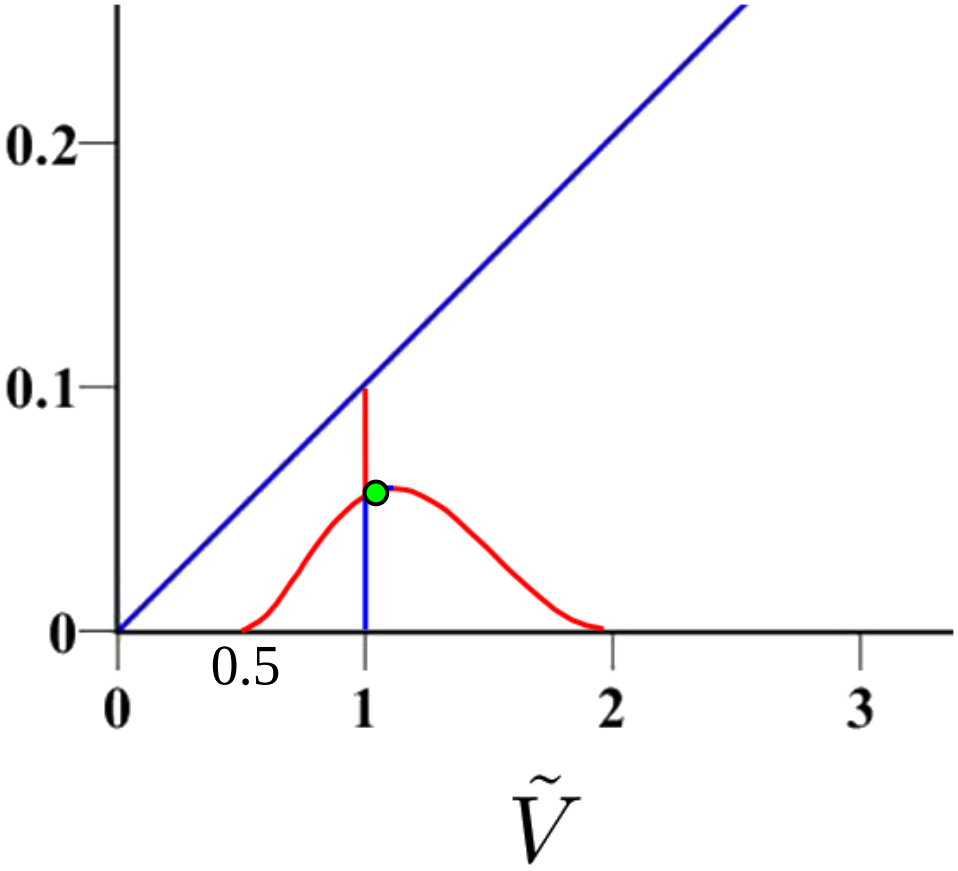

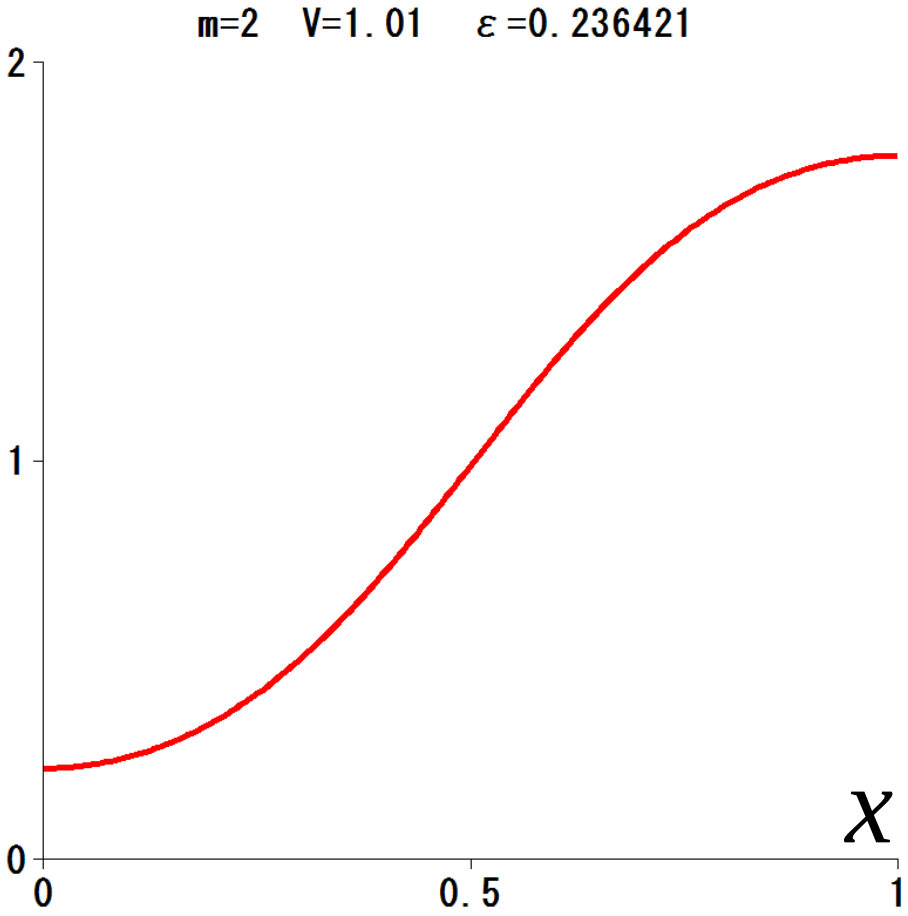

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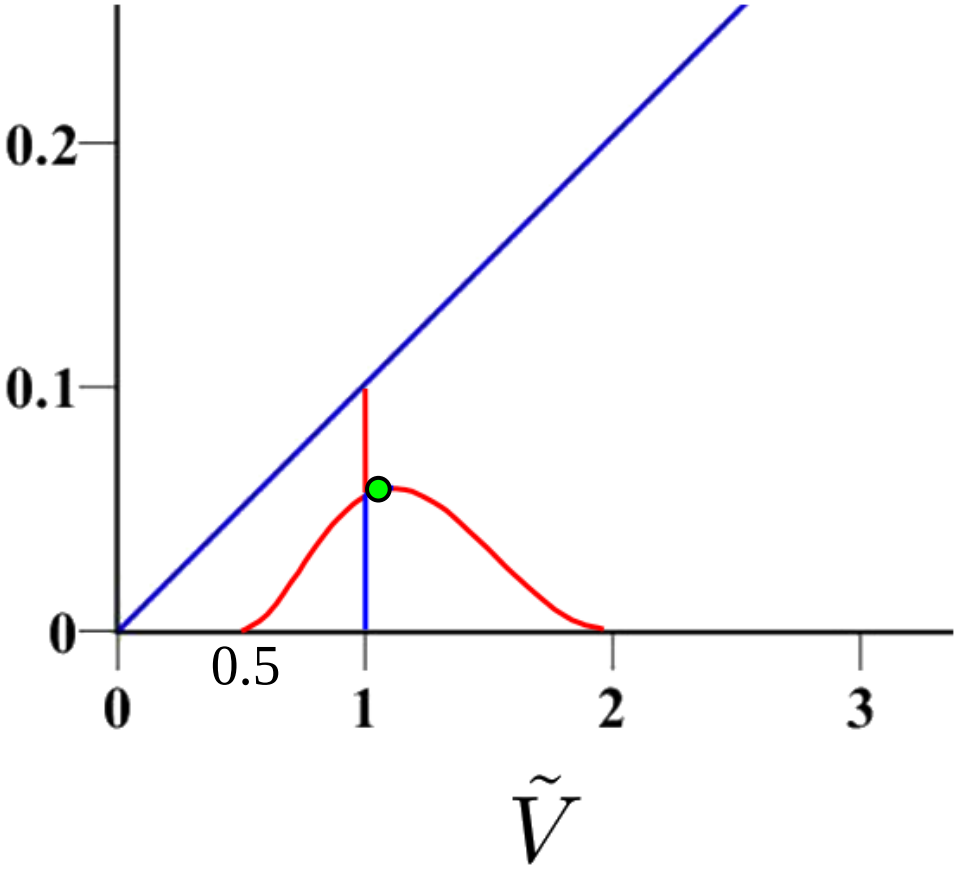

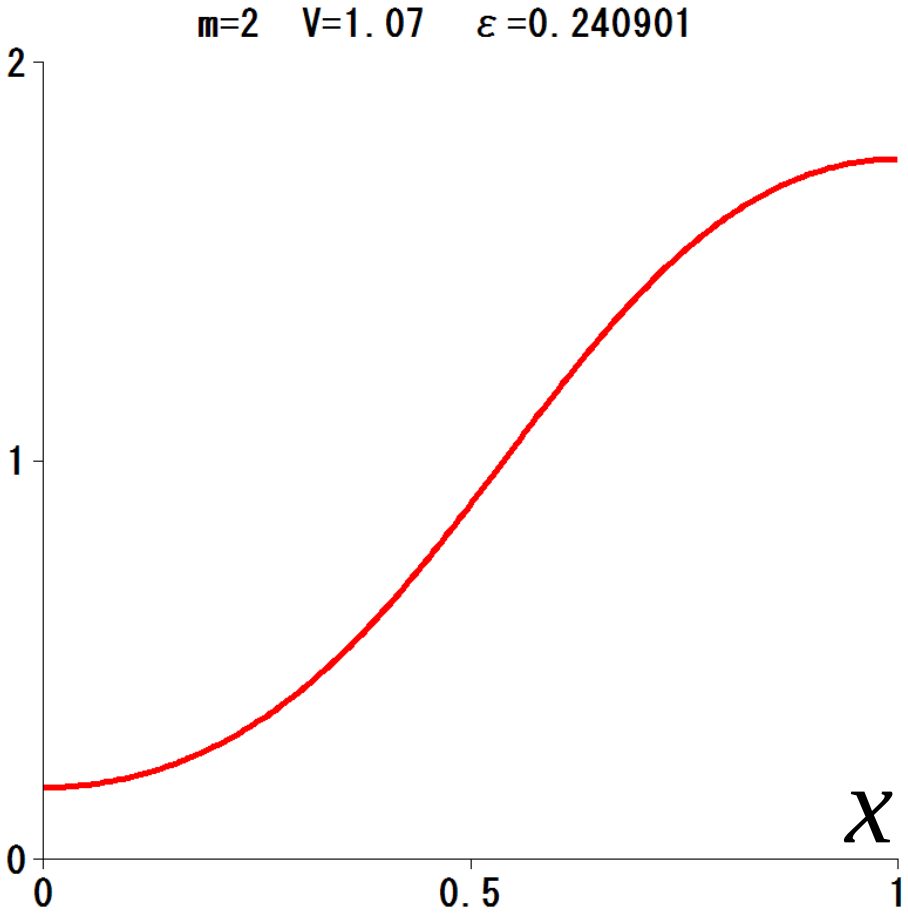

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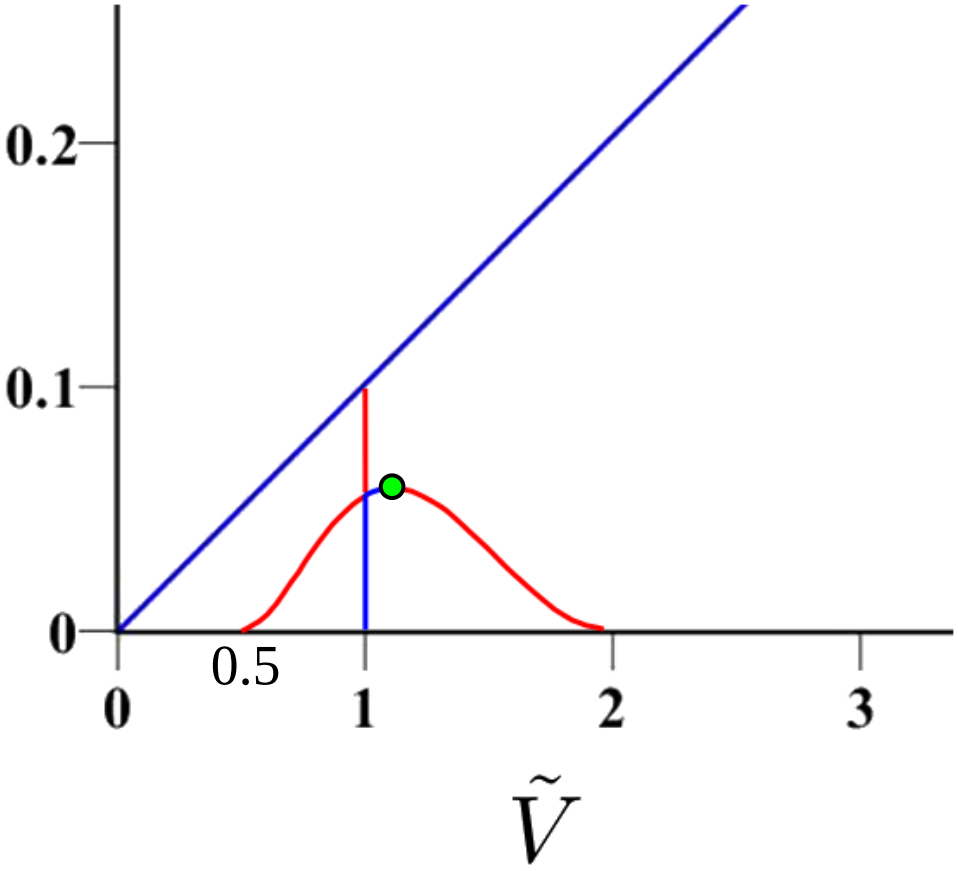

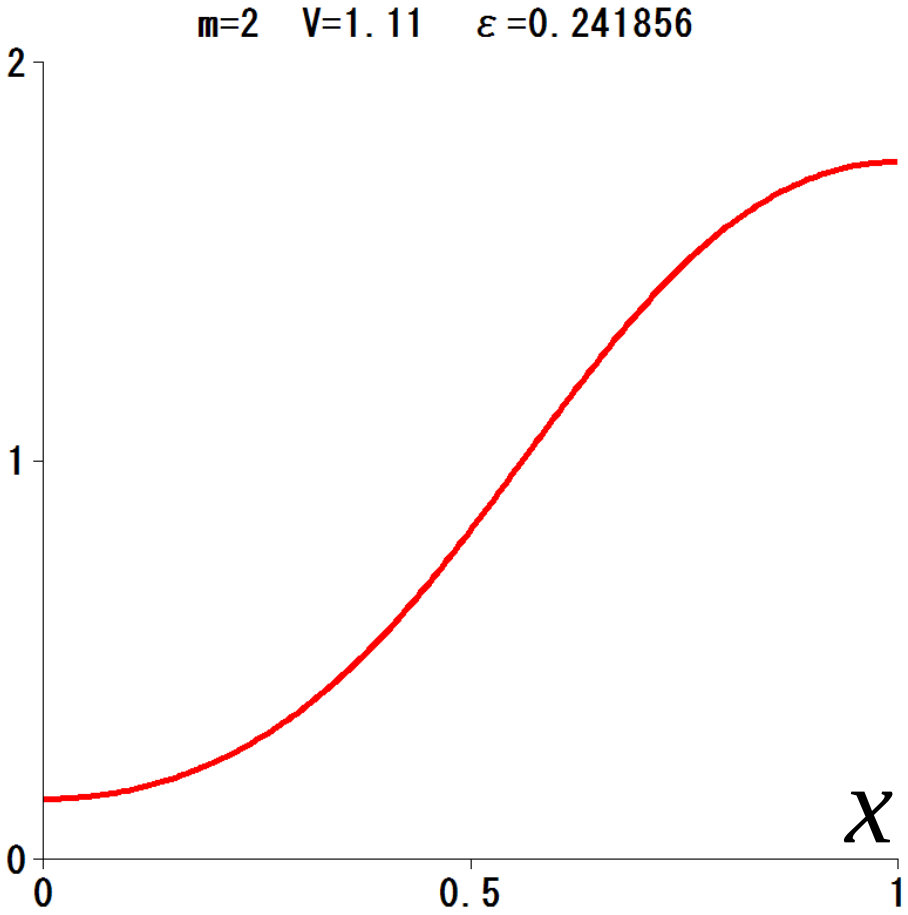

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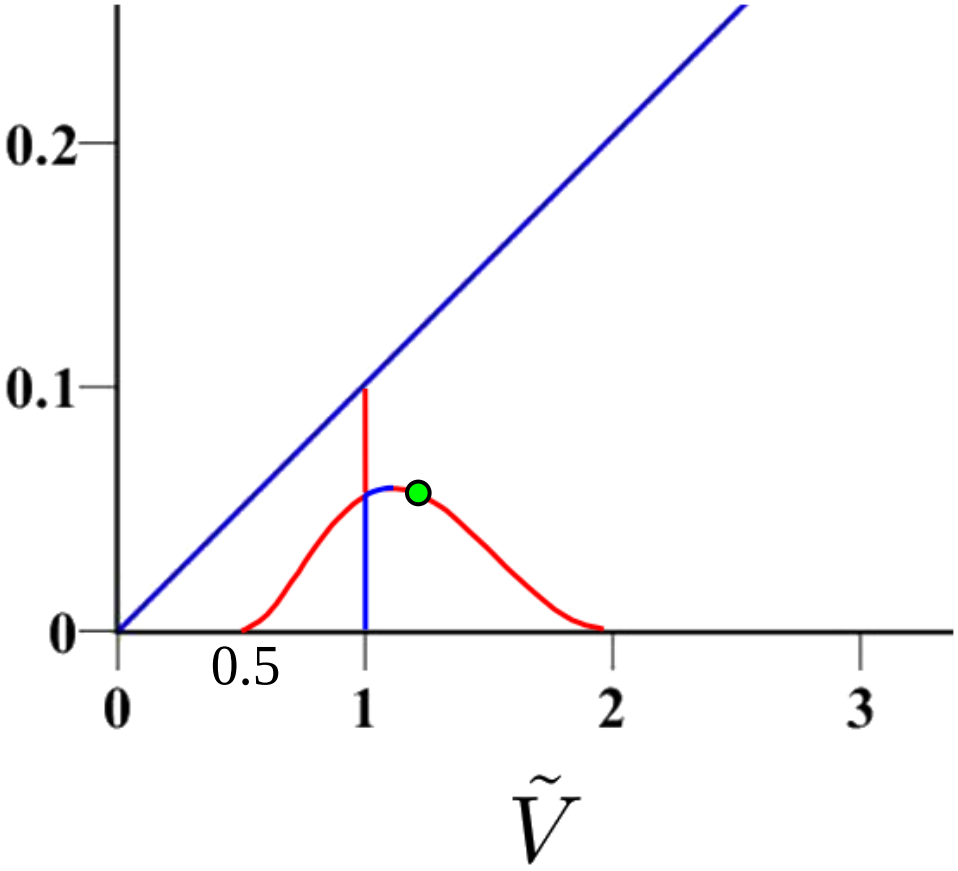

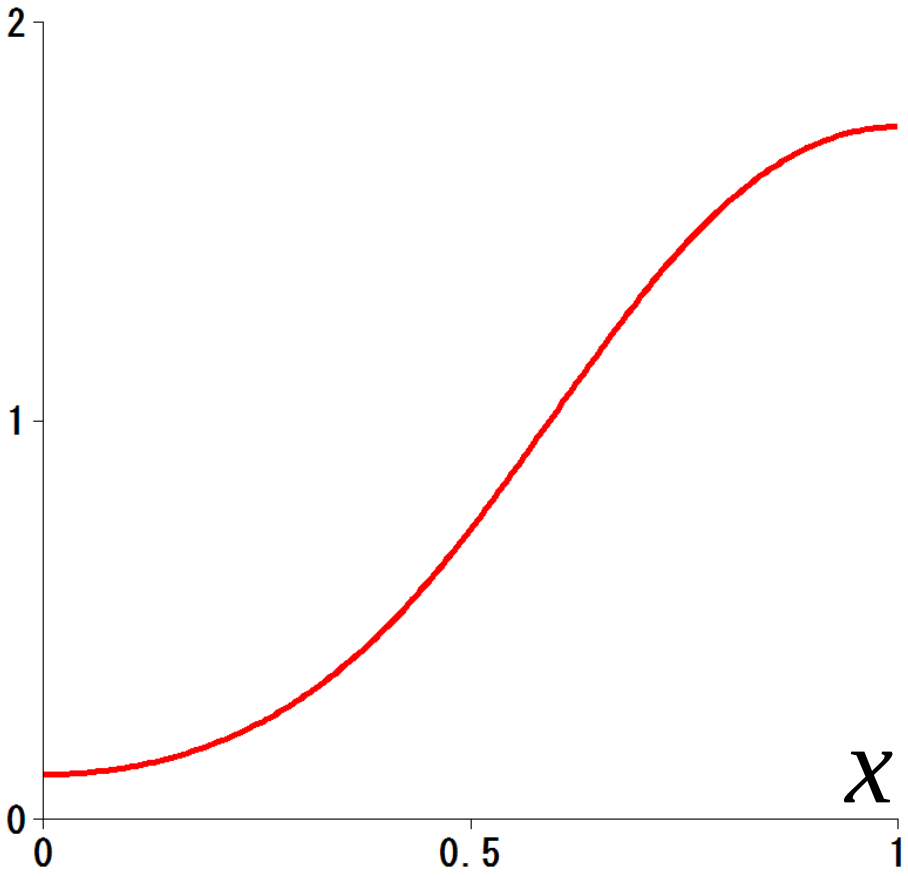

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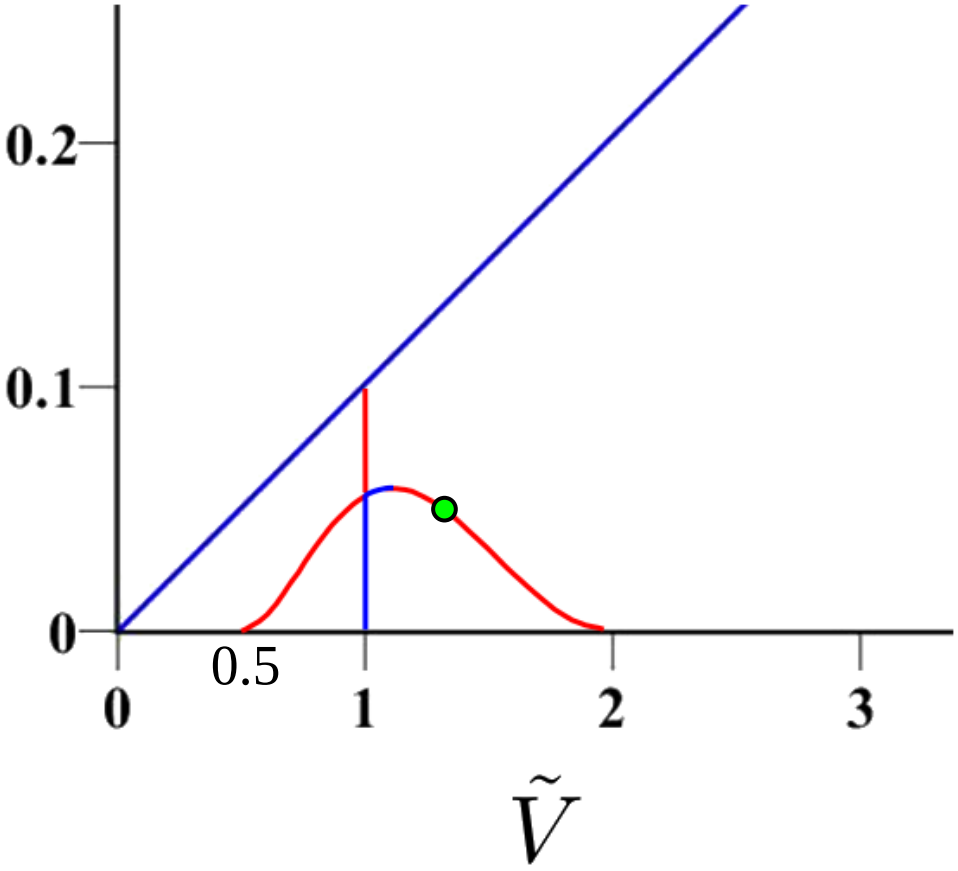

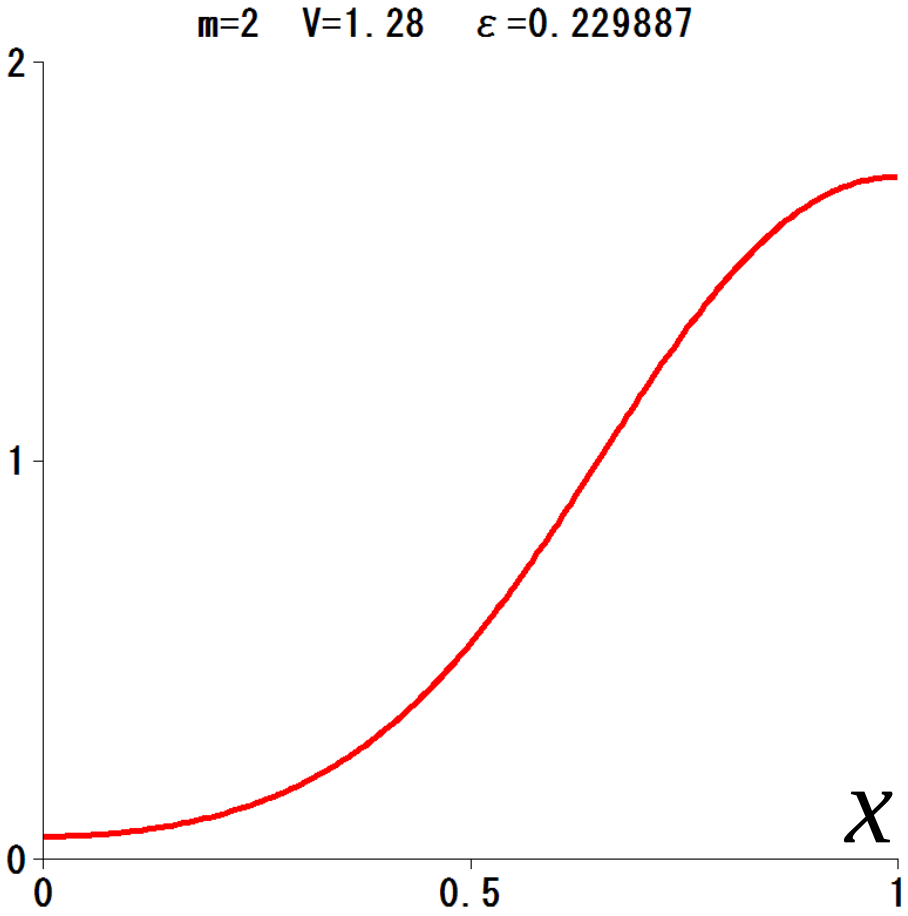

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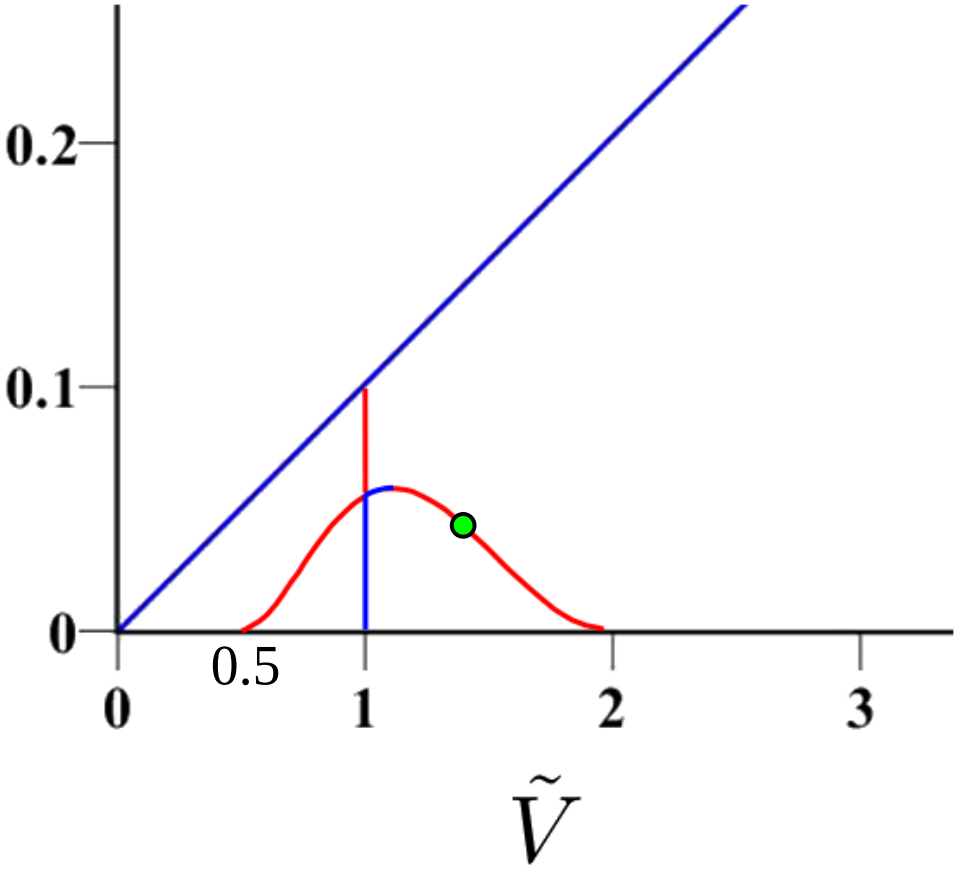

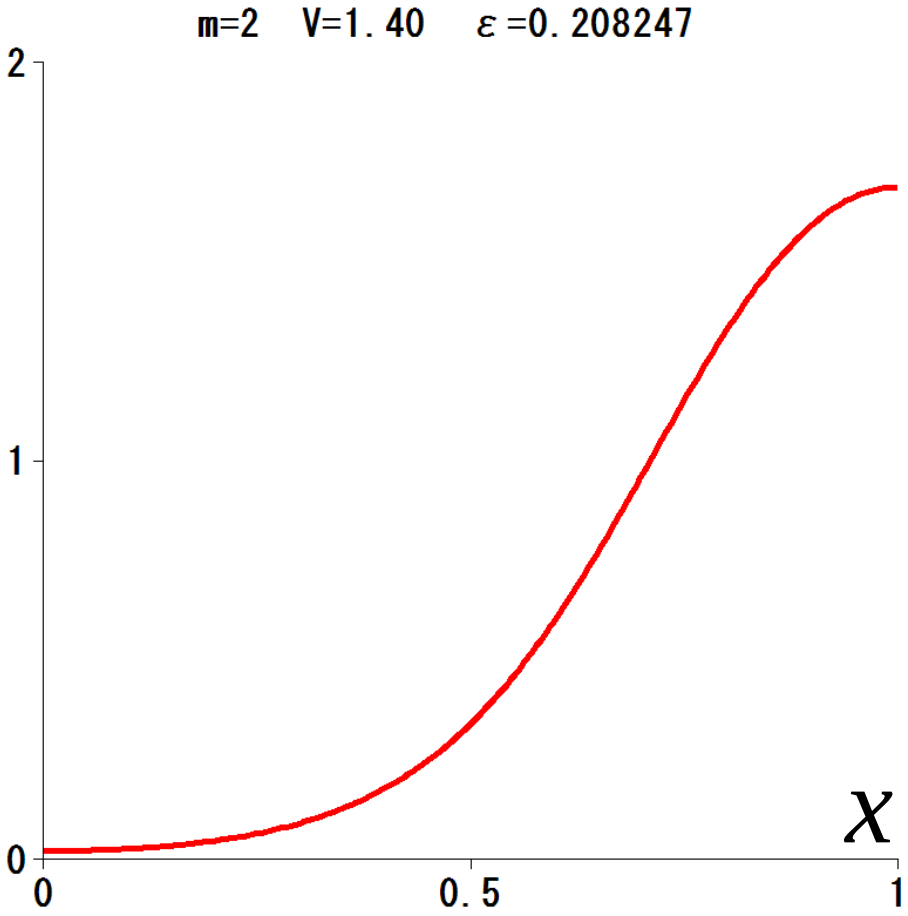

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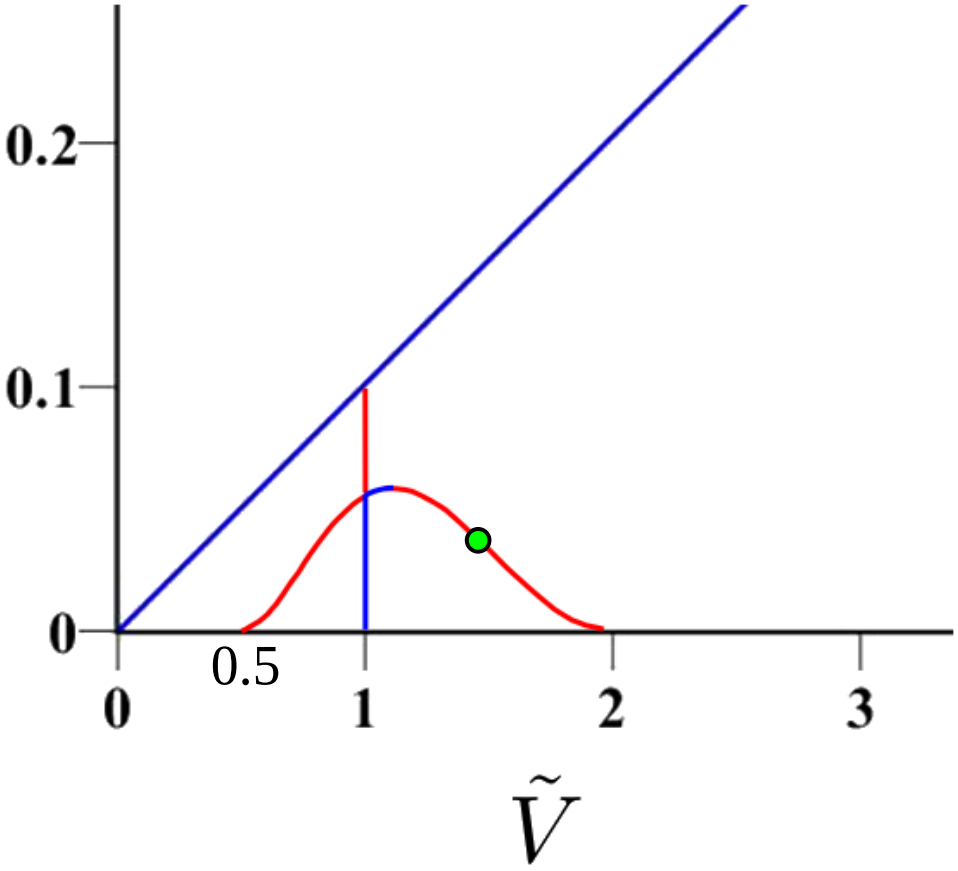

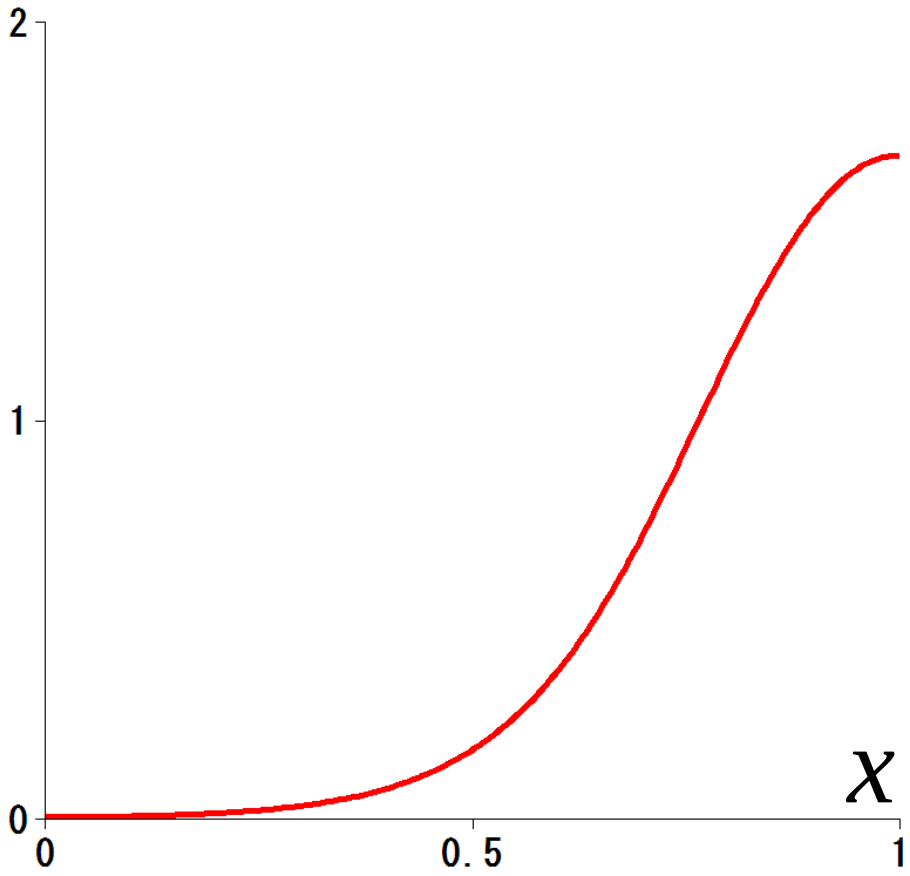

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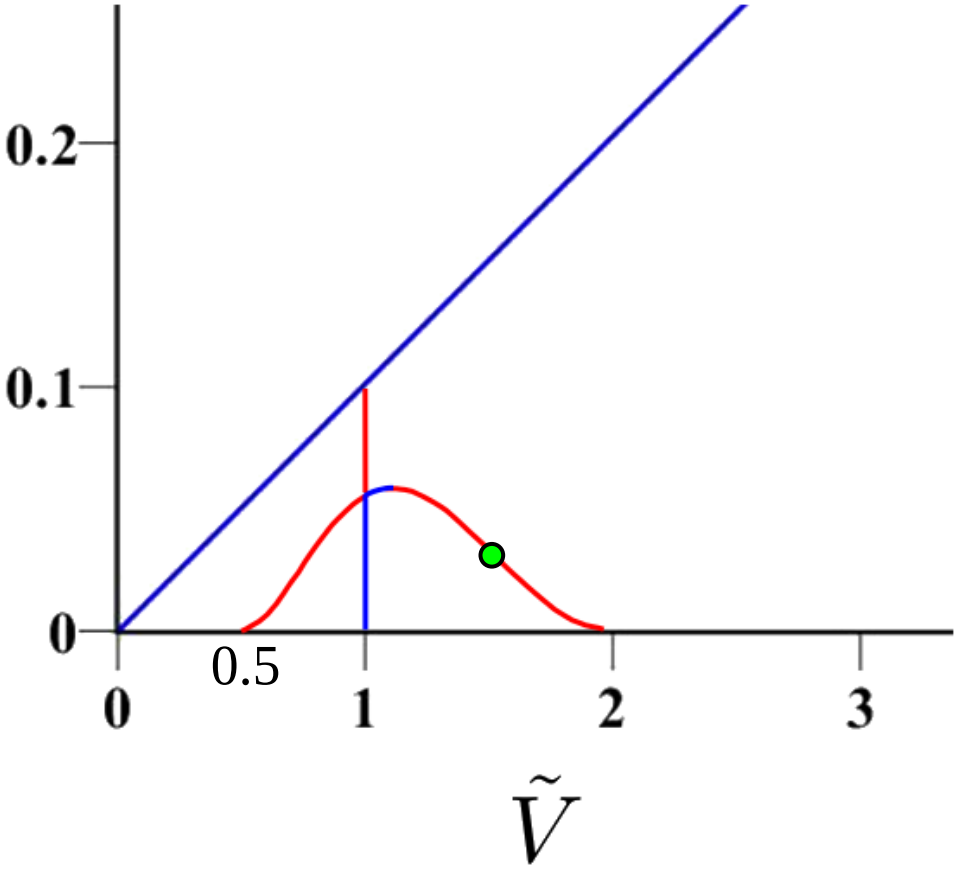

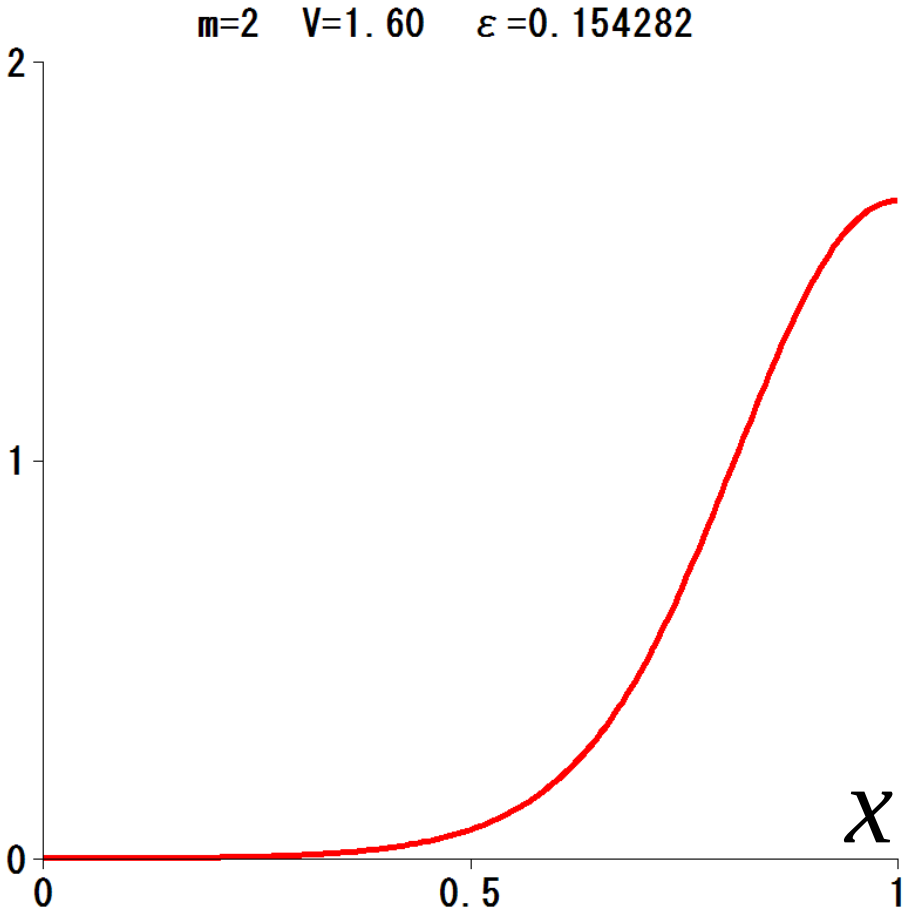

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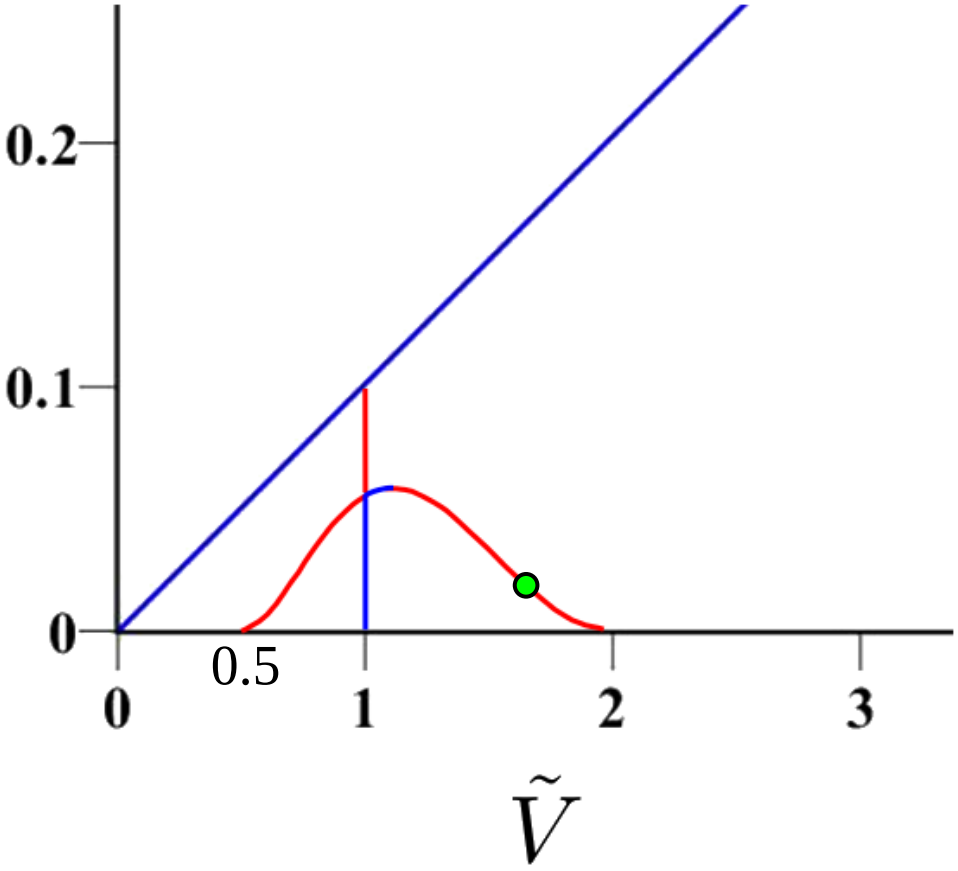

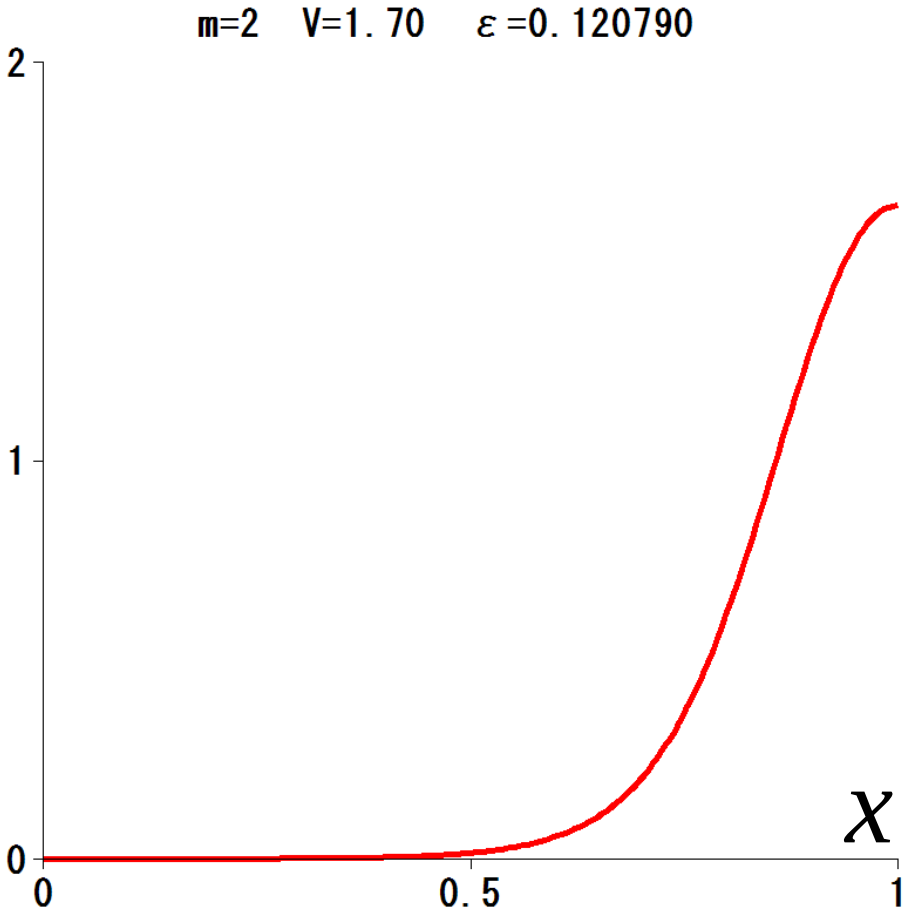

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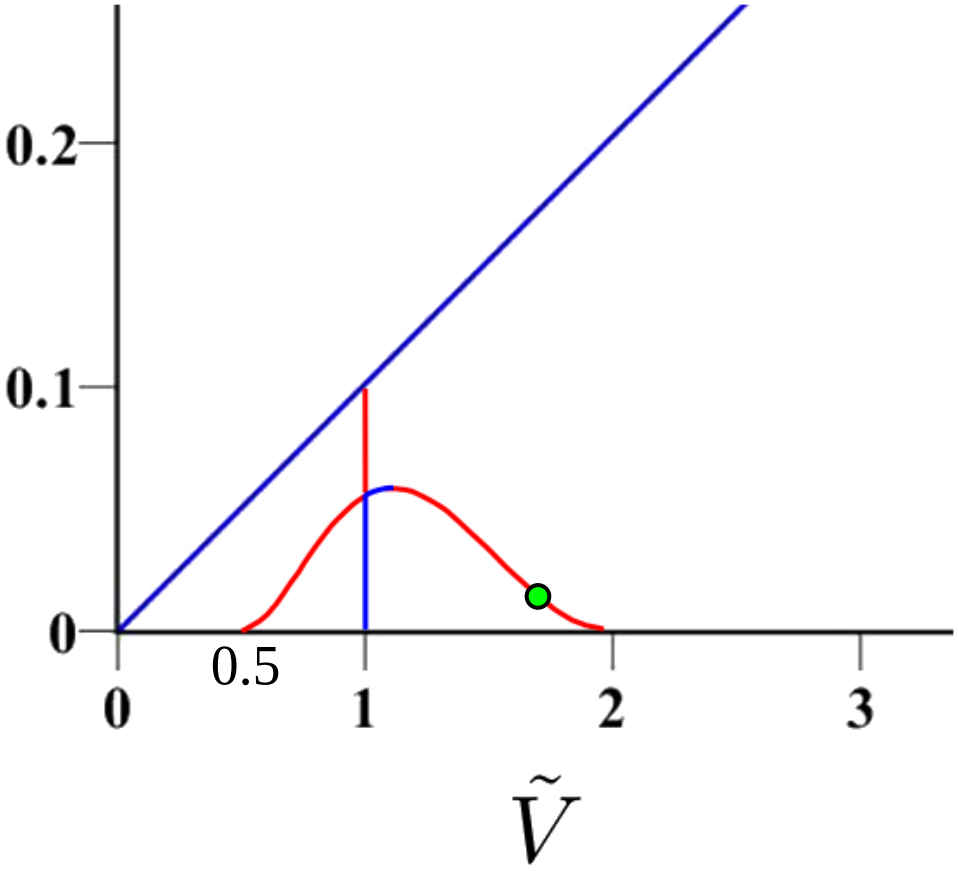

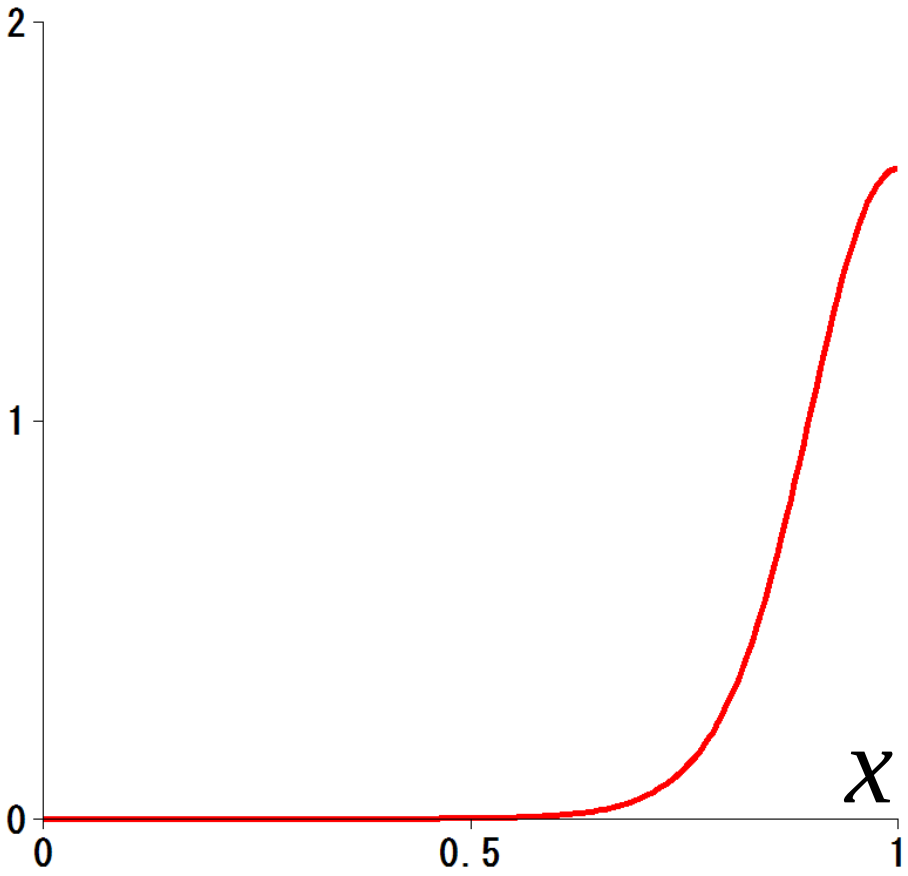

$$(SLP)\begin{cases} \varepsilon^2 W_{xx} + W(W-1)(\tilde{V}+1-W) = 0, \\ W_x(0) = W_x(1) = 0, \\ W'(x) > 0, \\ \int_0^1 W(x)\,dx + \tilde{V} = m. \end{cases}$$

不安定

$\tilde{V} \to 2,\ \varepsilon^2 \to 0$ の形状

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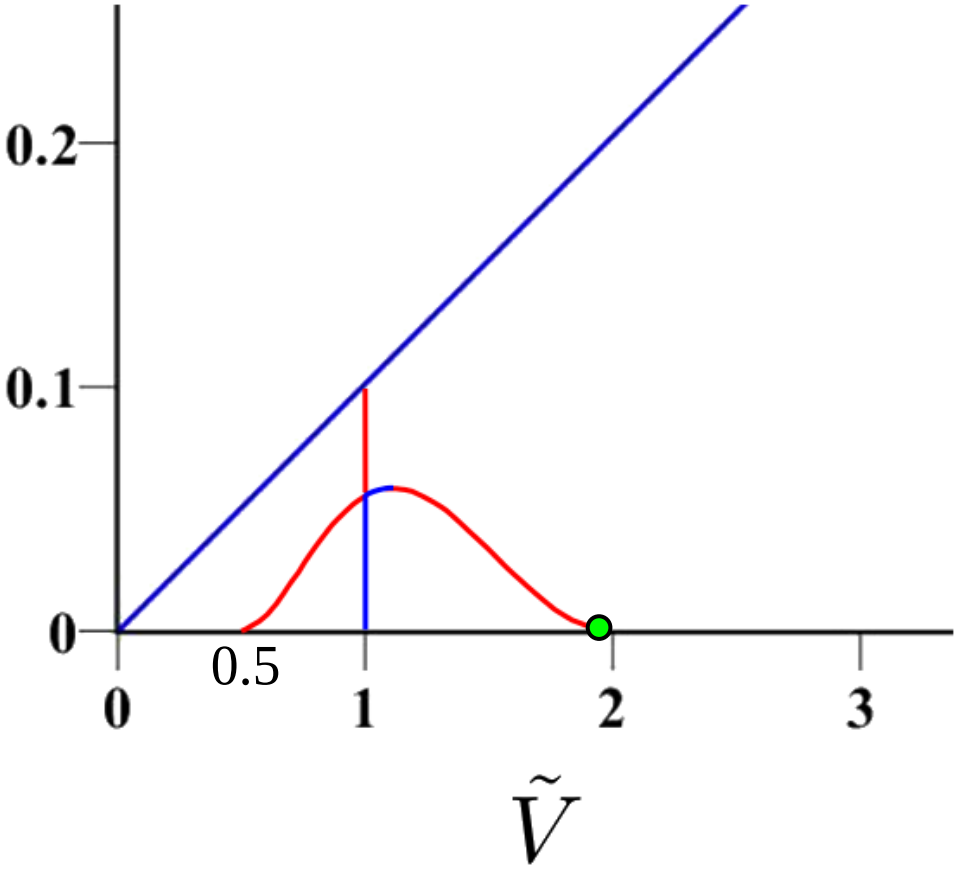
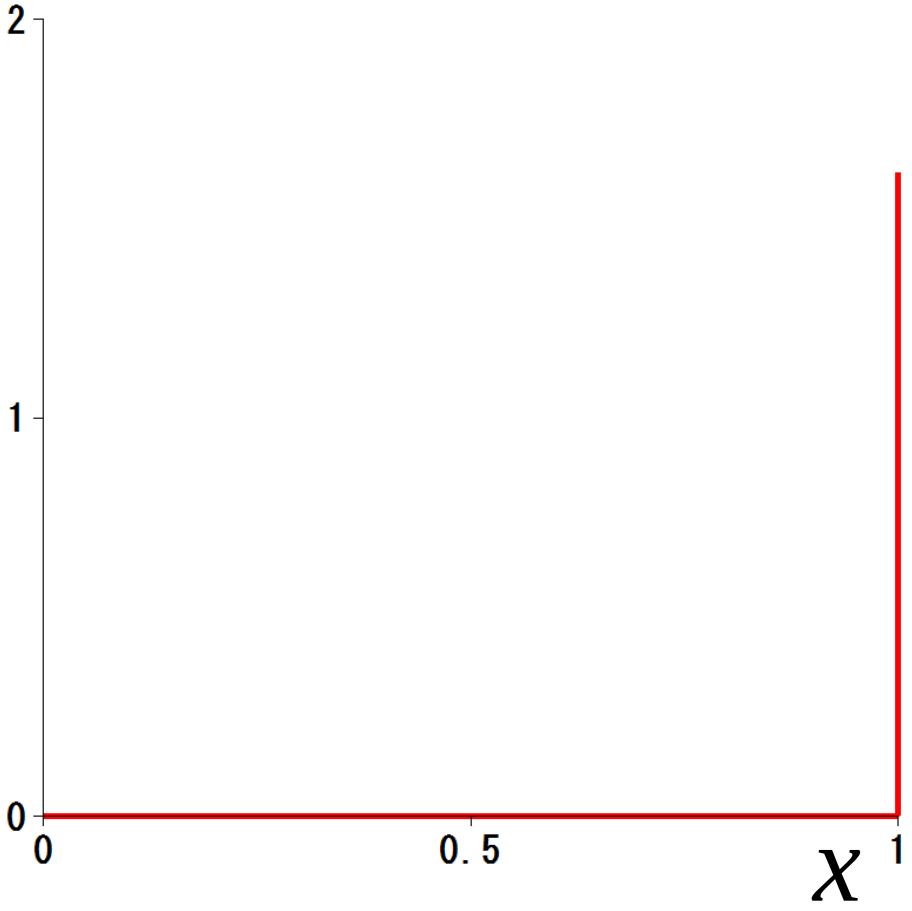


m=2 $\tilde{V} \to 2$ $\varepsilon \to 0$



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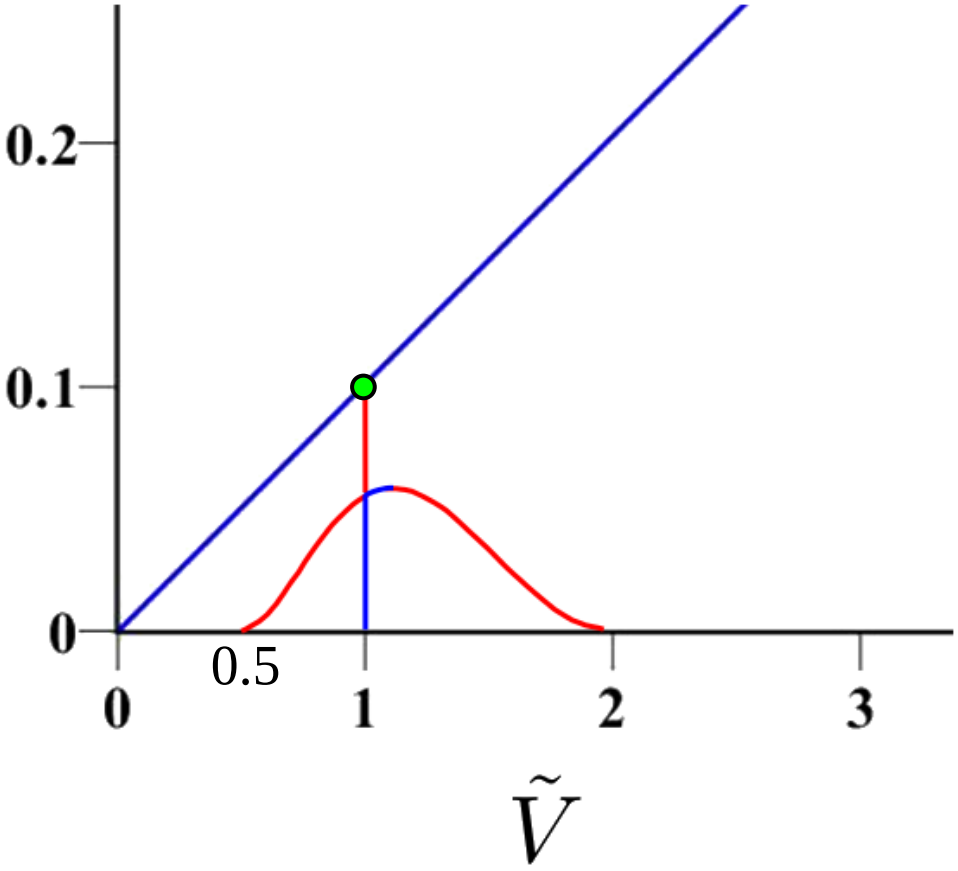

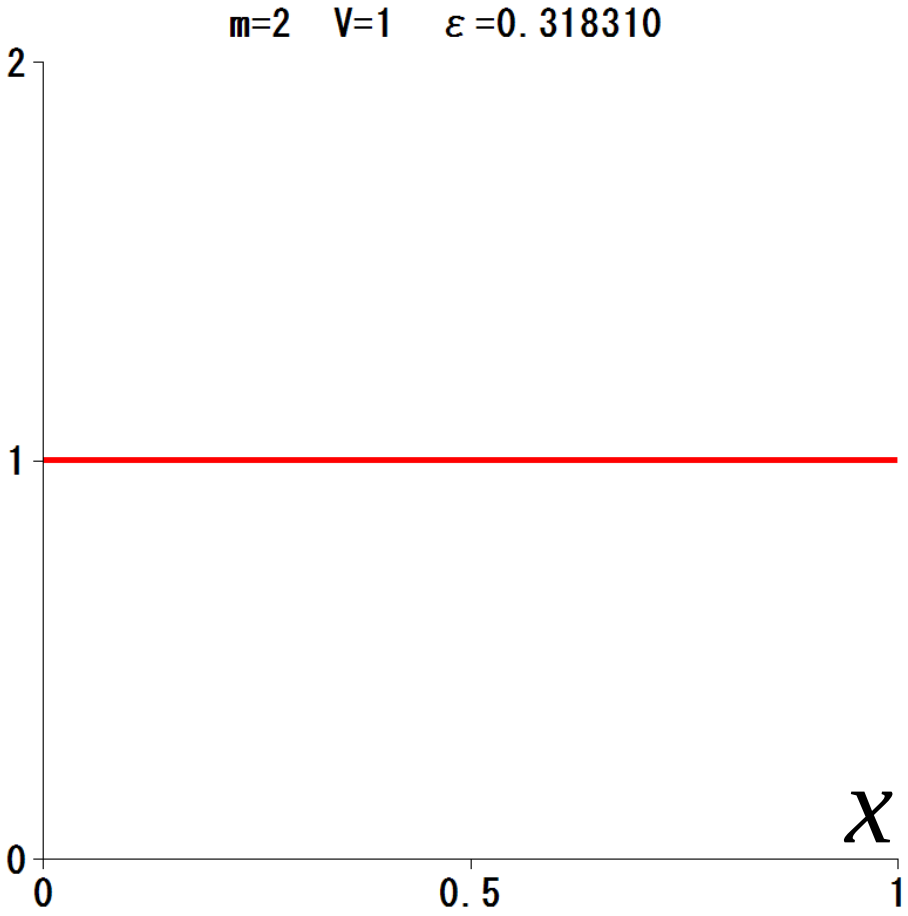

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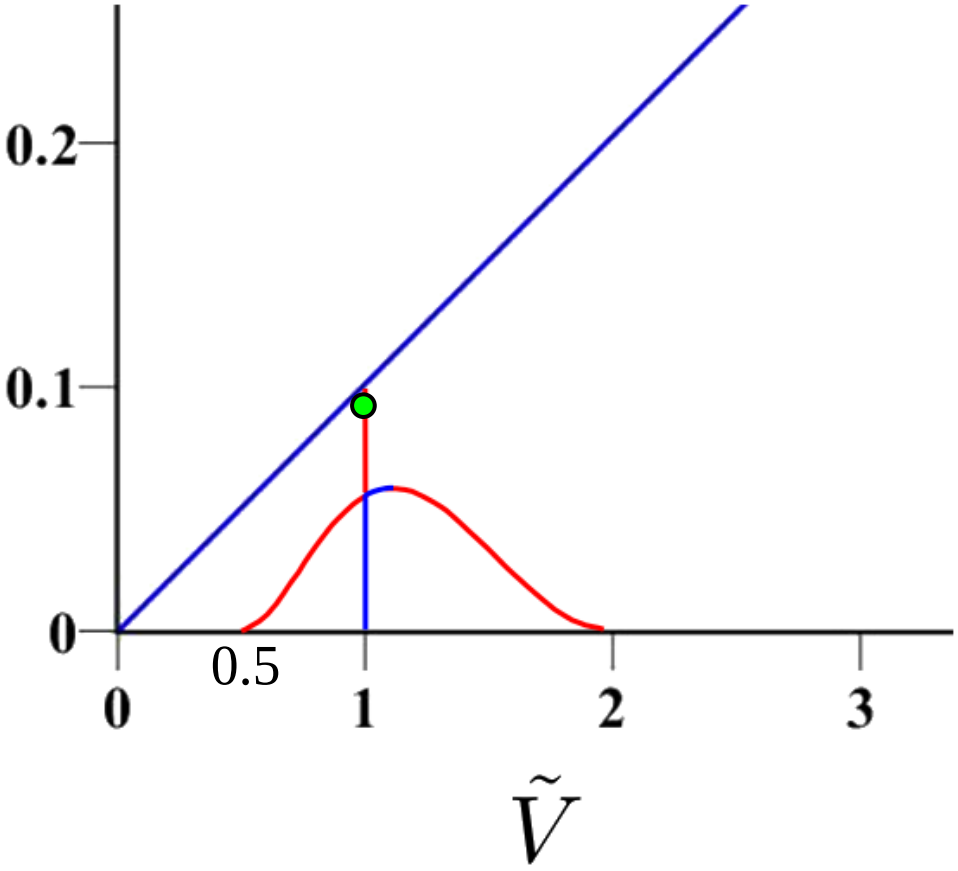

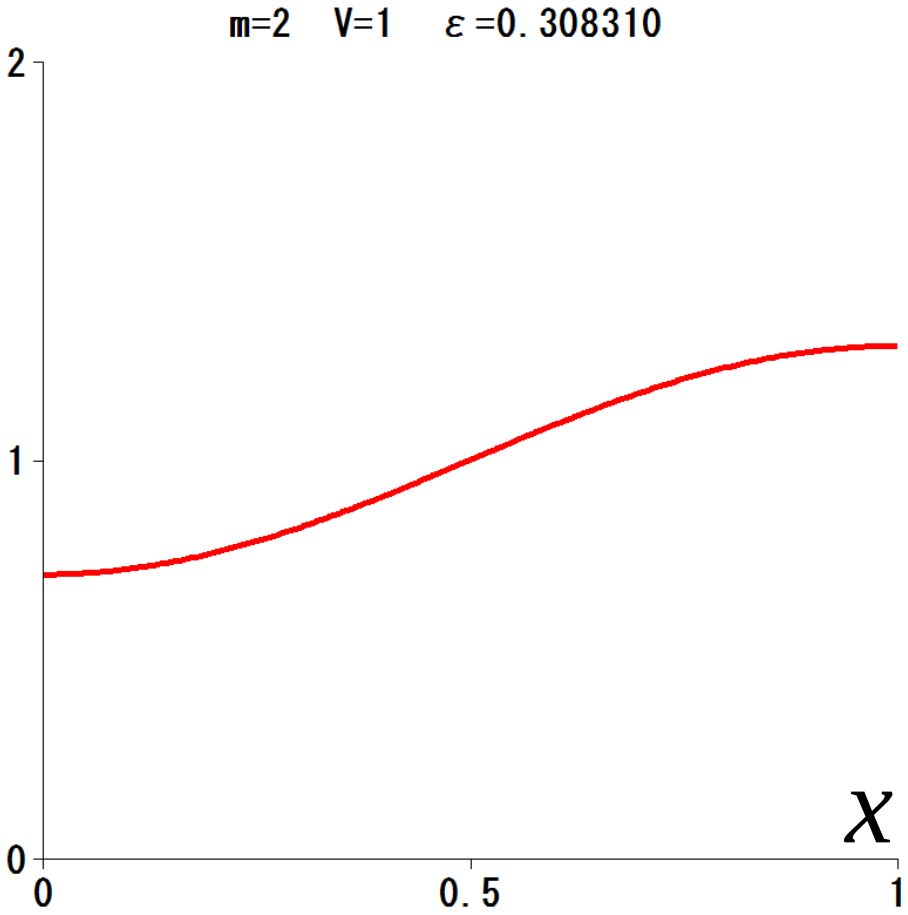

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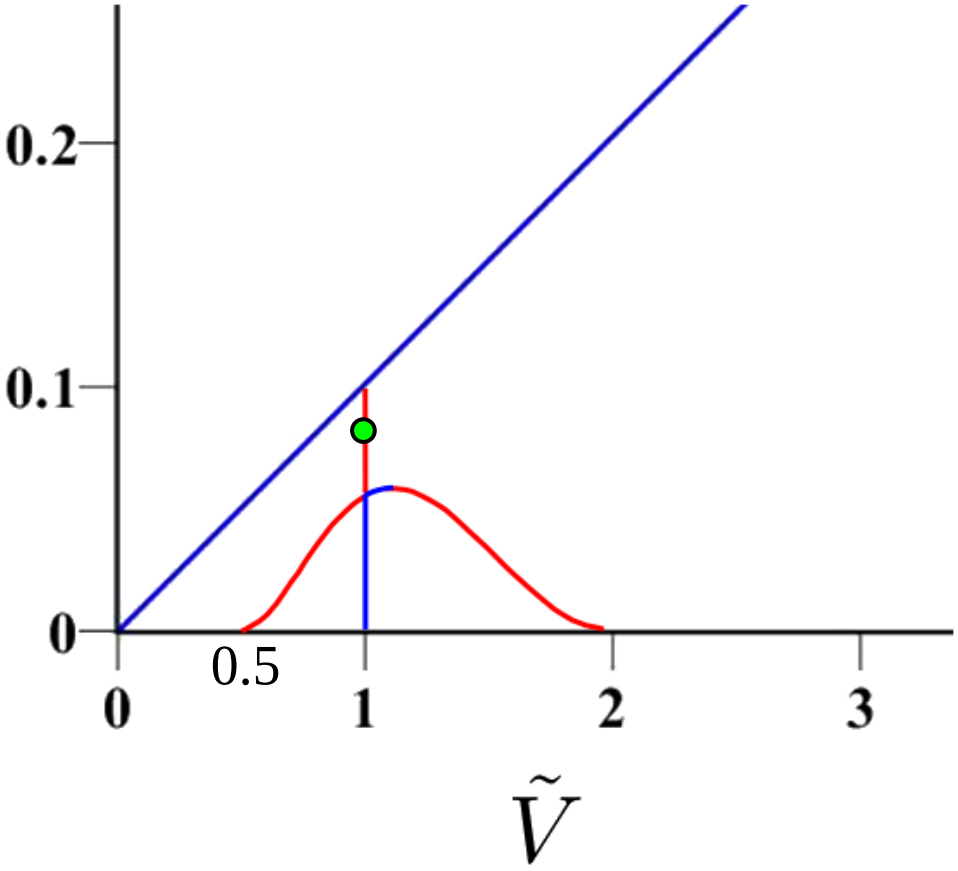

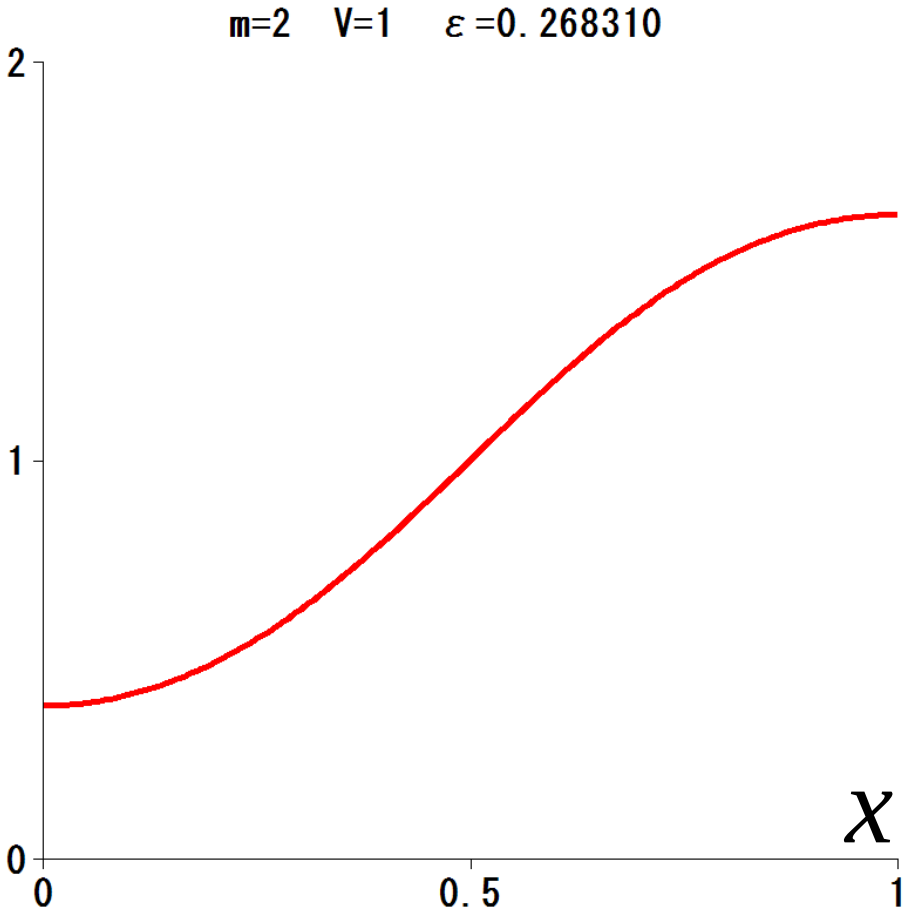

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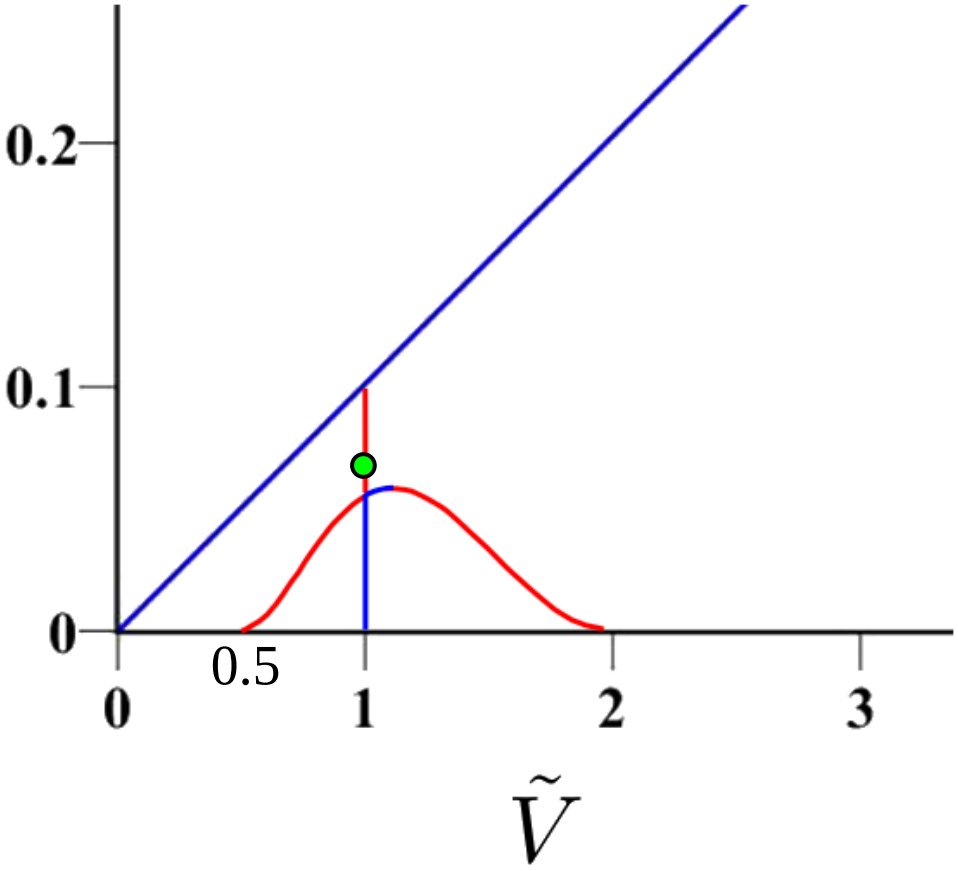

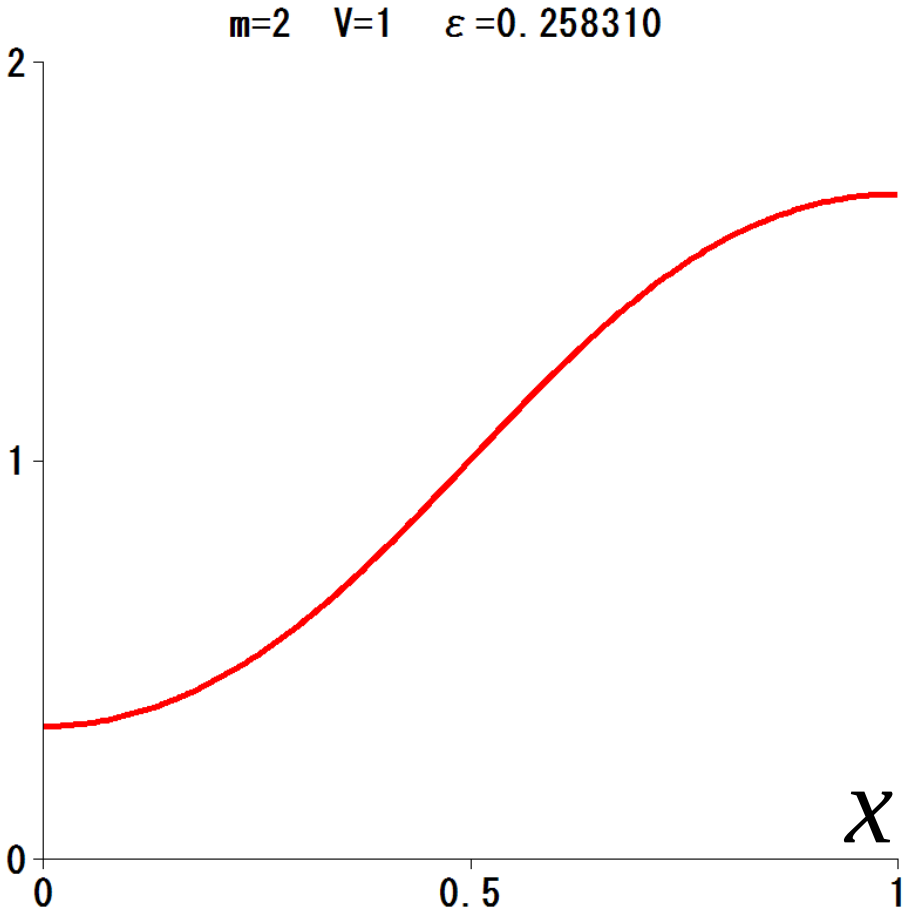

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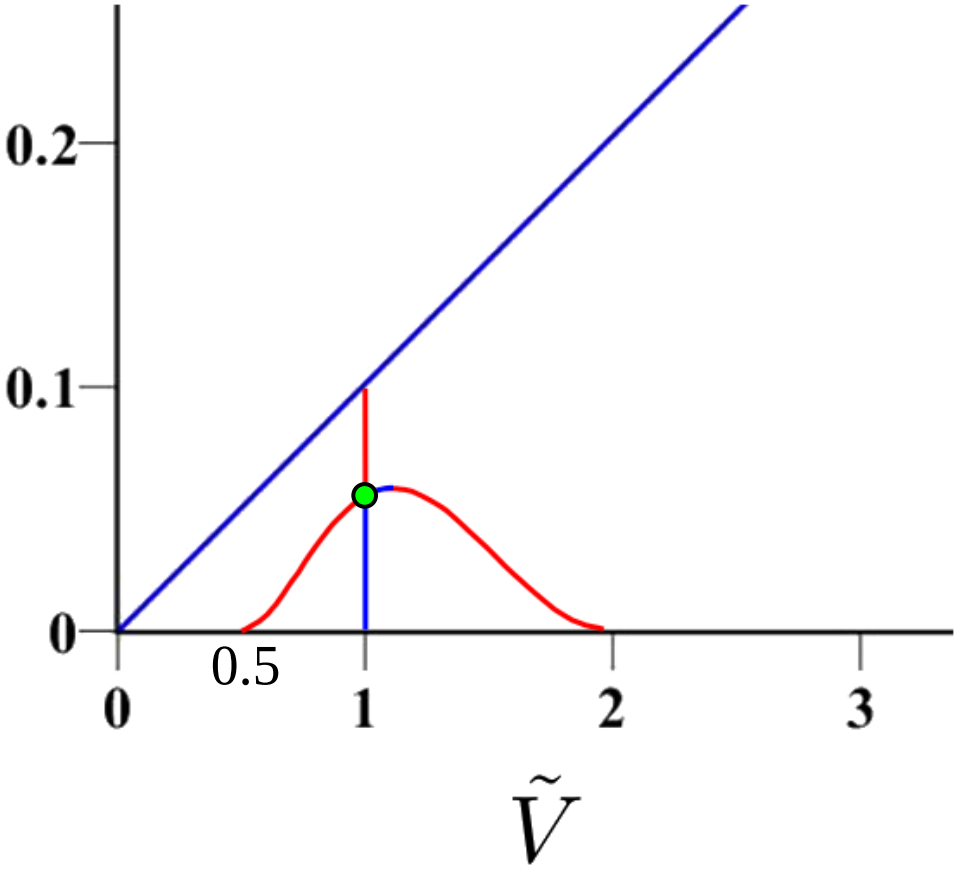

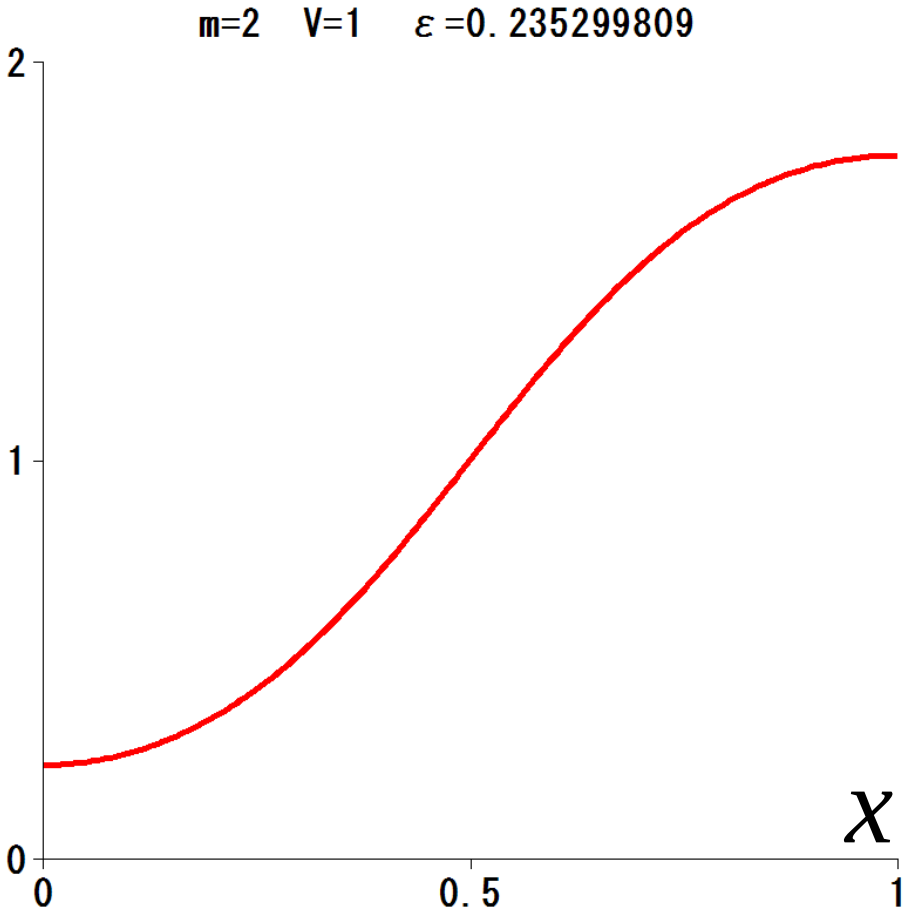

2 次分岐点

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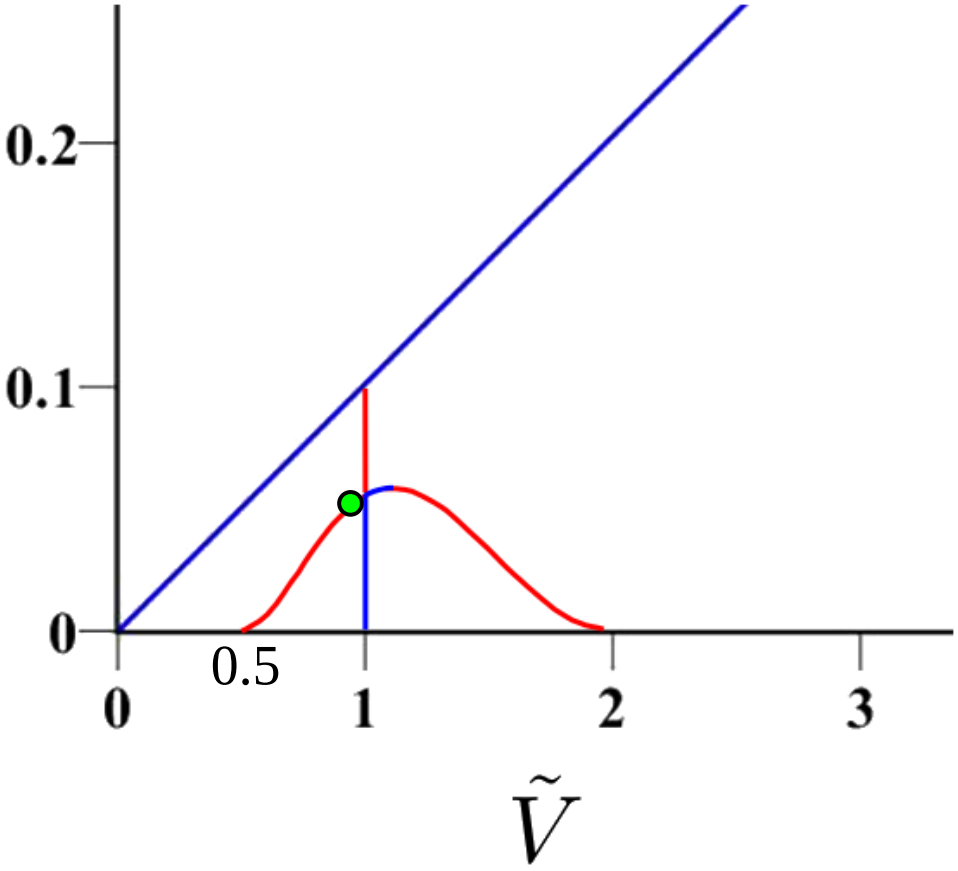

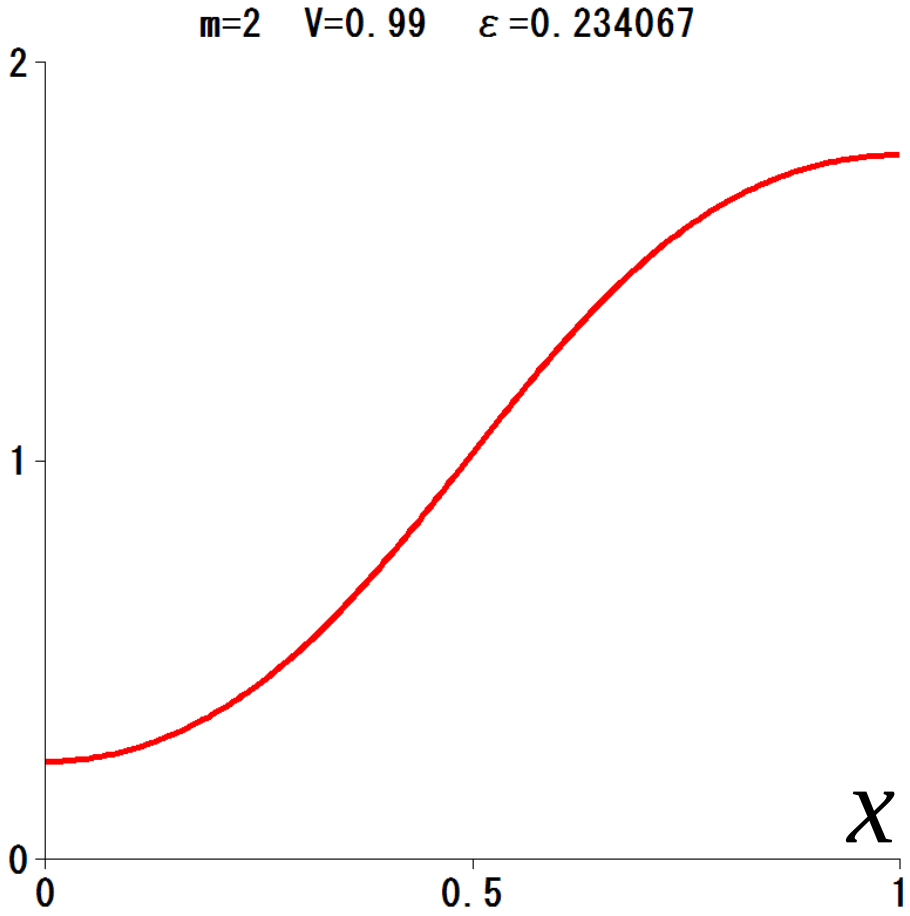

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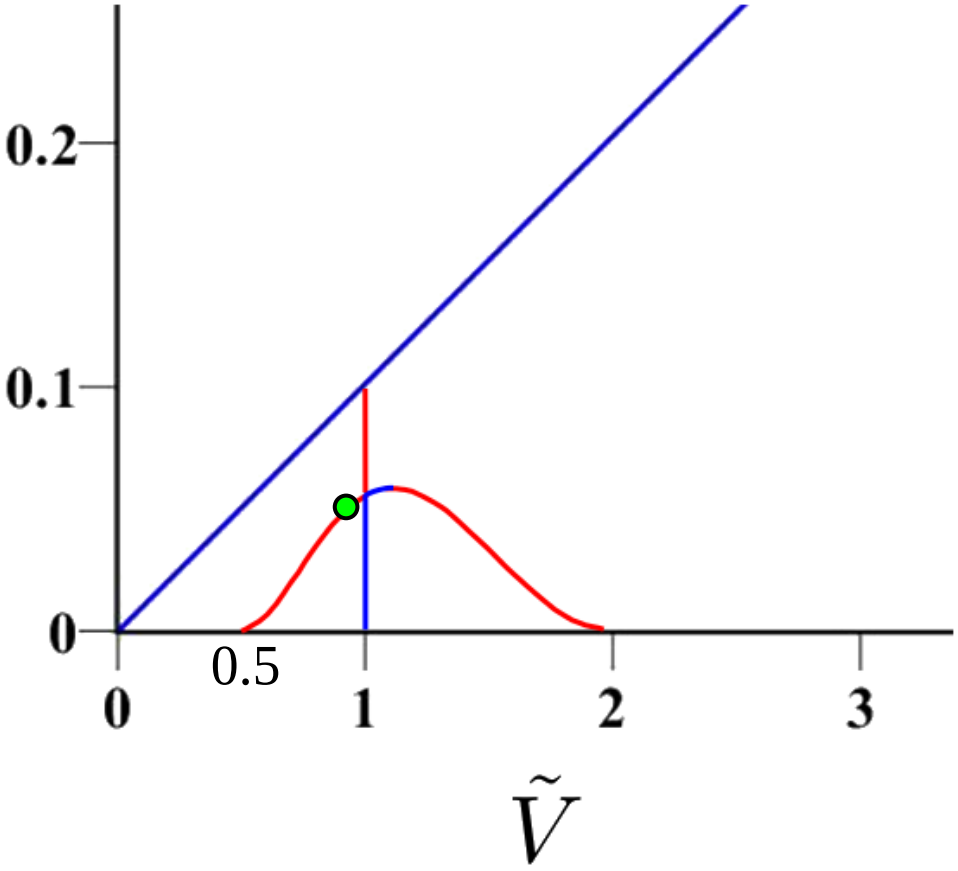

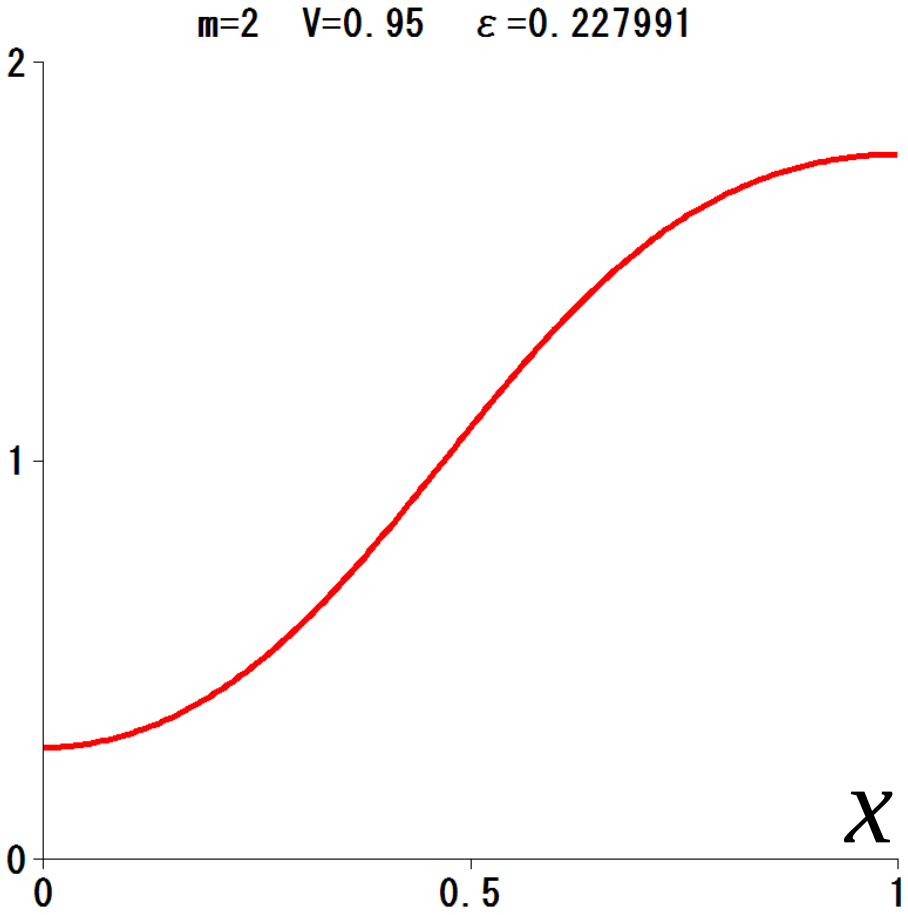

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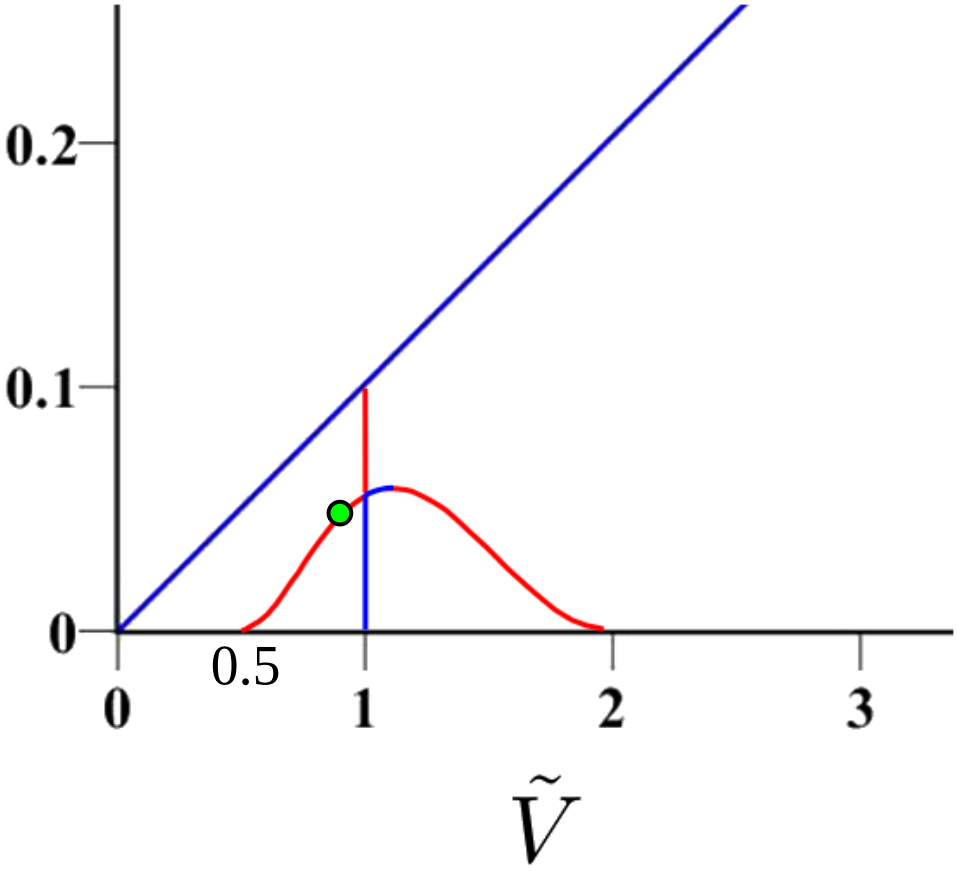

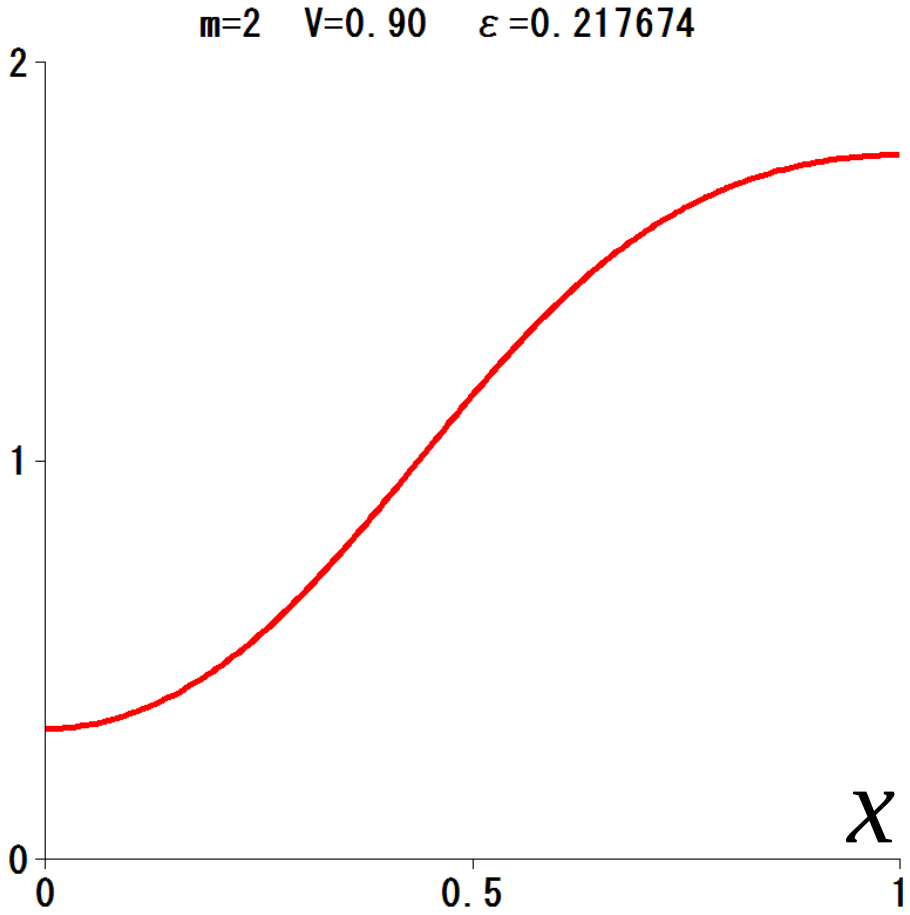

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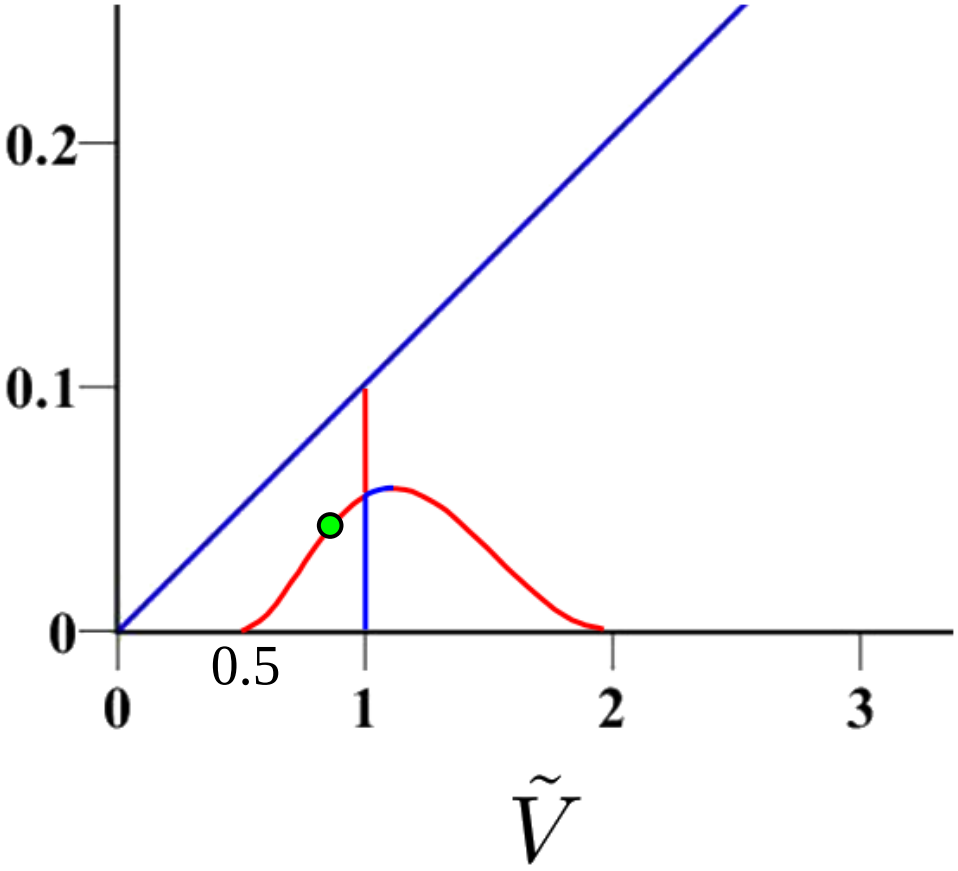

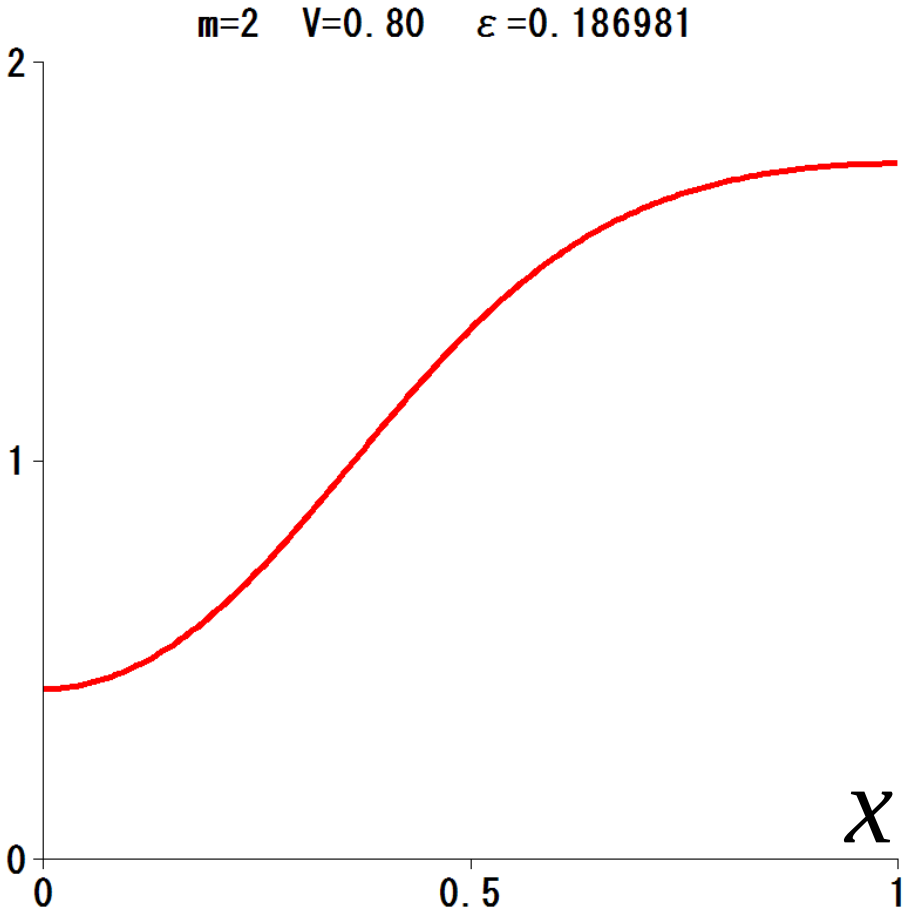

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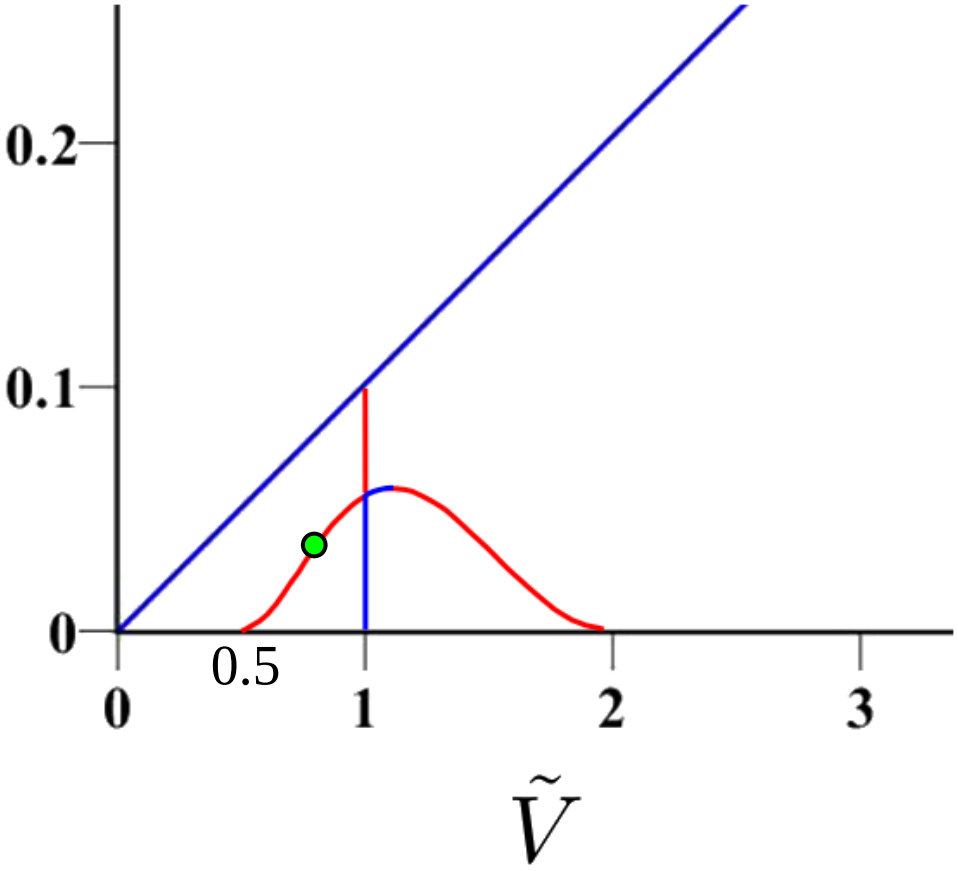

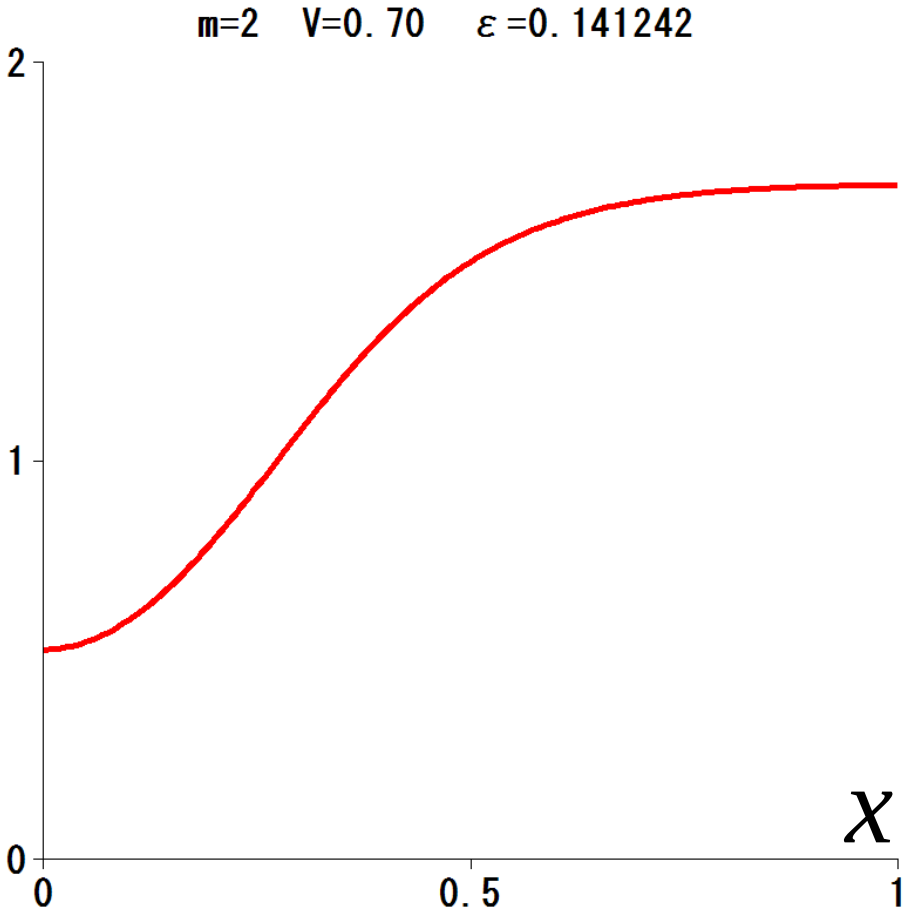

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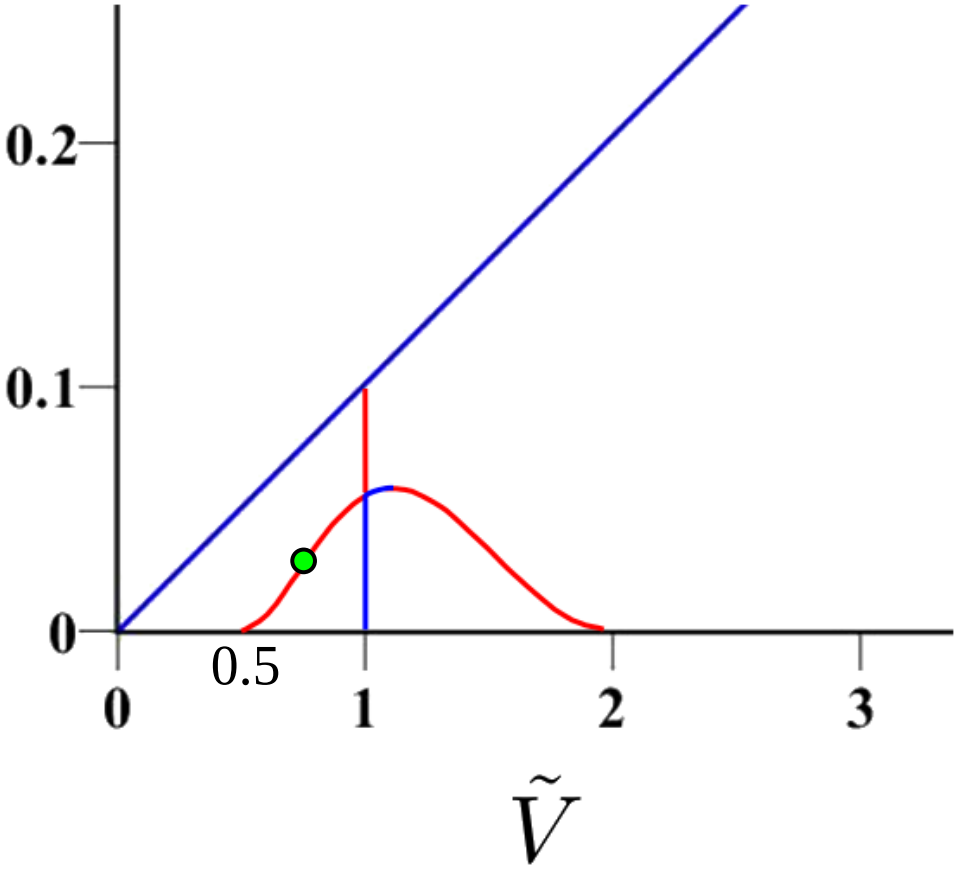

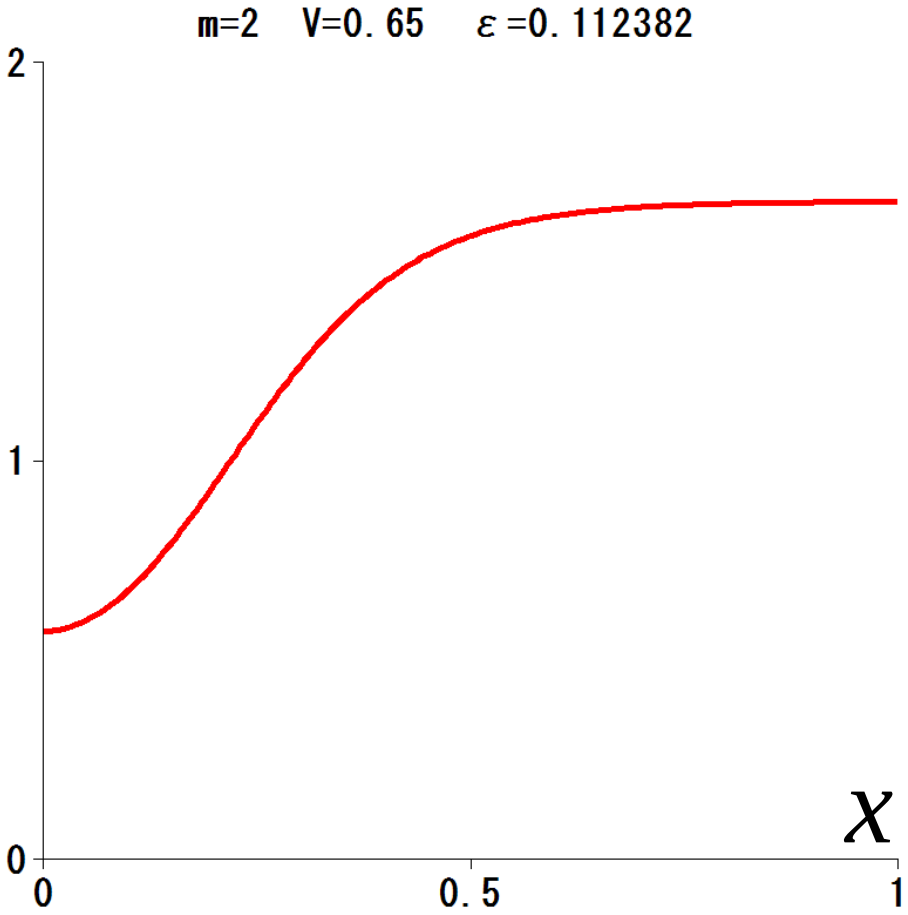

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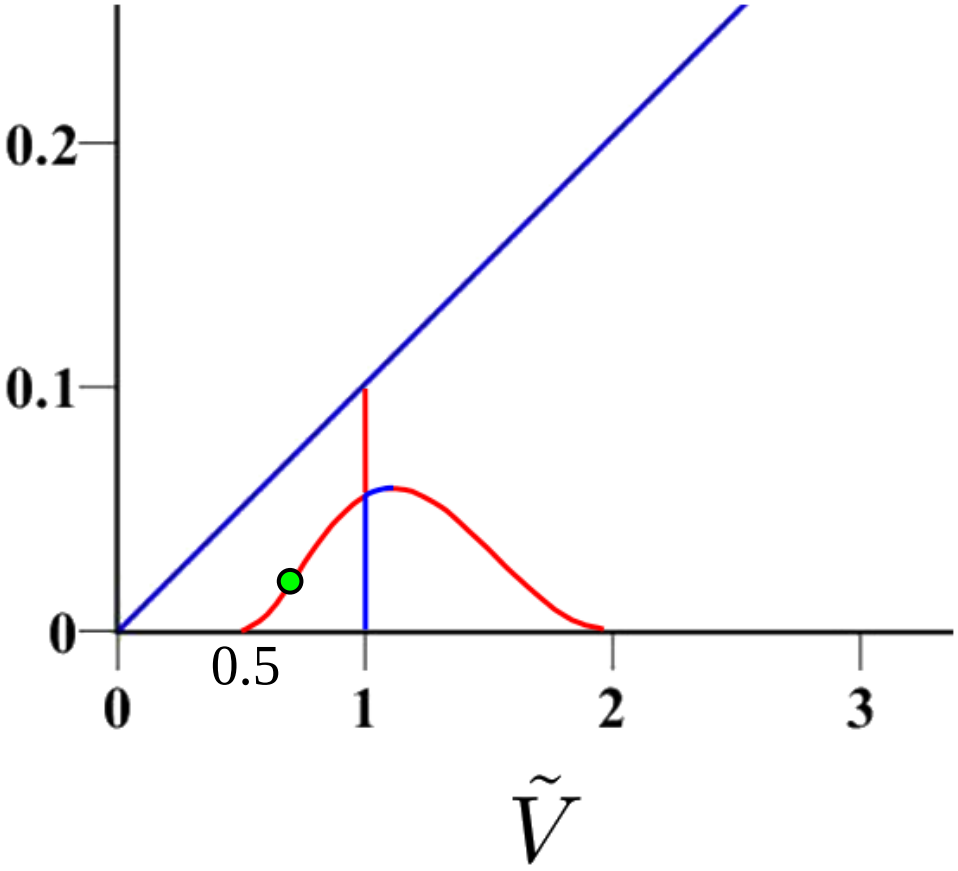

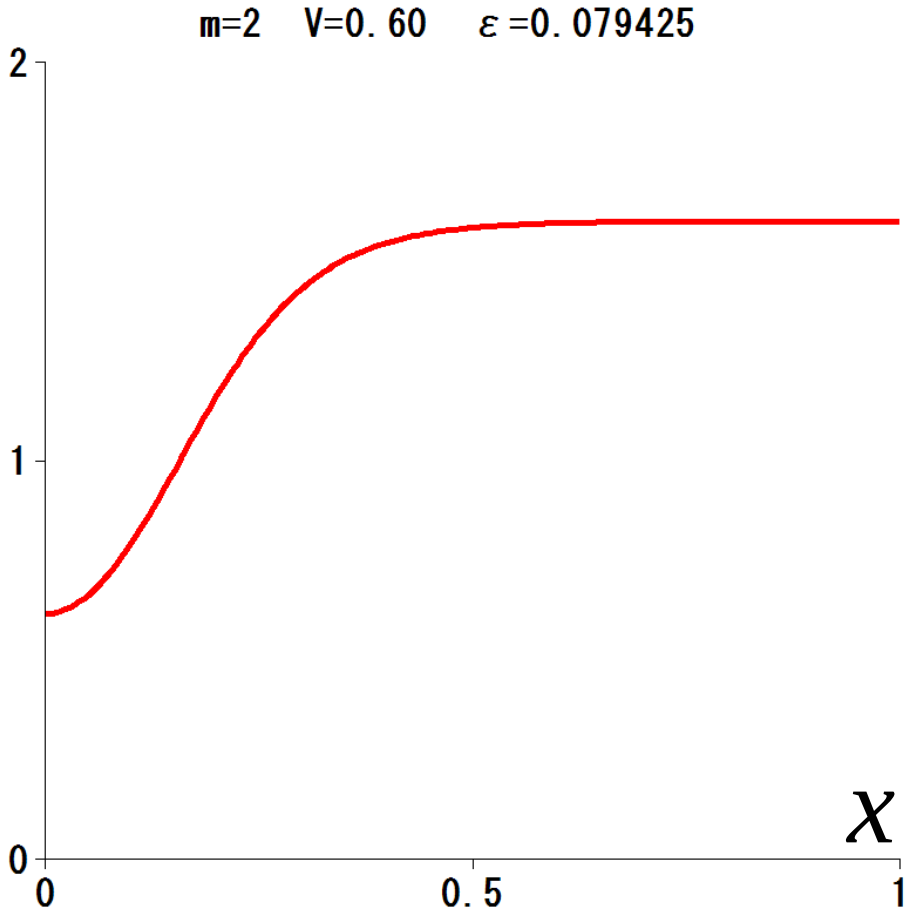

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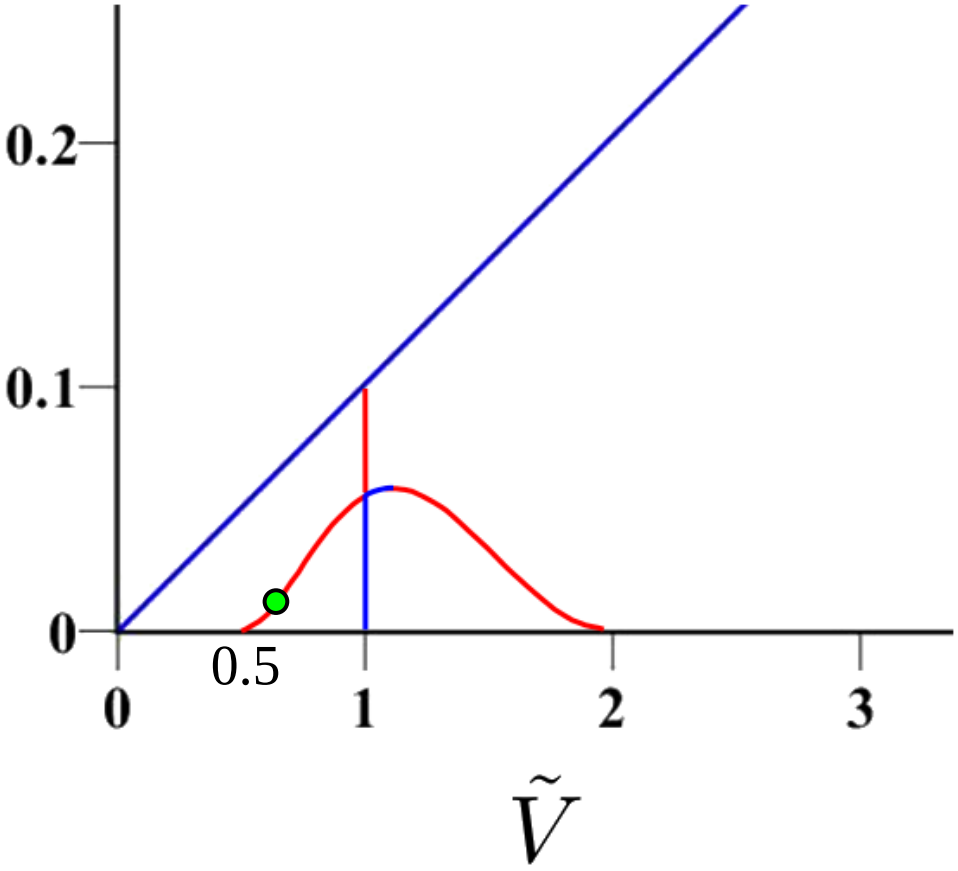

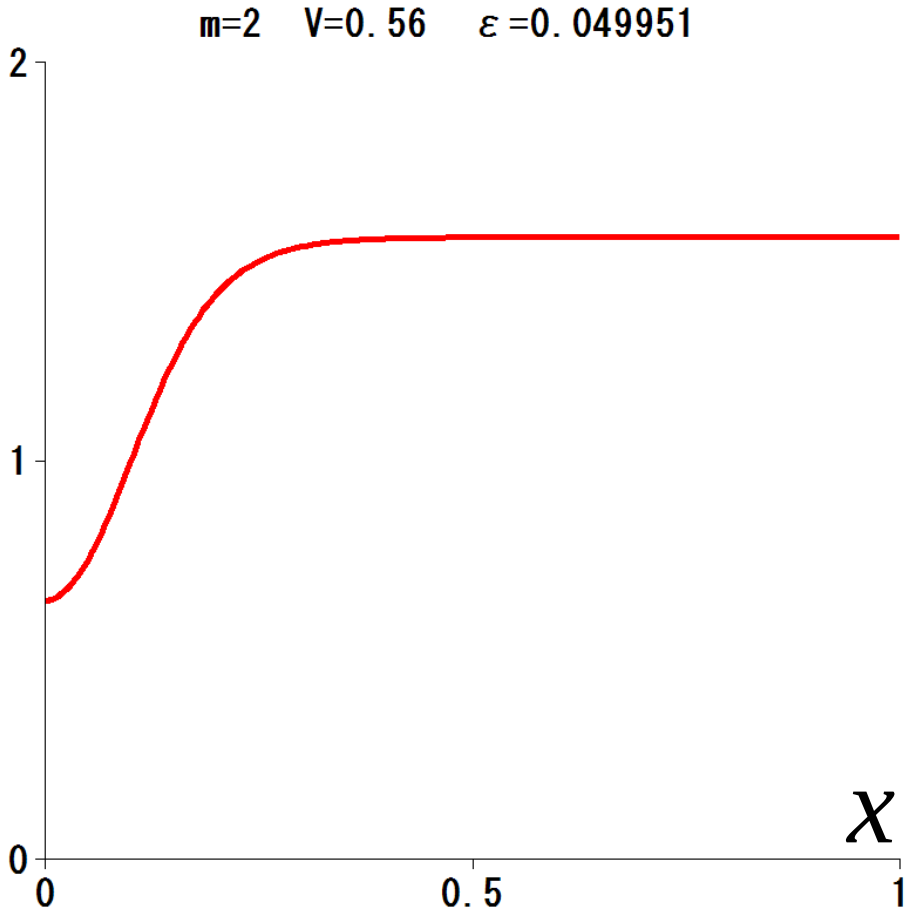

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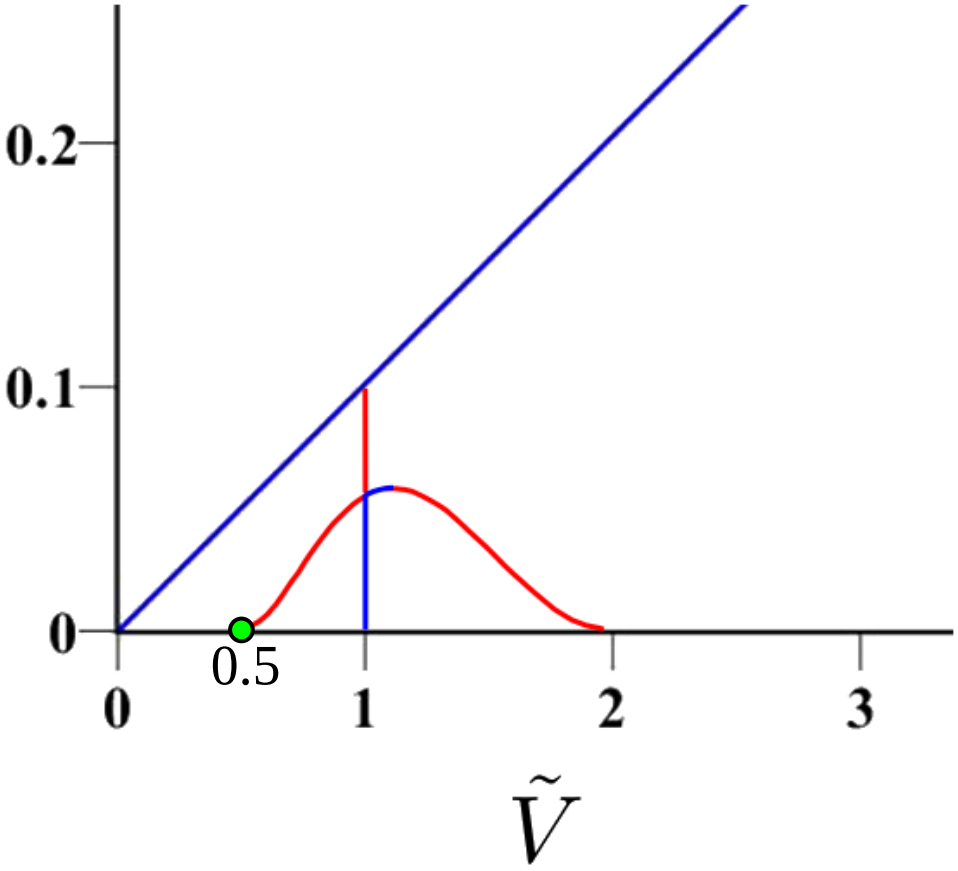

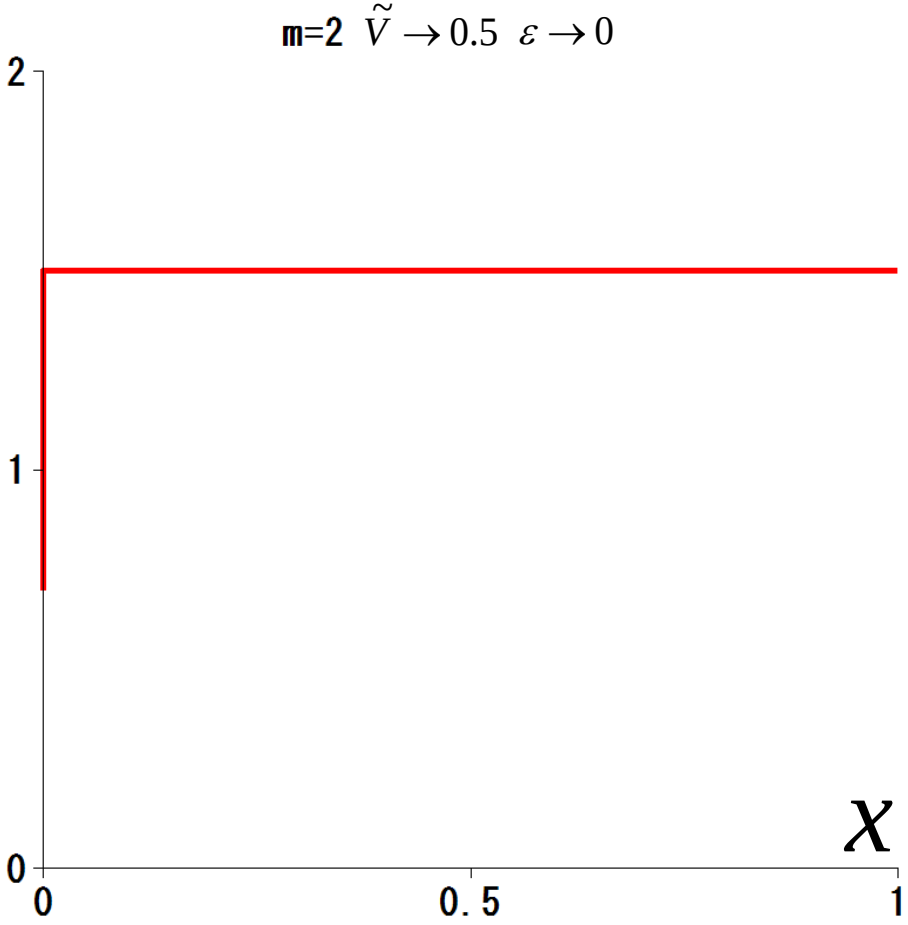

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(SLP)を満たす $(\tilde{V}, \varepsilon^2)$ および解 $W(x)$ をすべて求めよ.

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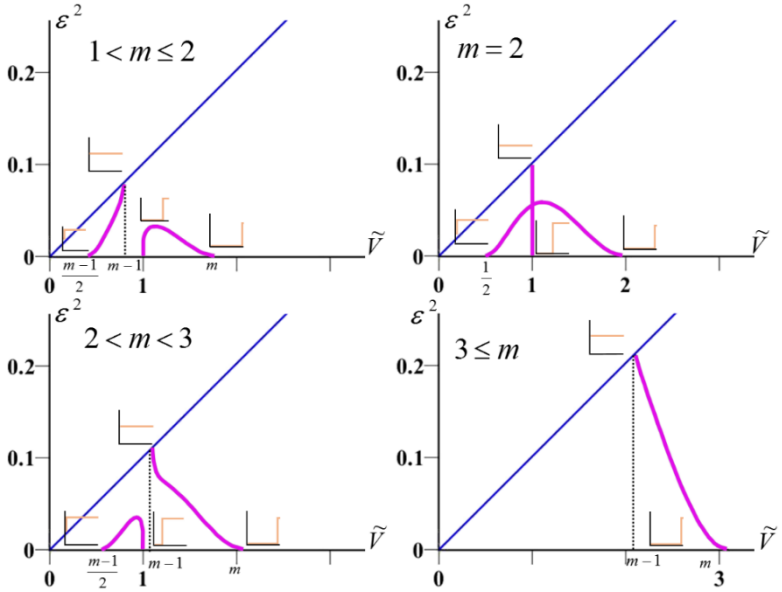
Numerical computation

Y.Mori, A. Jilkine and L. Edelstein-Keshet,
SIAM J, Appl. Math 71(2011), 1401-1427.



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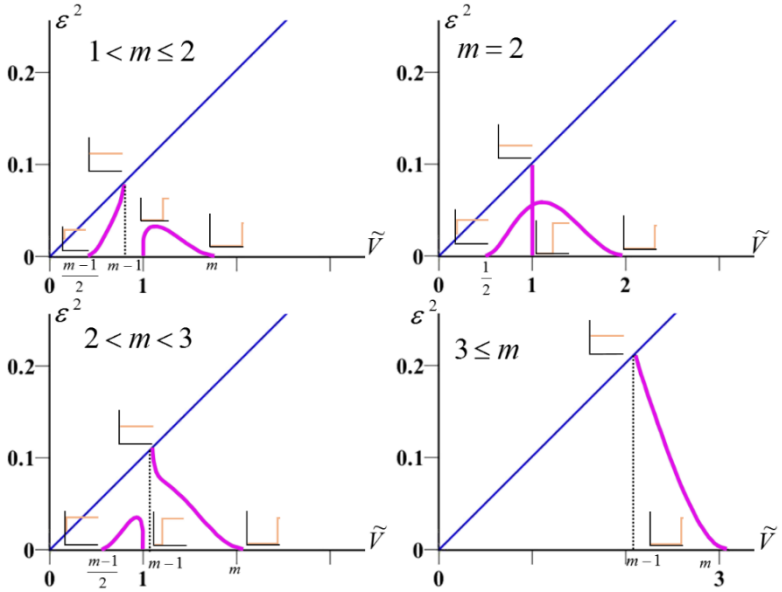

Mathematical results

Existence

K.Kuto and T.Tsujikawa,
DCDS Supplement 2013, 467-476.

some partial results

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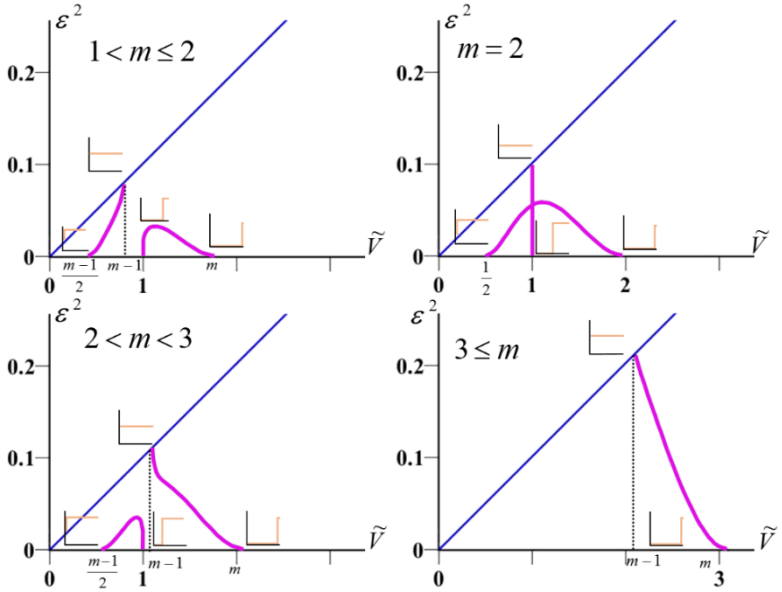
Y.Mori, A. Jilkine and L. Edelstein-Keshet,
SIAM J, Appl. Math 71(2011), 1401-1427.



Mathematical results

Existence

K.Kuto and T.Tsujikawa,
DCDS Supplement 2013, 467-476.

some partial results

Exact multiplicity　no results!

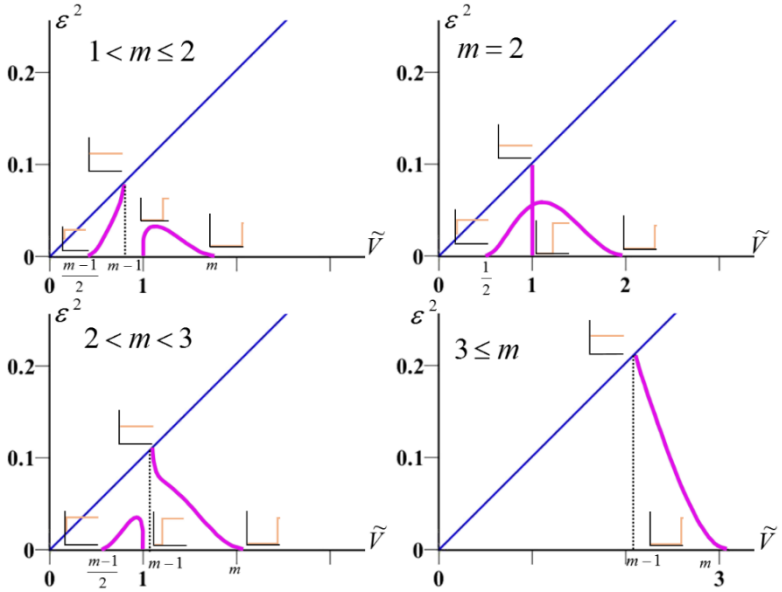
問題 正数 m が与えられたとき，

$\tilde{V}$ = 未知定数, $W(x)$ = 未知関数

$$(SLP)\begin{cases} \varepsilon^2 W_{xx} + W(W-1)(\tilde{V}+1-W) = 0, & \text{in } (0,1), \\ W_x(0) = W_x(1) = 0, \\ W(x) > 0, \quad W'(x) > 0 \quad \text{in } (0,1), \qquad \tilde{V} > 0, \\ \int_0^1 W(x)\,dx + \tilde{V} = m. \end{cases}$$

(SLP)を満たす $(\tilde{V}, \varepsilon^2)$ および解 $W(x)$ をすべて求めよ.

Mathematical results



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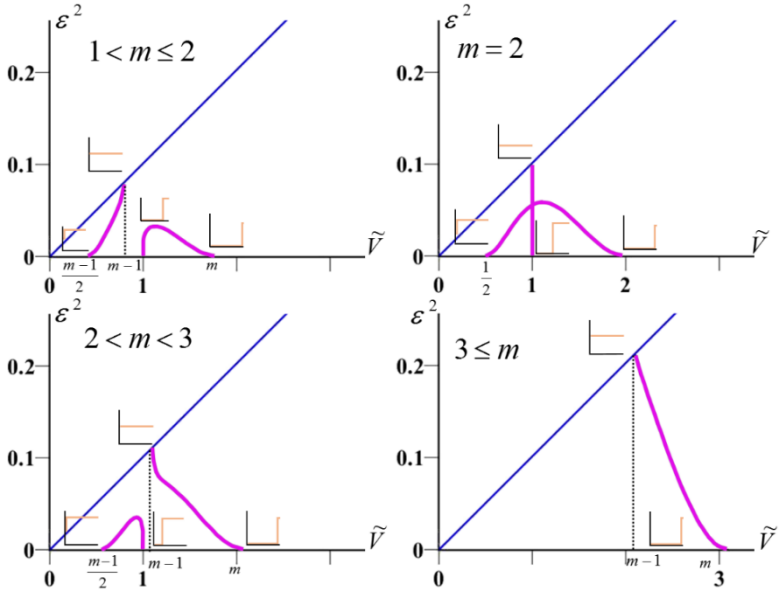

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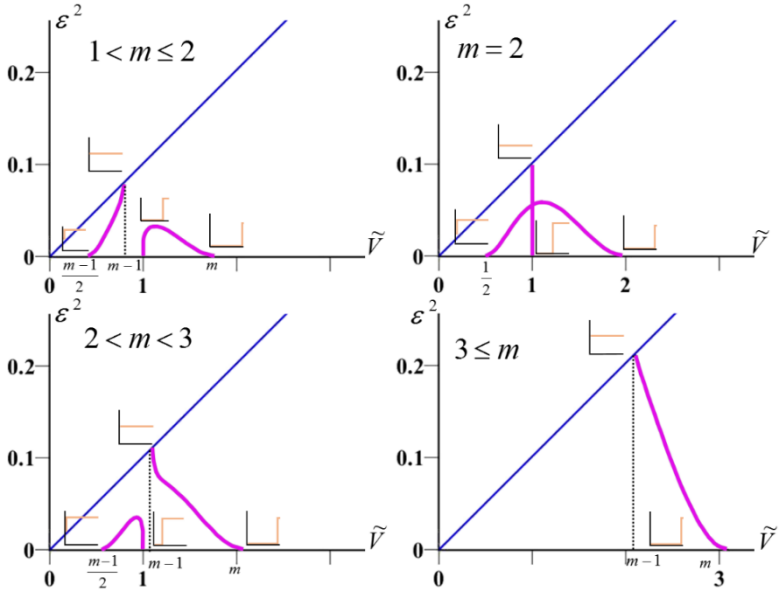
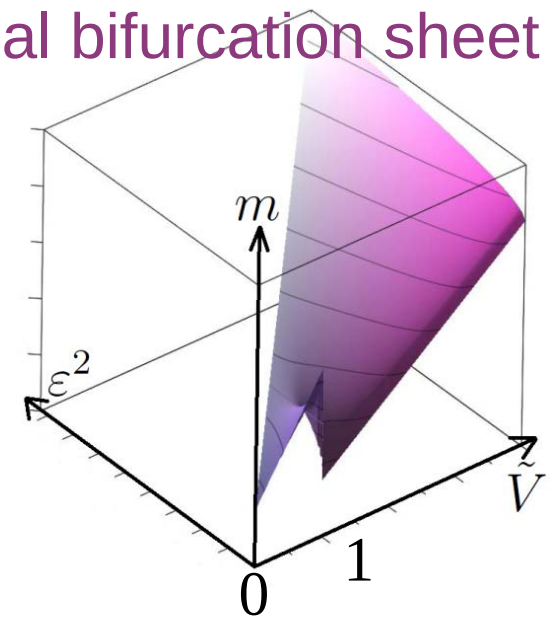


Exact multiplicity

global bifurcation sheet の表示式

global bifurcation sheet



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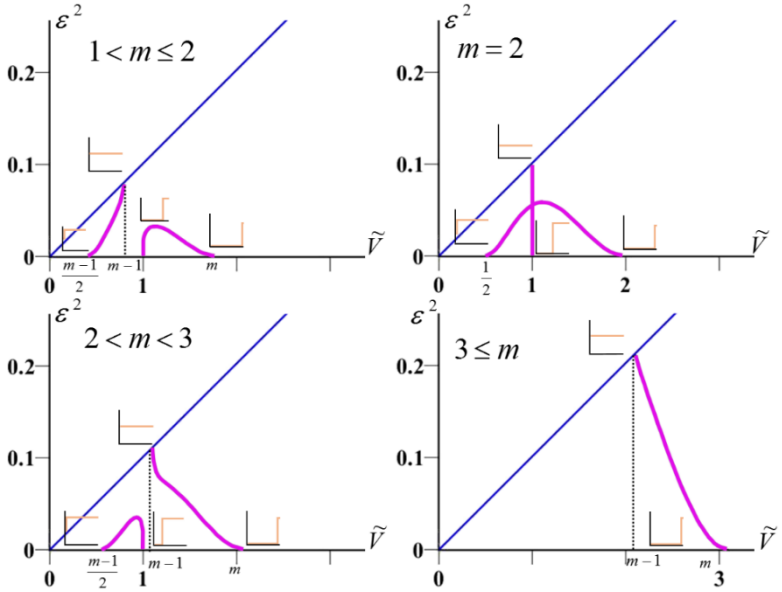

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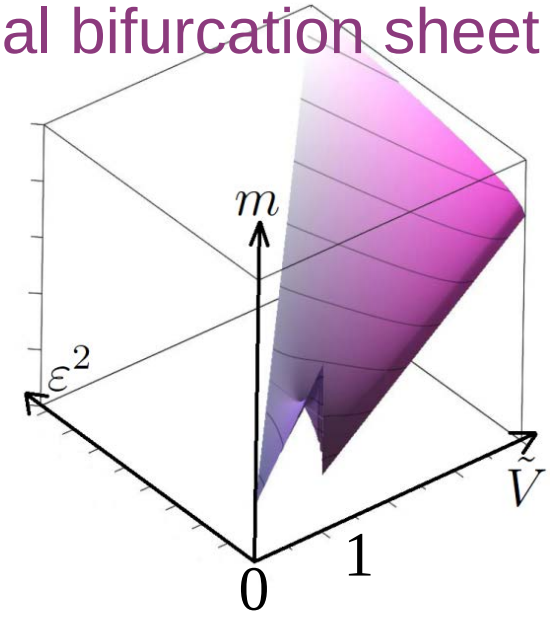

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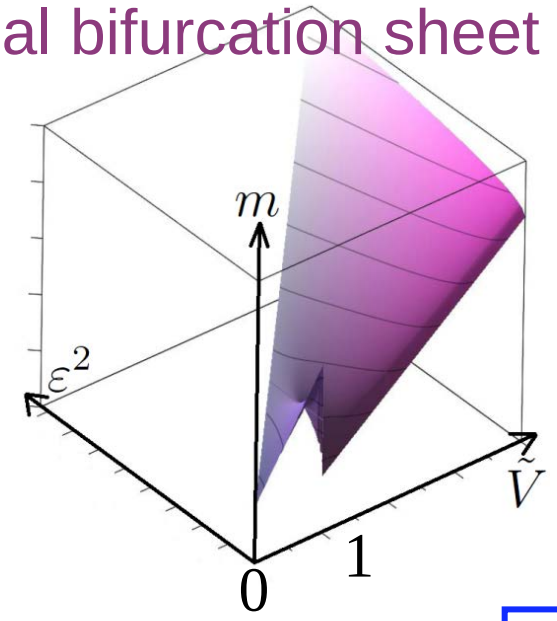

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$\varepsilon, \tilde{V}$: 与えられた正数, $W(x)$ = 未知関数

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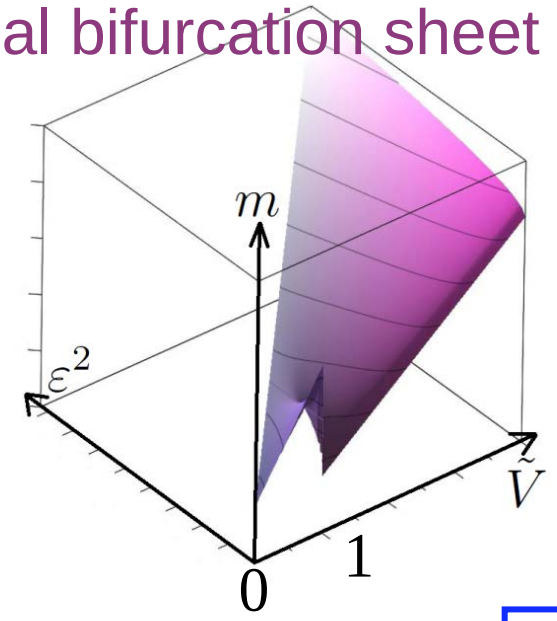

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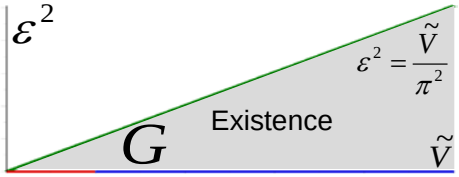
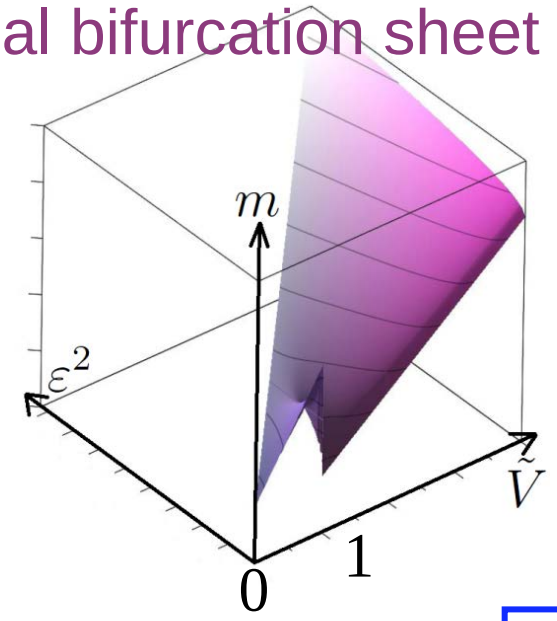


$$G := \left\{(\tilde{V}, \varepsilon^2) : 0 < \varepsilon^2 < \frac{\tilde{V}}{\pi^2}\right\}.$$

・この問題が解を持つ $\Leftrightarrow$ $(\tilde{V}, \varepsilon^2) \in G$. さらに解は一意である. これを $W(x;\tilde{V},\varepsilon^2)$ とかく

global bifurcation sheet



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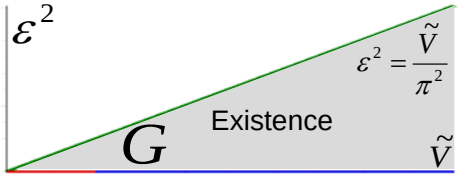

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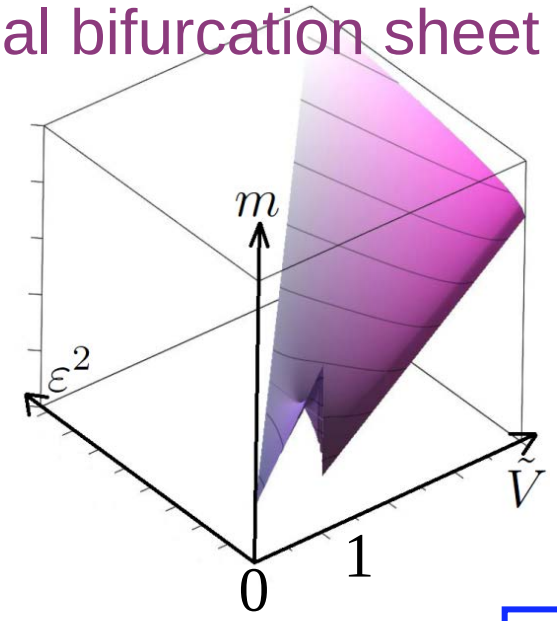
global bifurcation sheet



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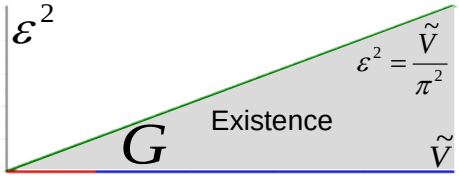

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定義 : (SLP)の global bifurcation sheet $:=\left\{\left(\tilde{V},\varepsilon^2,m(\tilde{V},\varepsilon^2)\right):(\tilde{V},\varepsilon^2)\in G\right\}$

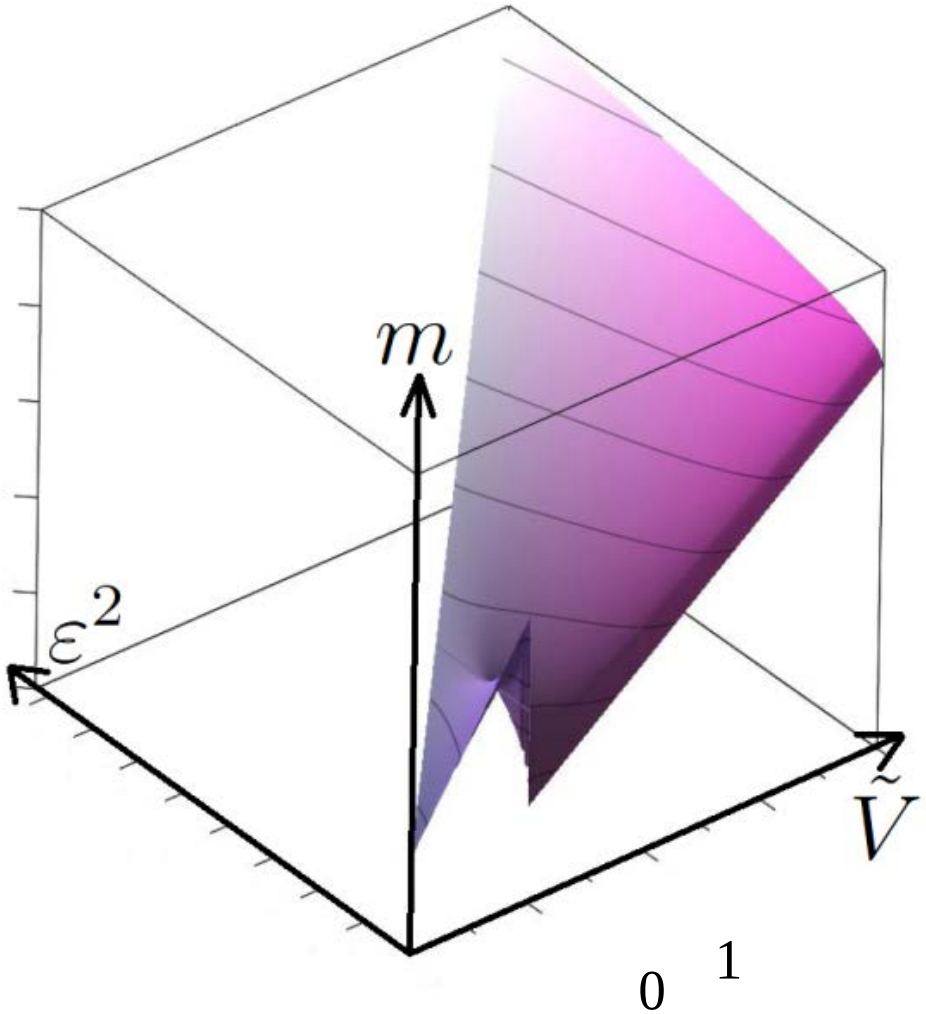
Remark. *Let* $W(x;\tilde{V},\varepsilon^2)$ *be the unique solution of* $(AP;\tilde{V})$, *and*

$$m(\tilde{V},\varepsilon^2) := \int_0^1 W(x;\tilde{V},\varepsilon^2)\,dx + \tilde{V},$$

then

$$m(\tilde{V},\varepsilon^2) = 2\tilde{V} + 2 - \tilde{V}\cdot m\left(\frac{1}{\tilde{V}}, \frac{\varepsilon^2}{\tilde{V}^2}\right) \qquad \text{for any} \quad \tilde{V} > 0, \quad \varepsilon > 0.$$

In particular, $m(1,\varepsilon^2) = 2$ *for any* $\varepsilon > 0$.

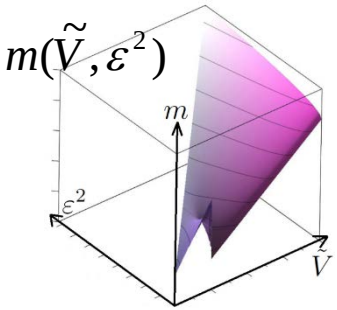


Theorem A （ $m(\tilde{V},\varepsilon^2)$ の表示式 ）.

T.Mori, K.Kuto, T.Tsujikawa, M.Nagayama and S.Yotsutani, Proc. of AIMS conf. 2014, submitted.

$m(\tilde{V},\varepsilon^2)$ *is represented by*

$$m(\tilde{V},\varepsilon^2) := \frac{4\tilde{V}+2}{3} + \frac{1}{\sqrt{3}} \cdot \sqrt{\tilde{V}^2+\tilde{V}+1} \cdot M(h,s),$$



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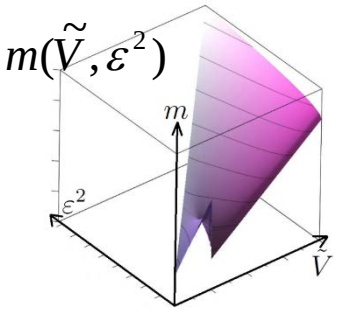

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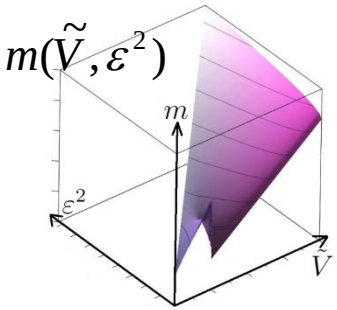

where $(h,\,s)=(h(\tilde{V},\,\varepsilon^2),\,s(\tilde{V},\,\varepsilon^2))$ *is the unique solution of the following system of transcendental equations*

$$\begin{cases} E(h,s) := \dfrac{\sqrt{2s(1-s)(1-sh)}\big/K(\sqrt{h})}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}} = \sqrt{3}\cdot\dfrac{\varepsilon}{\sqrt{\tilde{V}^2+\tilde{V}+1}}, \\ A(h,s) := \dfrac{2(hs^2-sh+1)(hs^2-2s+1)(1-hs^2)}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}^{\,3}} = \dfrac{1}{3\sqrt{3}}\cdot\dfrac{(1-\tilde{V})(2\tilde{V}+1)(\tilde{V}+2)}{\sqrt{\tilde{V}^2+\tilde{V}+1}^{\,3}}, \\ 0<h<1,\quad 0<s<1. \end{cases}$$

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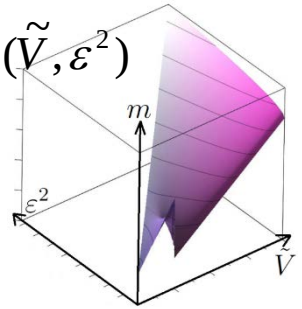

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Here, $K(\cdot)$ *is the complete elliptic integral of the first kind,*

$\Pi(\cdot,\cdot)$ *is the complete elliptic integral of the third kind.*

$$K(k) := \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}, \qquad \Pi(\mu,k) := \int_0^{\frac{\pi}{2}} \frac{d\varphi}{(1+\mu\sin^2\varphi)\sqrt{1-k^2\sin^2\varphi}}.$$

(SLP)の大域的分岐シート

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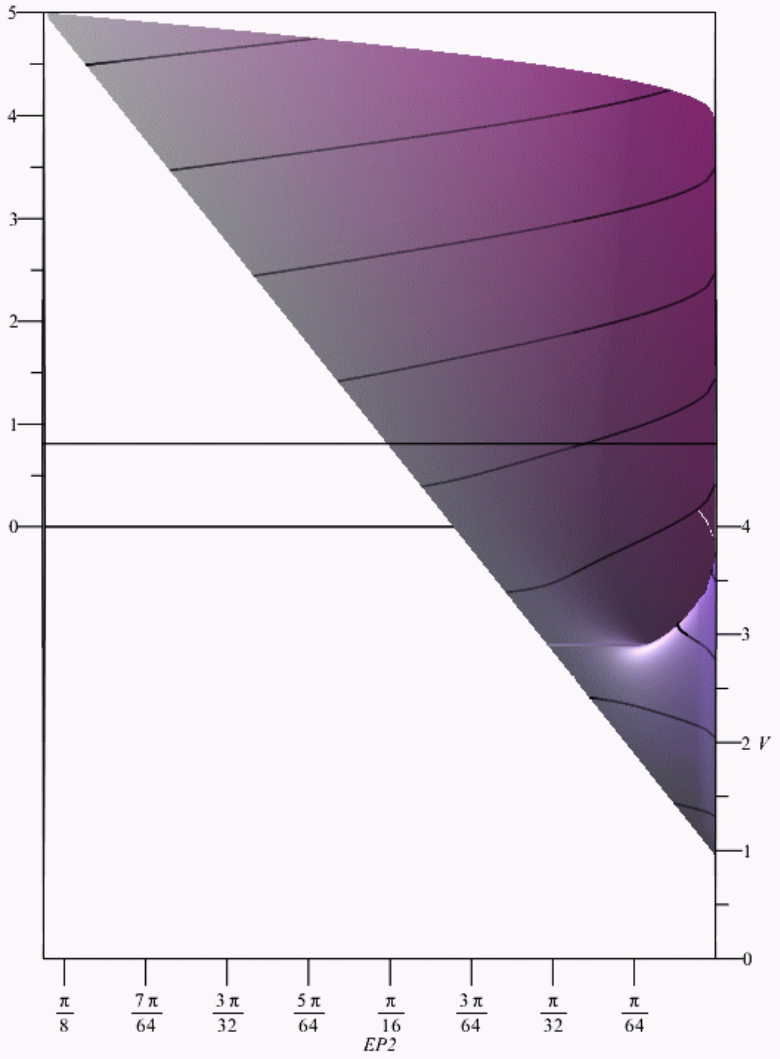
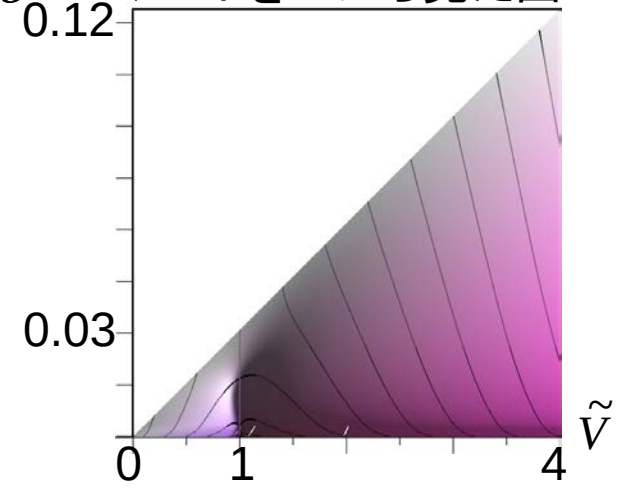
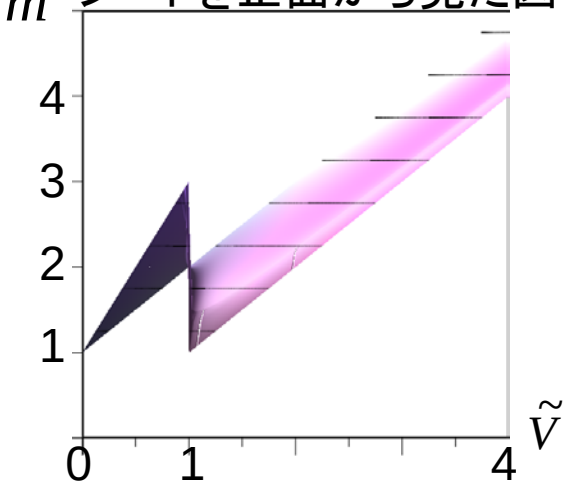


シートを上から見た図



シートを正面から見た図

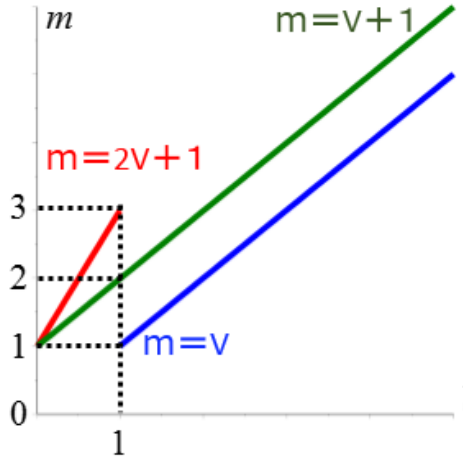


TheoremB *Let* $\tilde{V} > 0$ *be fixed. then*

(i) For $0 < \tilde{V} < 1$, $m(\tilde{V}, \varepsilon^2)$ *is decreasing in* $\varepsilon^2 \in \left(0, \tilde{V}/\pi^2\right)$, *and*

$$\lim_{\varepsilon^2 \downarrow 0} m(\tilde{V}, \varepsilon^2) = 2\tilde{V} + 1, \qquad \lim_{\varepsilon^2 \downarrow 0} W(\tilde{V}, \varepsilon^2) = \tilde{V} + 1 \quad \text{in } (0,1],$$

$$\lim_{\varepsilon^2 \uparrow \tilde{V}/\pi^2} m(\tilde{V}, \varepsilon^2) = \tilde{V} + 1, \qquad \lim_{\varepsilon^2 \uparrow \tilde{V}/\pi^2} W(\tilde{V}, \varepsilon^2) = 1, \quad \text{in } [0,1].$$



(ii) For $1 < \tilde{V}$, $m(\tilde{V}, \varepsilon^2)$ *is increasing in* $\varepsilon^2 \in \left(0, \tilde{V}/\pi^2\right)$, *and*

$$\lim_{\varepsilon^2 \downarrow 0} m(\tilde{V}, \varepsilon^2) = \tilde{V}, \qquad \lim_{\varepsilon^2 \downarrow 0} W(\tilde{V}, \varepsilon^2) = 0 \quad \text{in } [0,1),$$

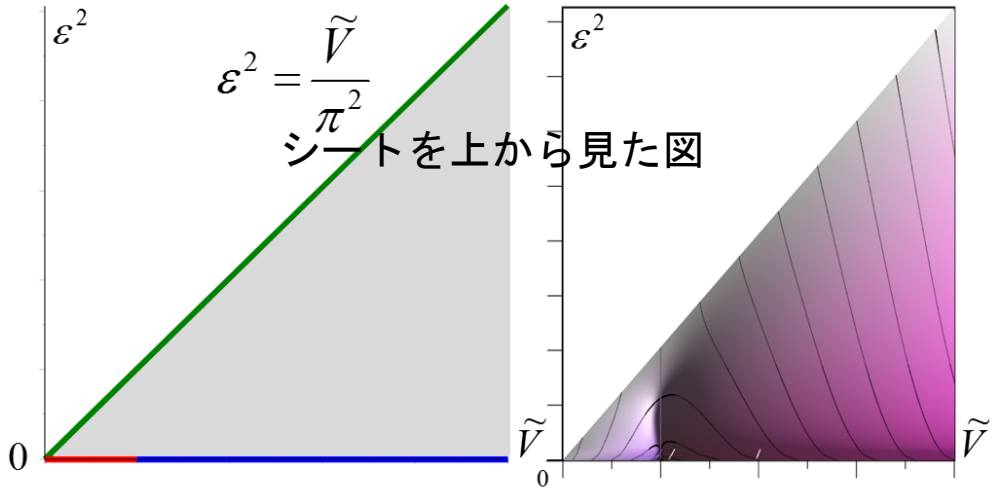
$$\lim_{\varepsilon^2 \uparrow \tilde{V}/\pi^2} m(\tilde{V}, \varepsilon^2) = \tilde{V} + 1. \qquad \lim_{\varepsilon^2 \uparrow \tilde{V}/\pi^2} W(\tilde{V}, \varepsilon^2) = 1 \quad \text{in } [0,1].$$

Remark

$$m(1, \varepsilon^2) = 2 \quad \textit{for} \quad \varepsilon^2 \in \left(0, \sqrt{\tilde{V}}/\pi^2\right),$$

$$\lim_{\varepsilon^2 \downarrow 0} W(x; 1, \varepsilon^2) = 0 \ \left(0 \le x < 1/2\right),\ 1 \ (x = 1/2),\ 2 \ \left(1/2 < x \le 1\right).$$

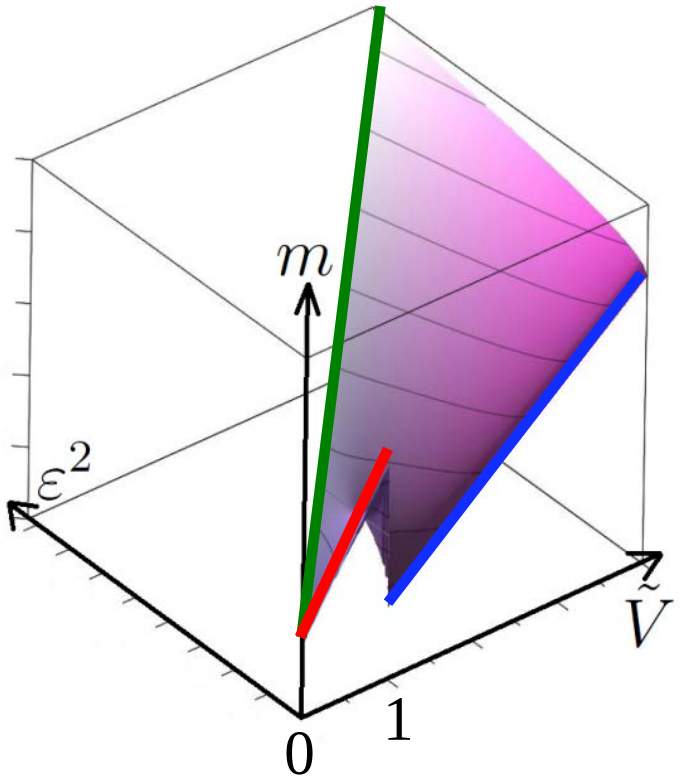
Theorem B.　（ $m(\tilde{V},\varepsilon^2)$ の端点での極限値 ）.



シートを上から見た図

global bifurcation sheet

$$\left\{\left(\tilde{V},\ \varepsilon^2,\ m(\tilde{V},\varepsilon^2)\right):(\tilde{V},\ \varepsilon^2)\in G\right\}$$

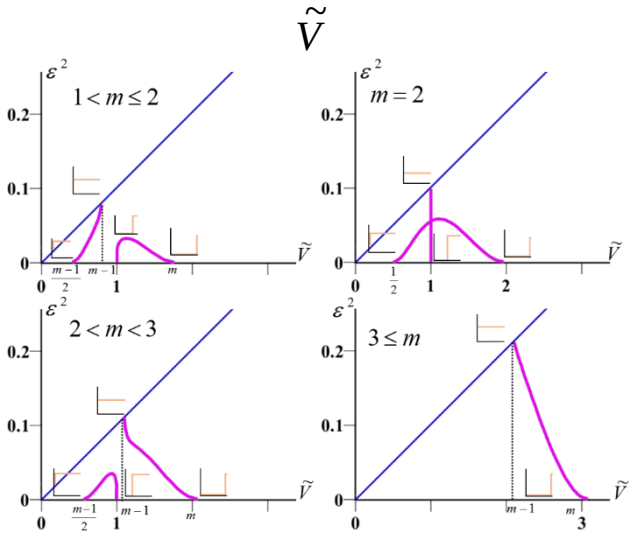


Bifurcation curve

Front view of the sheet

Movie
bifurcation curve for each m

Move m Increasing 1 to 4

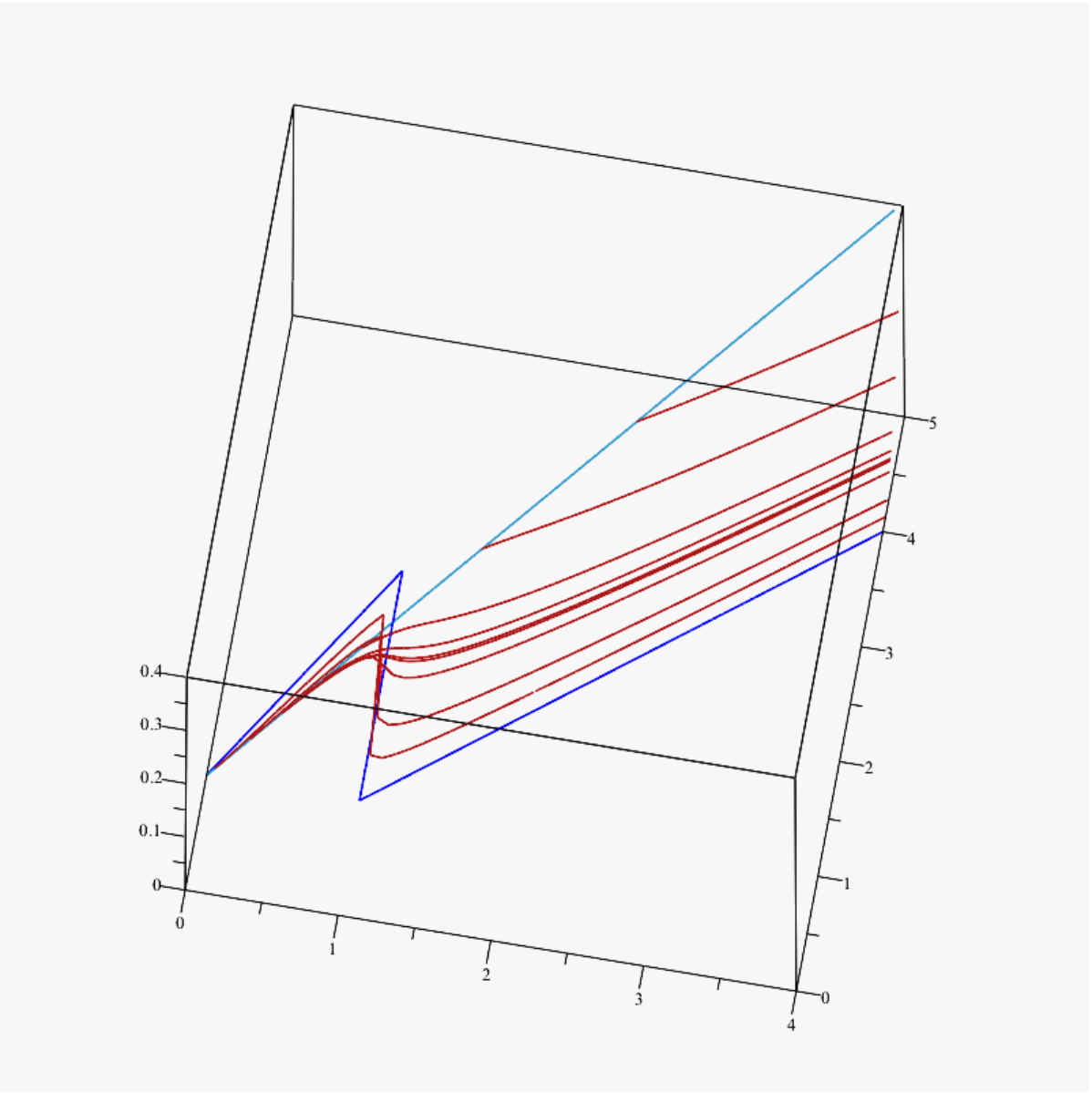


$$\tilde{V}$$

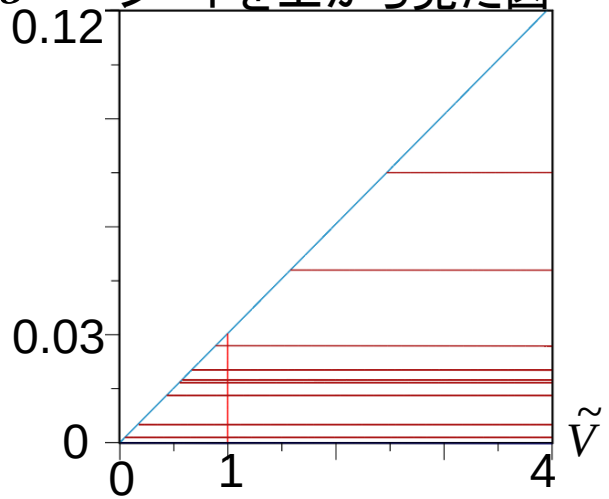
$$(SLP)\begin{cases} \varepsilon^2 W_{xx} + W(W-1)(\tilde{V}+1-W) = 0, \\ W_x(0) = W_x(1) = 0, \\ W'(x) > 0, \\ \int_0^1 W(x)\,\mathrm{dx} + \tilde{V} = m. \end{cases}$$

(SLP)の大域的分岐シート

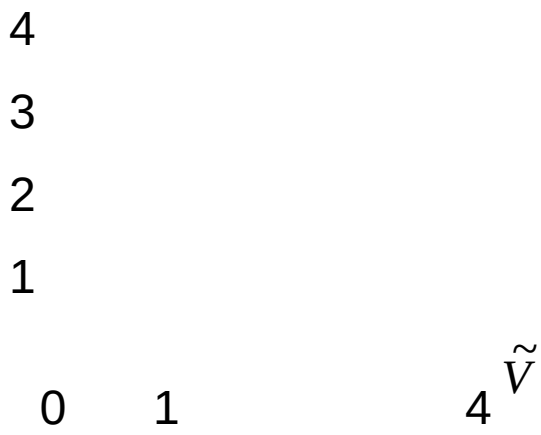
$$(SLP)\begin{cases}\varepsilon^2 W_{xx}+W(W-1)(\tilde{V}+1-W)=0,\\ W_x(0)=W_x(1)=0,\\ W'(x)>0,\\ \int_0^1 W(x)\,dx+\tilde{V}=m.\end{cases}$$



ε^2 シートを上から見た図



m シートを正面から見た図



(SLP)の大域的分岐シート

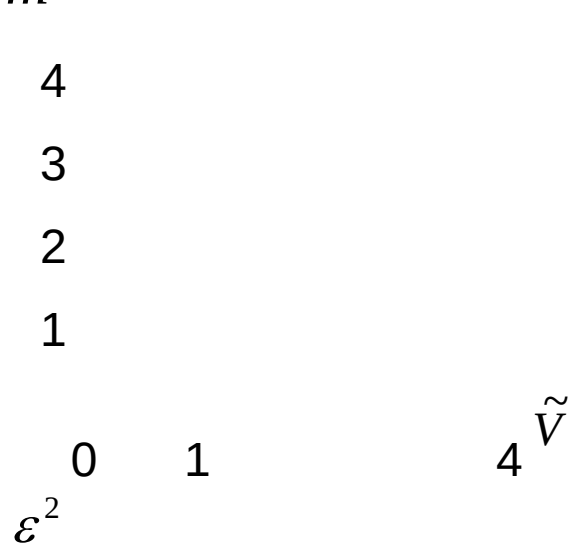
各 ε ごとに分岐シートを正面から見た動画
0.07から0.45まで動かす

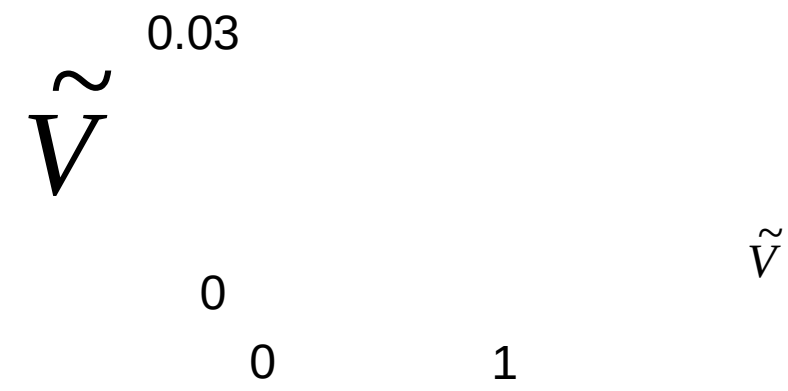
$$\varepsilon^2 \qquad ^2$$

$$m$$

$$(SLP)\begin{cases} \varepsilon^2 W_{xx} + W(W-1)(\tilde{V}+1-W) = 0, \\ W_x(0) = W_x(1) = 0, \\ W'(x) > 0, \\ \int_0^1 W(x)\,\mathrm{dx} + \tilde{V} = m. \end{cases}$$

m シートを正面から見た図





(SLP)の大域的分岐シート

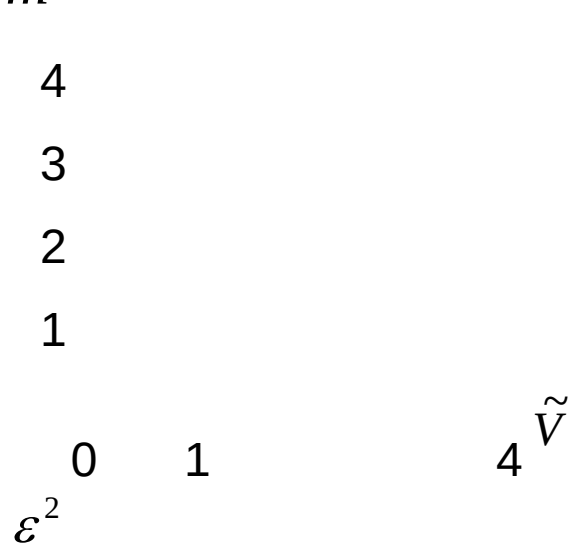
各 ε ごとに分岐シートを正面から見た動画
0.07から0.45まで動かす

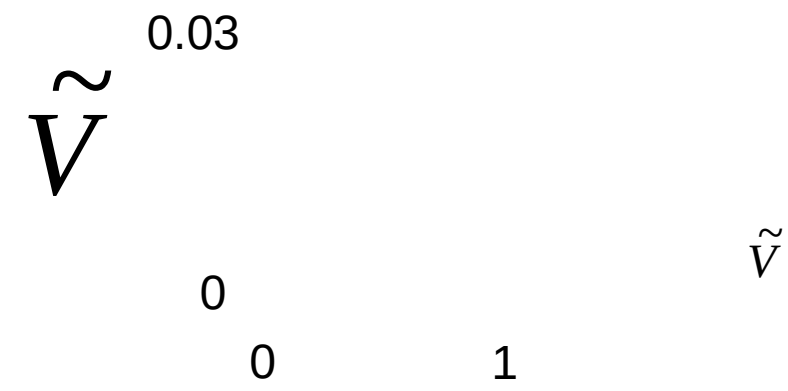
$$\varepsilon^2 \qquad ^2$$

$$m$$

$$(SLP)\begin{cases} \varepsilon^2 W_{xx} + W(W-1)(\tilde{V}+1-W) = 0, \\ W_x(0) = W_x(1) = 0, \\ W'(x) > 0, \\ \int_0^1 W(x)\,\mathrm{dx} + \tilde{V} = m. \end{cases}$$

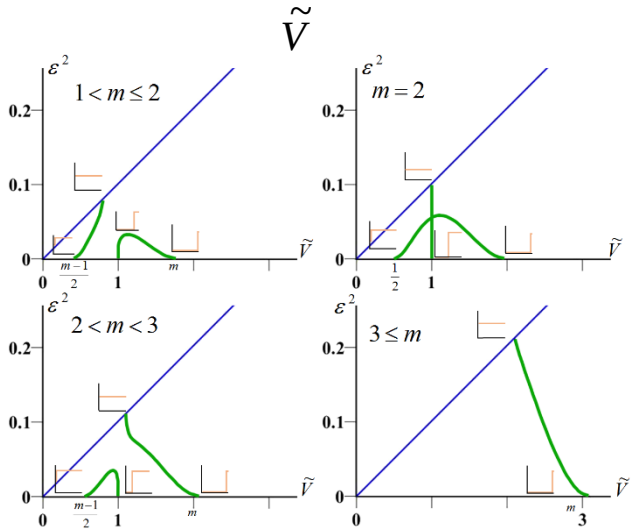
m シートを正面から見た図





分岐曲線

各mごとの分曲線図の動画
mを1から4まで動かす



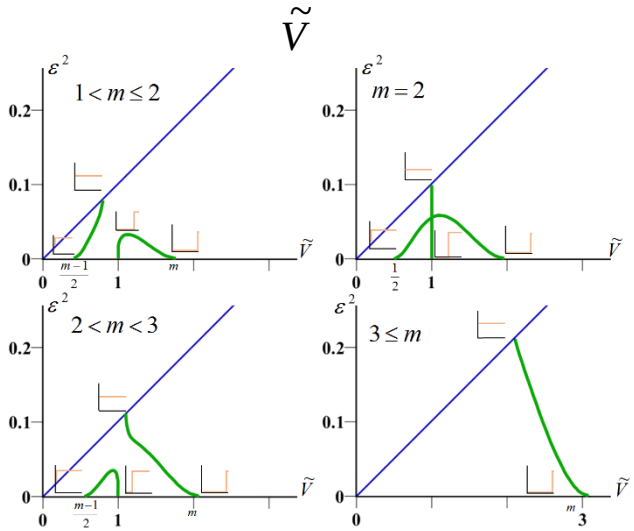
分岐シートの正面

$$\tilde{V}$$

$$(SLP)\begin{cases}\varepsilon^2 W_{xx}+W(W-1)(\tilde{V}+1-W)=0,\\ W_x(0)=W_x(1)=0,\\ W'(x)>0,\\ \int_0^1 W(x)\,\mathrm{dx}+\tilde{V}=m.\end{cases}$$

分岐曲線

各mごとの分曲線図の動画
mを1から4まで動かす



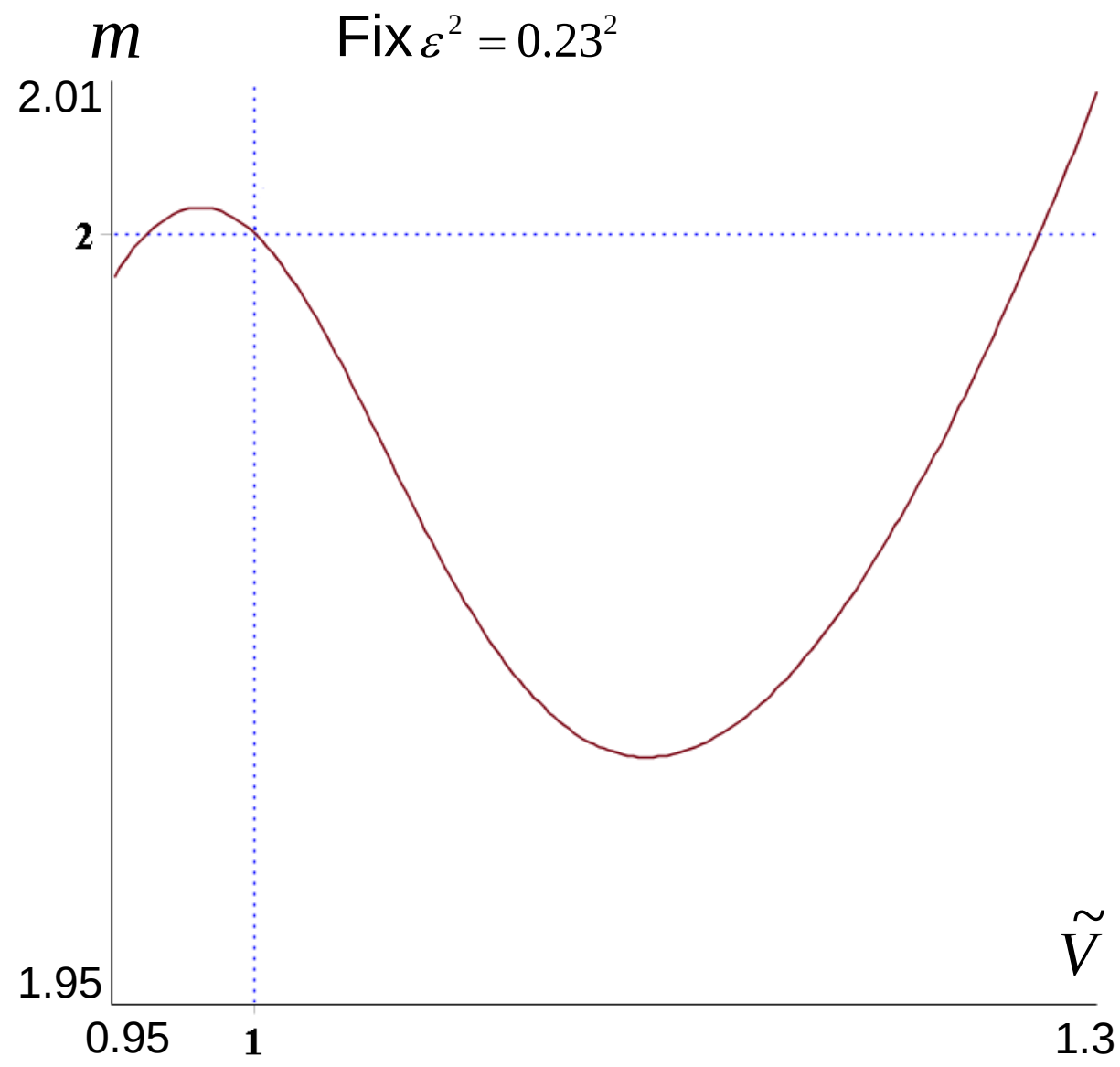
分岐シートの正面

$$\tilde{V}$$

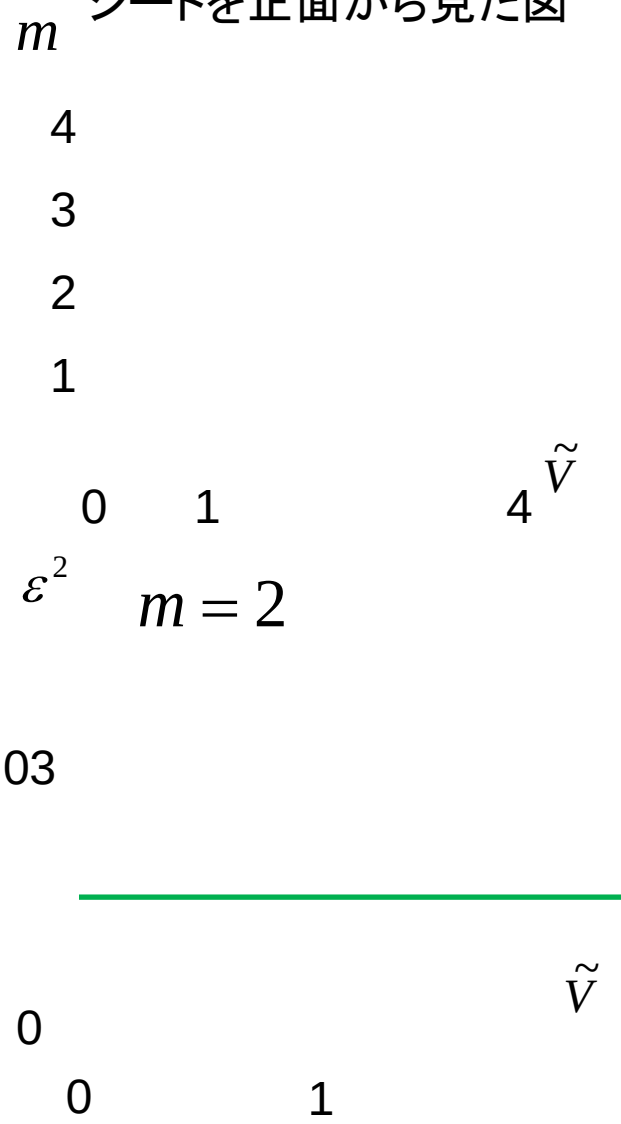
$$(SLP)\begin{cases}\varepsilon^2 W_{xx}+W(W-1)(\tilde{V}+1-W)=0,\\ W_x(0)=W_x(1)=0,\\ W'(x)>0,\\ \int_0^1 W(x)\,\mathrm{dx}+\tilde{V}=m.\end{cases}$$

(SLP)の大域的分岐シート

Zoom up $(\tilde{V}, m) \in [0.95, 1.3] \times [1.95, 2.01]$

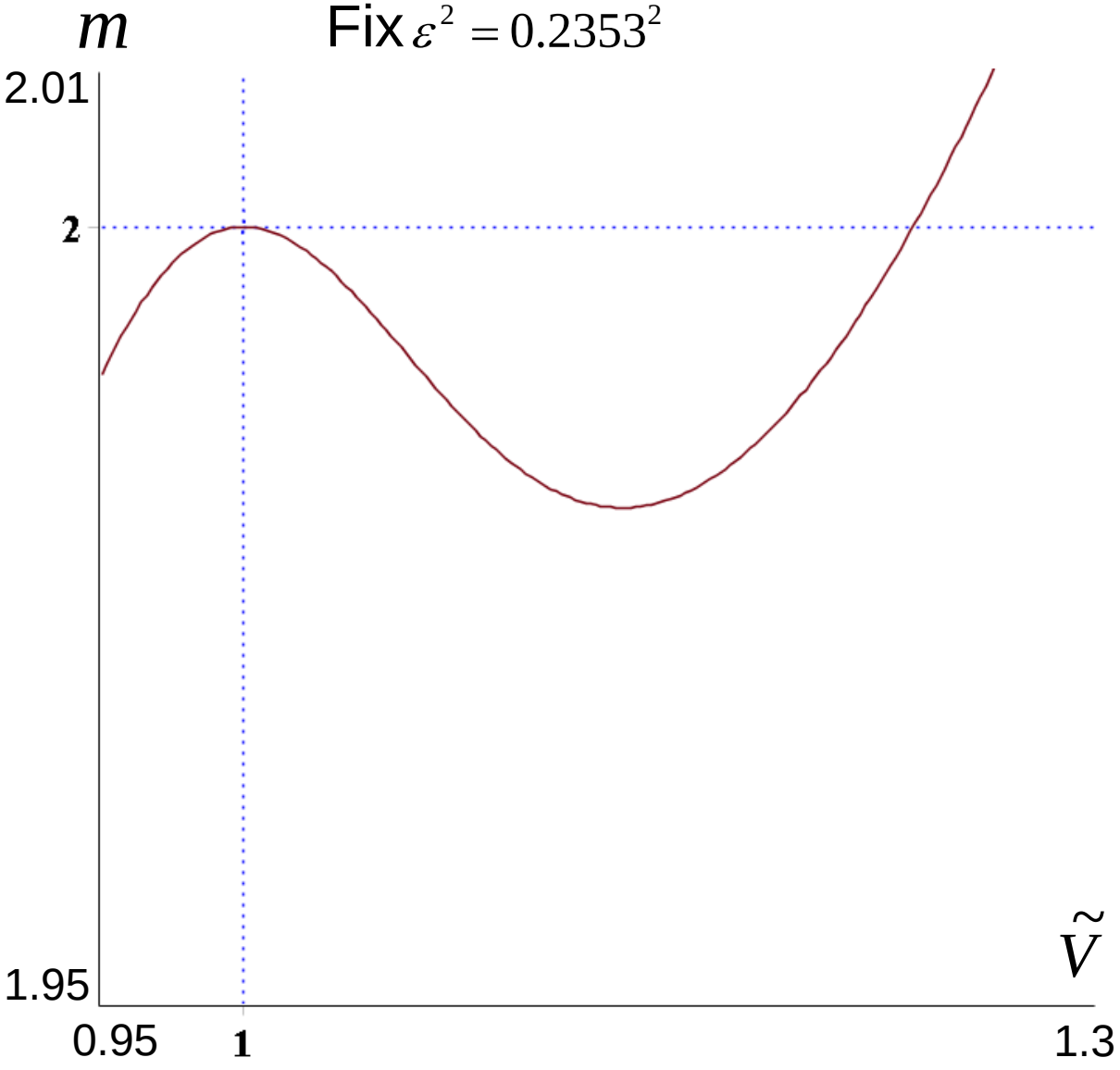


シートを正面から見た図

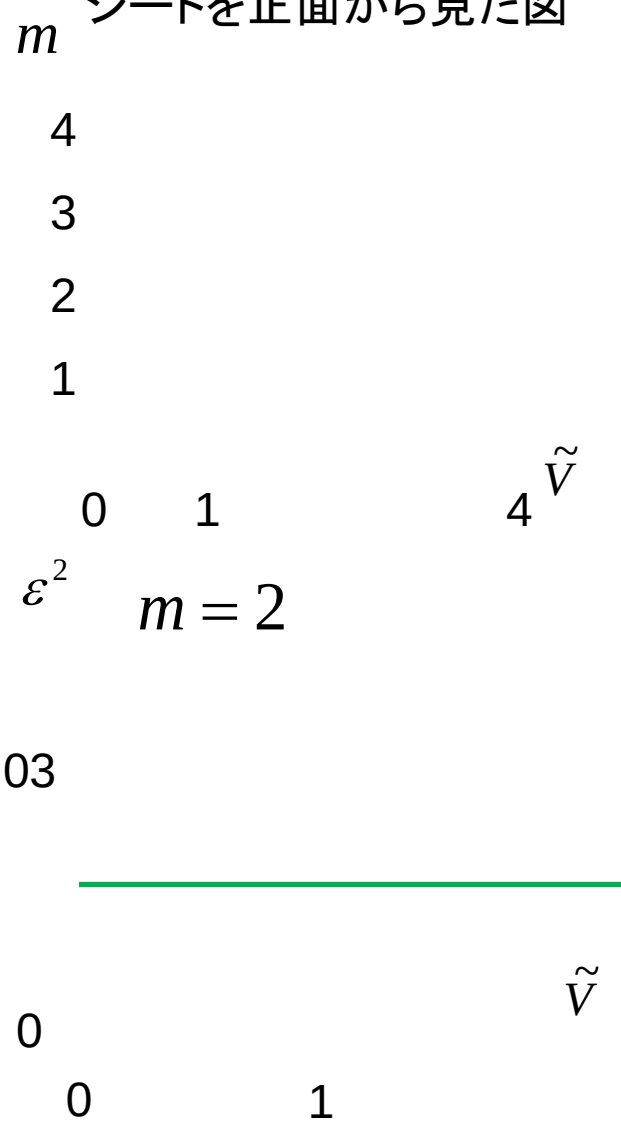


(SLP)の大域的分岐シート

Zoom up $(\tilde{V}, m) \in [0.95, 1.3] \times [1.95, 2.01]$

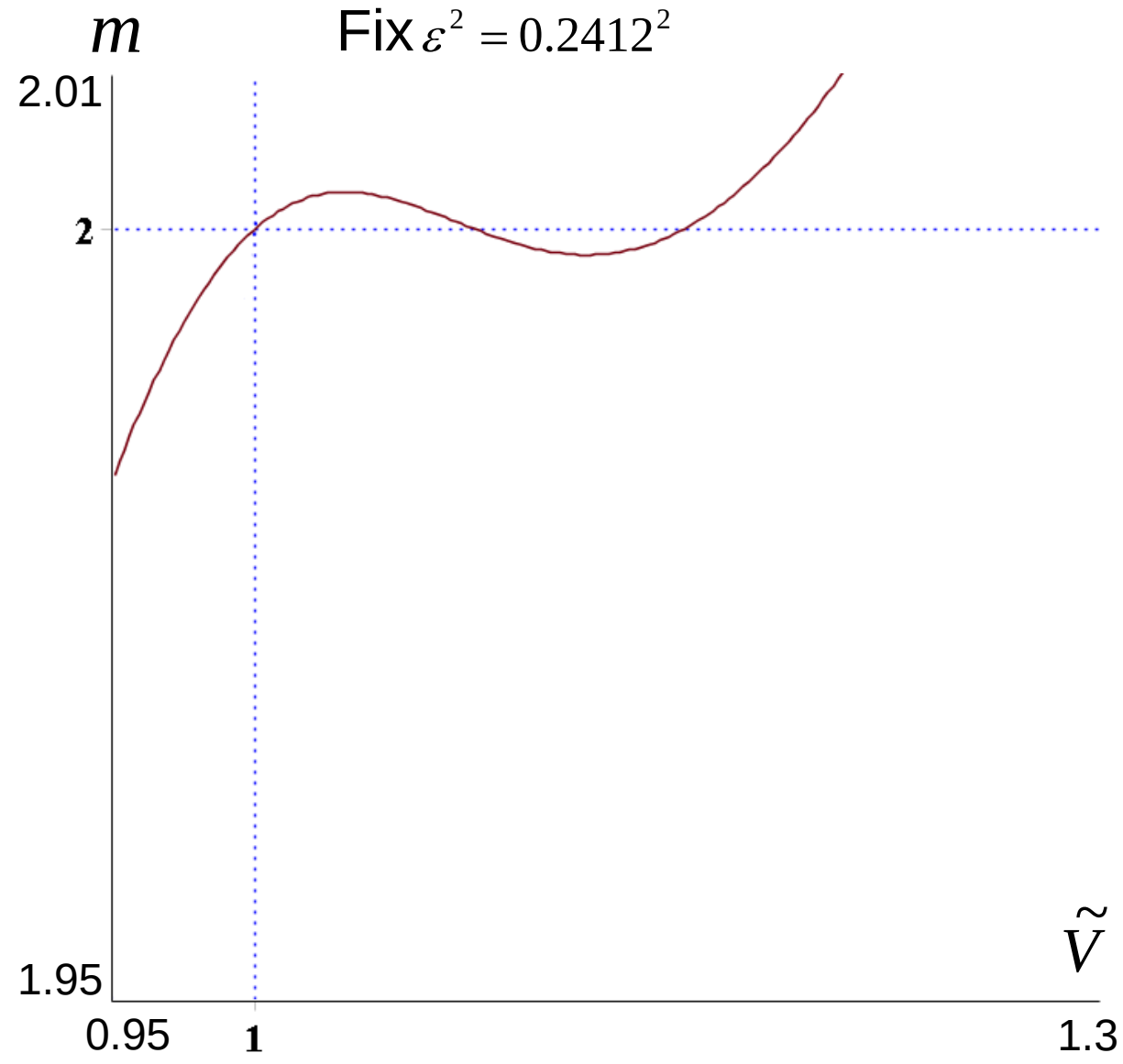


シートを正面から見た図

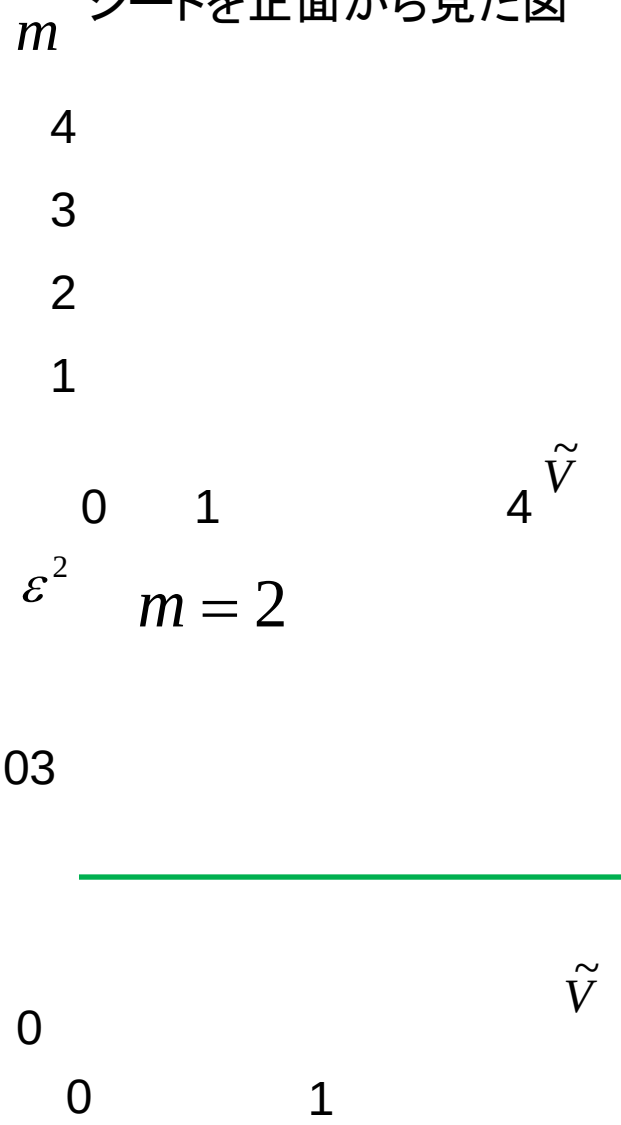


(SLP)の大域的分岐シート

Zoom up $(\tilde{V}, m) \in [0.95, 1.3] \times [1.95, 2.01]$

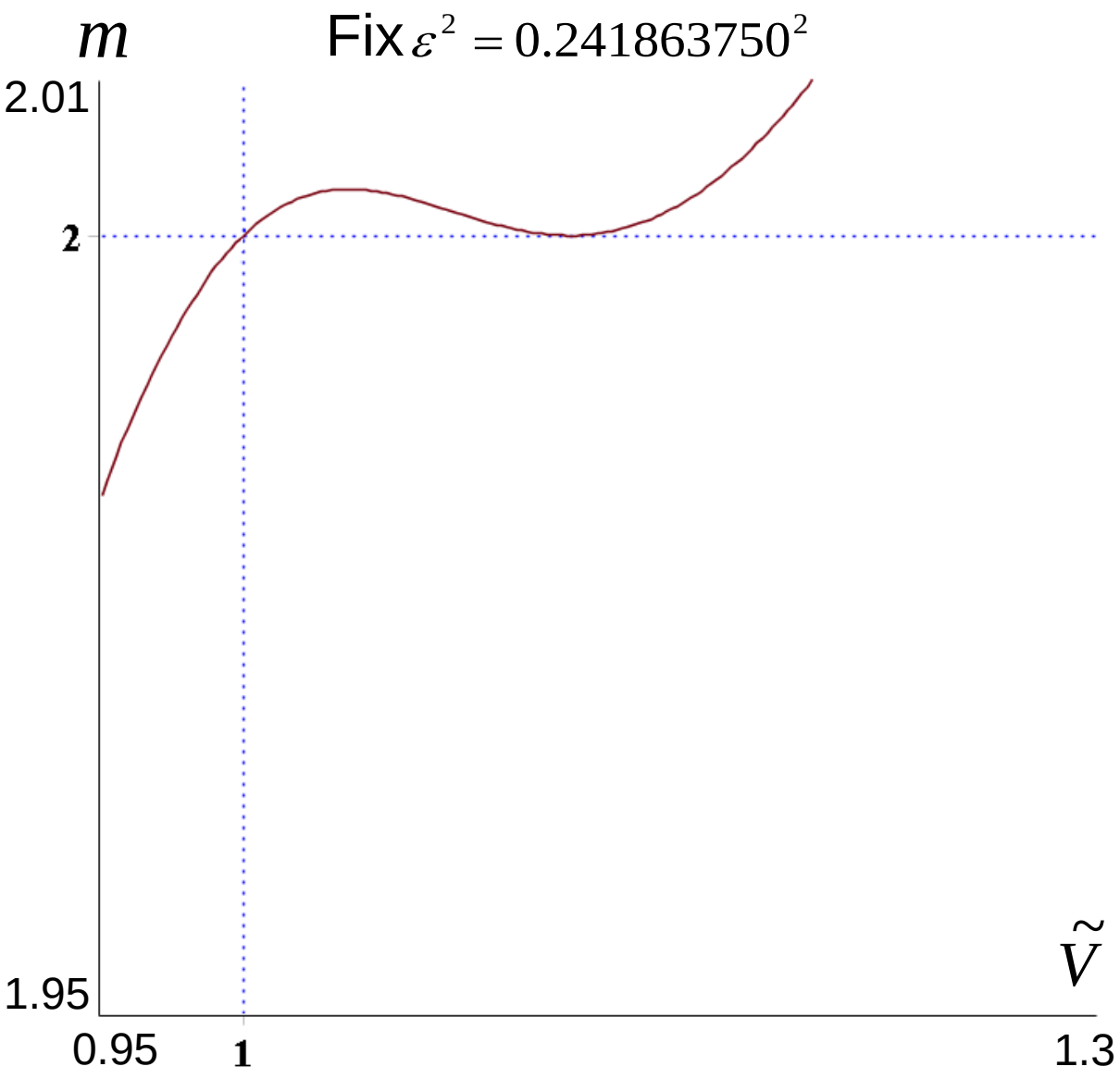


シートを正面から見た図

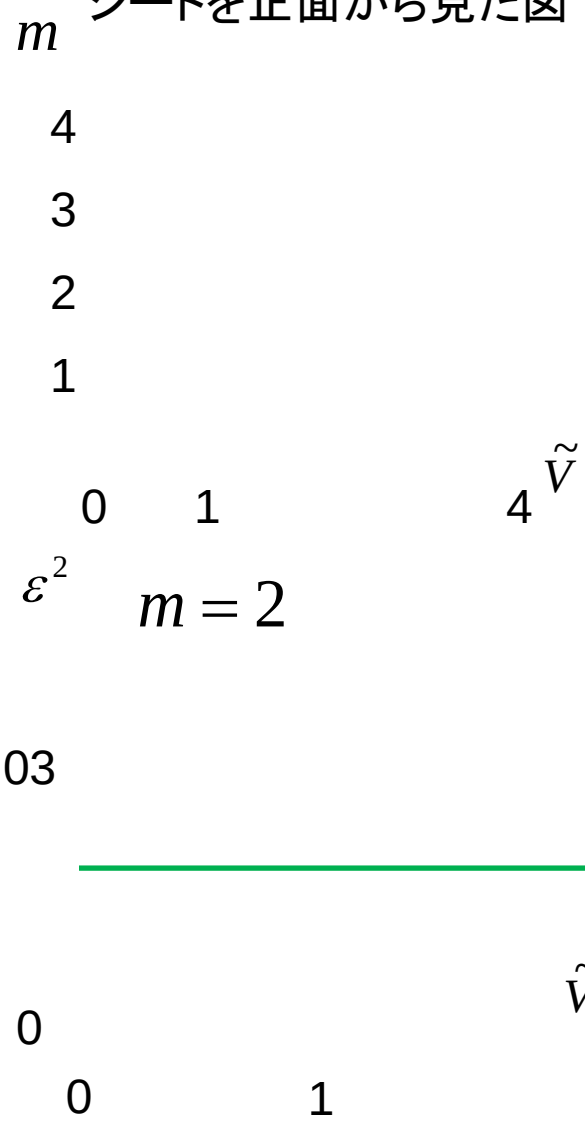


(SLP)の大域的分岐シート

Zoom up $(\tilde{V}, m) \in [0.95, 1.3] \times [1.95, 2.01]$

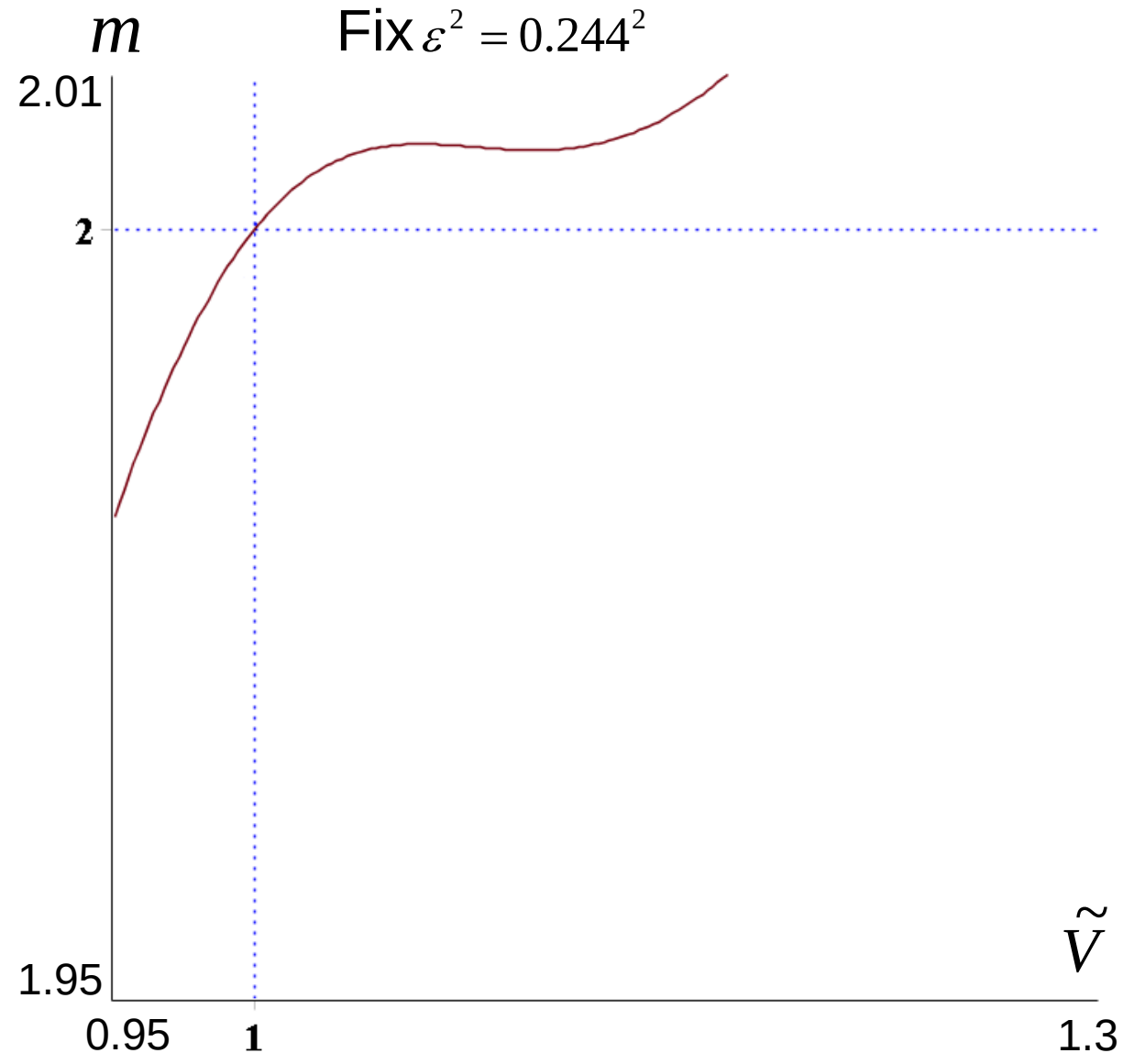


シートを正面から見た図

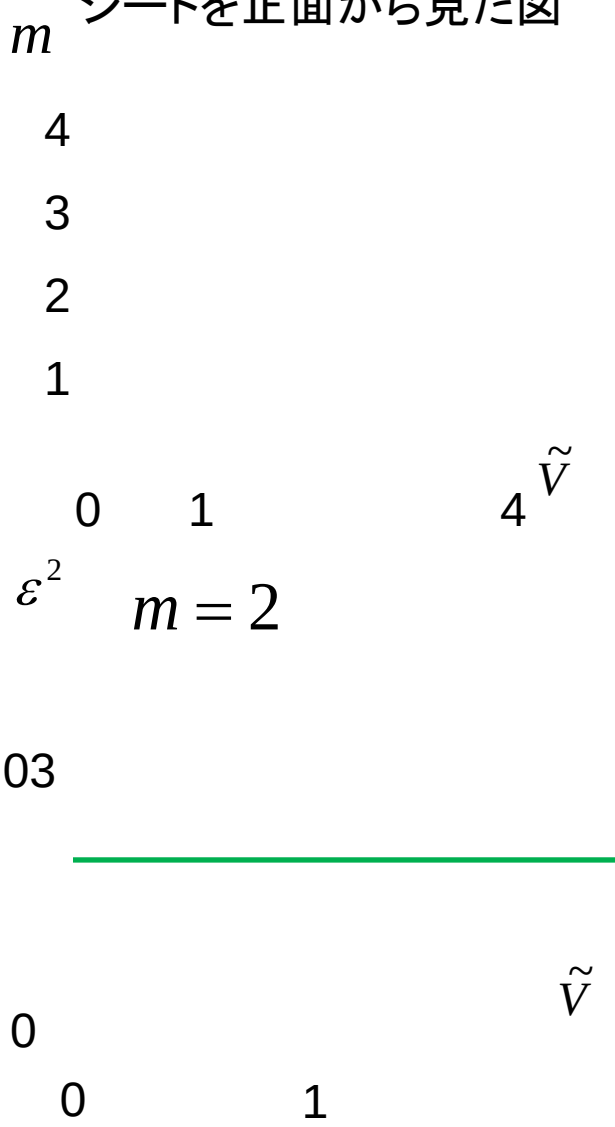


(SLP)の大域的分岐シート

Zoom up $(\tilde{V}, m) \in [0.95, 1.3] \times [1.95, 2.01]$



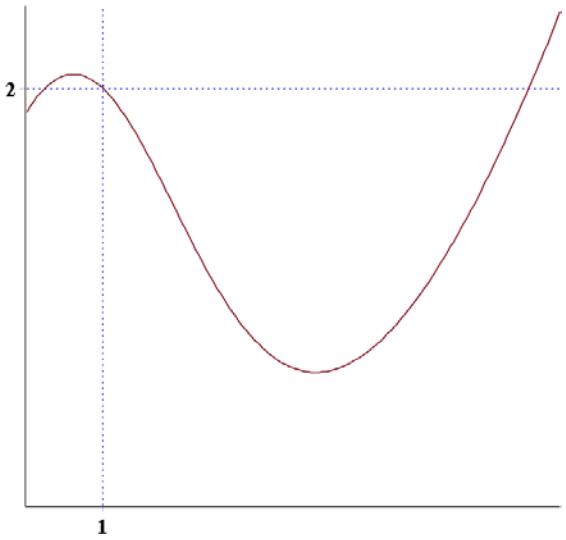
シートを正面から見た図



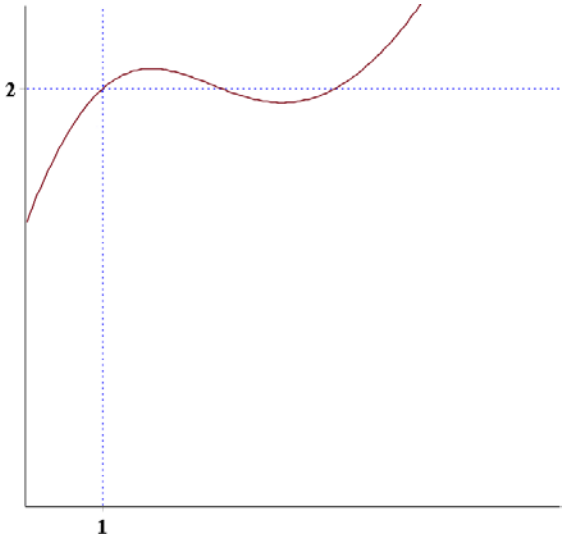
(SLP)の大域的分岐シート

Zoom up

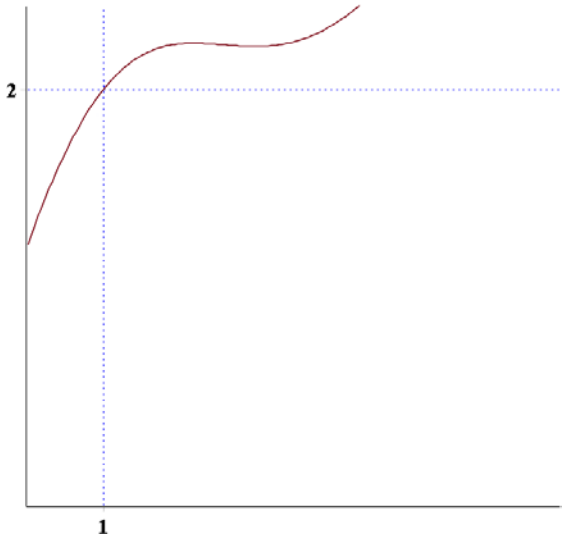
$\varepsilon^2 = 0.23^2$



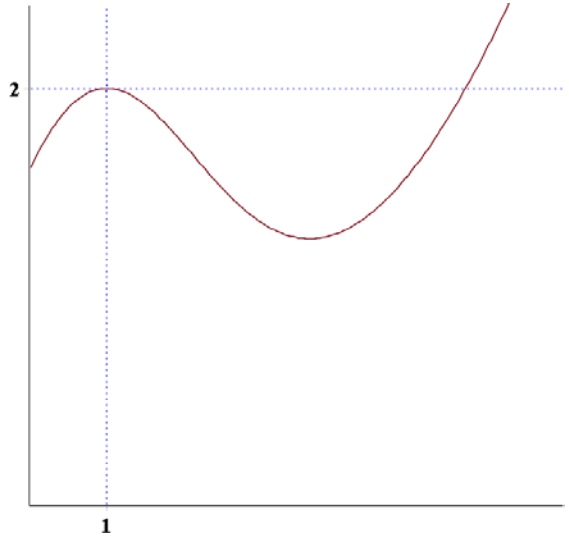
$\varepsilon^2 = 0.2412^2$



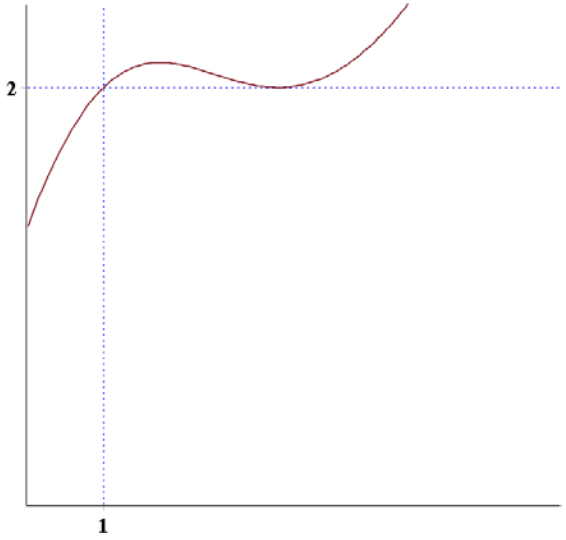
$\varepsilon^2 = 0.244^2$



$\varepsilon^2 = 0.2353^2$

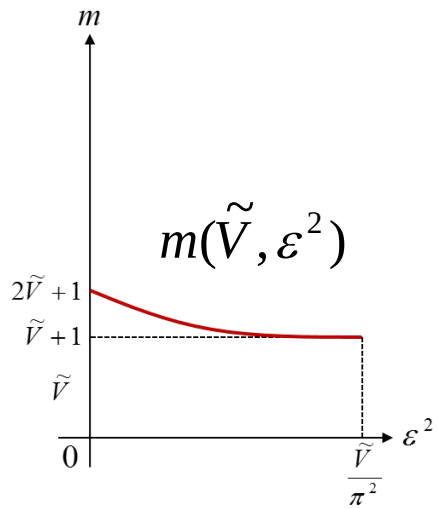


$\varepsilon^2 = 0.241863750^2$



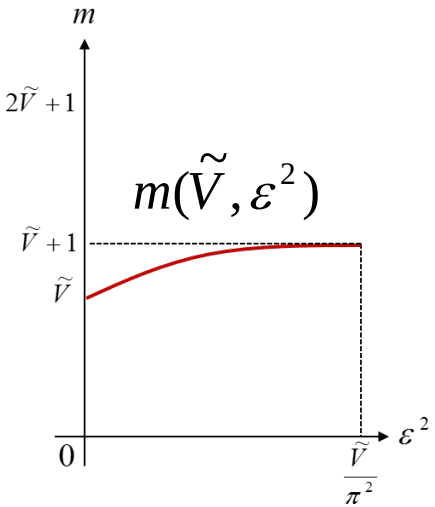
Theorem C *Let* $\tilde{V} > 0$ *be fixed. It holds that*

$$\frac{\partial m(\tilde{V}, \varepsilon^2)}{\partial \varepsilon} < 0 \quad \text{for} \quad \varepsilon^2 \in \left(0, \frac{\tilde{V}}{\pi^2}\right) \text{ with } \tilde{V} \in (0, 1),$$

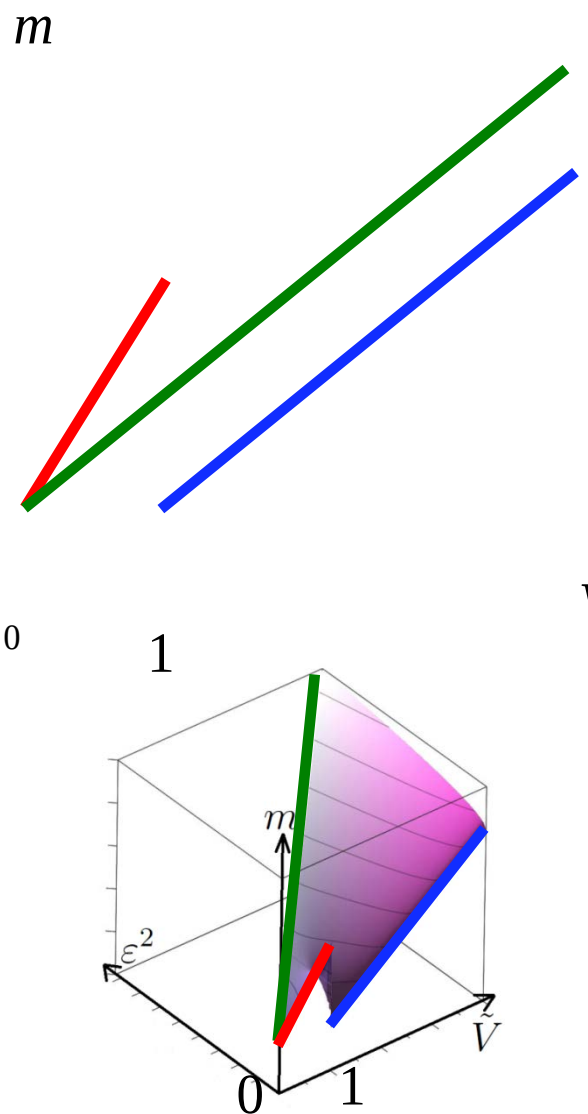


$\tilde{V} \in (0, 1)$ *be the fixed.*

$$\frac{\partial m(\tilde{V}, \varepsilon^2)}{\partial \varepsilon} > 0 \quad \text{for} \quad \varepsilon^2 \in \left(0, \frac{\tilde{V}}{\pi^2}\right) \text{ with } \tilde{V} \in (1, \infty).$$



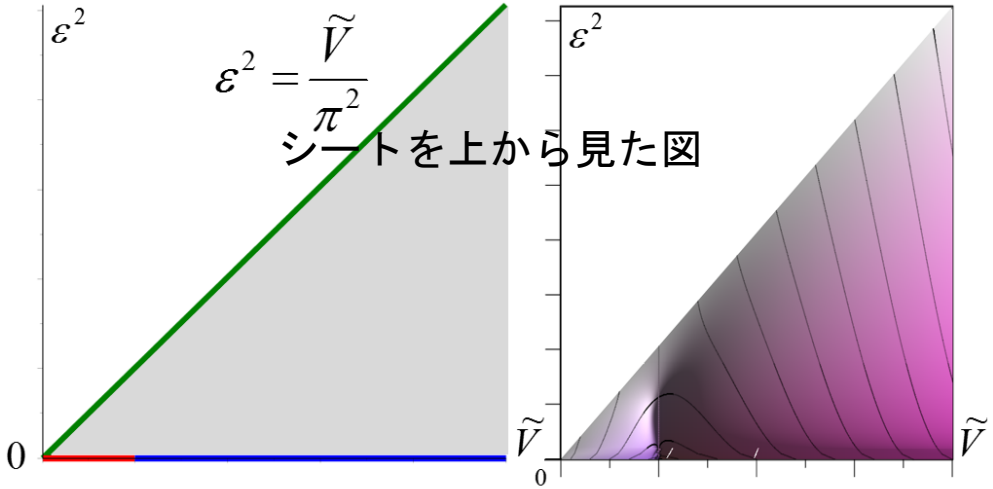
$\tilde{V} \in (1, \infty)$*be the fixed.*



Theorem C. *Let* $\tilde{V} > 0$ *be fixed. It holds that*

$\frac{\partial m(\tilde{V},\varepsilon^2)}{\partial \varepsilon} < 0$ *for* $0 < \tilde{V} < 1,\ 0 < \varepsilon^2 < \frac{\tilde{V}}{\pi^2}.$ $\quad$ $\frac{\partial m(\tilde{V},\varepsilon^2)}{\partial \varepsilon} > 0$ *for* $1 < \tilde{V},\ 0 < \varepsilon^2 < \frac{\tilde{V}}{\pi^2}.$

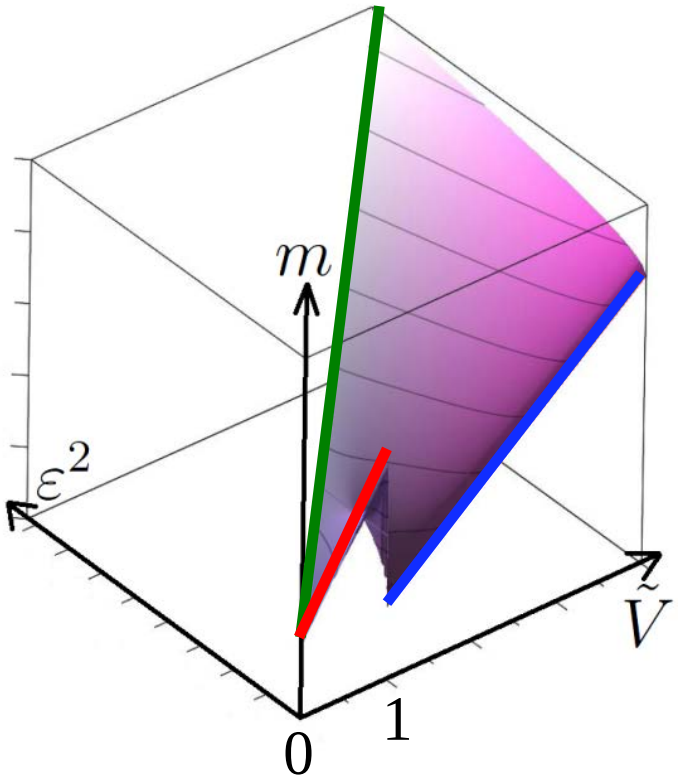
Theorem B. （ $m(\tilde{V},\varepsilon^2)$ の端点での極限値 ）.



シートを上から見た図

global bifurcation sheet

$\left\{\left(\tilde{V},\ \varepsilon^2,\ m(\tilde{V},\varepsilon^2)\right) : (\tilde{V},\ \varepsilon^2) \in G\right\}$

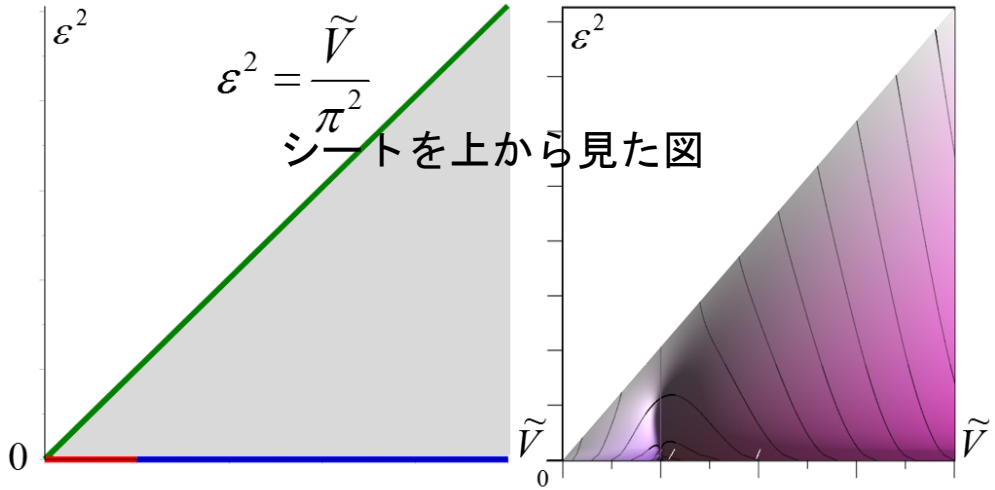


Theorem C. *Let* $\tilde{V} > 0$ *be fixed. It holds that*

$$\frac{\partial m(\tilde{V},\varepsilon^2)}{\partial \varepsilon} < 0 \text{ for } 0<\tilde{V}<1,\ 0<\varepsilon^2<\frac{\tilde{V}}{\pi^2}. \qquad \frac{\partial m(\tilde{V},\varepsilon^2)}{\partial \varepsilon} > 0 \text{ for } 1<\tilde{V},\ 0<\varepsilon^2<\frac{\tilde{V}}{\pi^2}.$$

これらより, 以下の Th. 1 - 5 に示すように, Exact multiplicityと

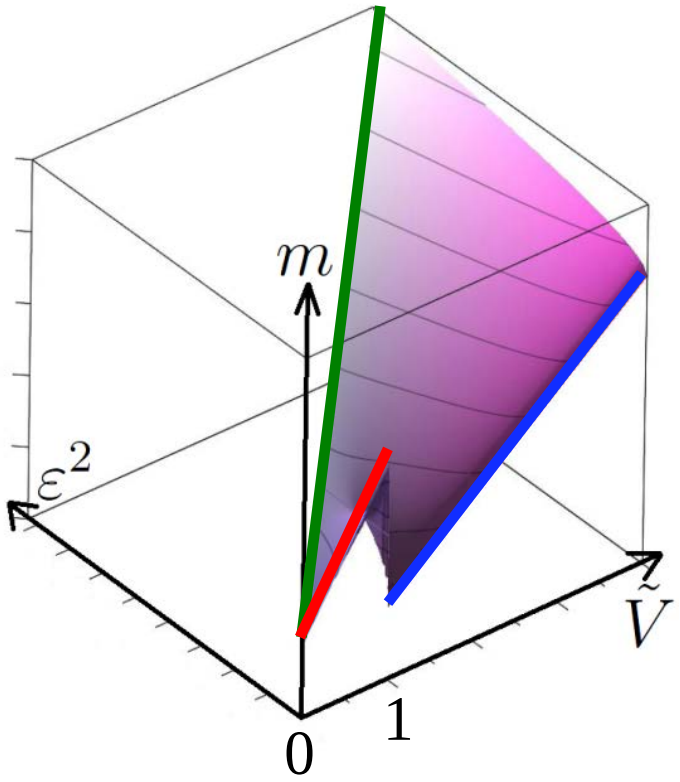
Theorem B. ($m(\tilde{V},\varepsilon^2)$ の端点での極限値) .大域的分岐曲線の存在・非存在・挙動がわかる.



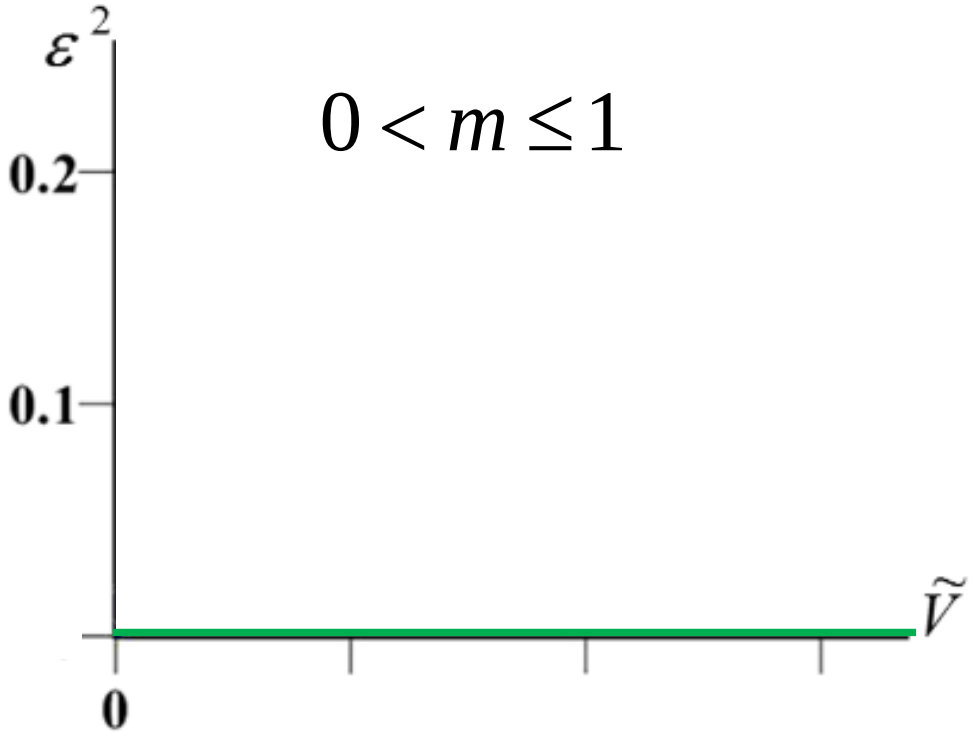
シートを上から見た図

global bifurcation sheet

$$\left\{\left(\tilde{V},\ \varepsilon^2,\ m(\tilde{V},\varepsilon^2)\right) : (\tilde{V},\ \varepsilon^2) \in G\right\}$$



Theorem 1. *Let $0 < m \leq 1$ be given. There exists no solution of (SLP).*



Theorem 2. *Let* $1 < m < 2$ *be given. The followings hold.*

(i) For $\tilde{V} \in \left(0, \frac{m-1}{2}\right] \cup [m-1, 1] \cup [m, \infty)$, *there exists no solution of (SLP).*

(ii) For $\tilde{V} \in \left(\frac{m-1}{2}, m-1\right)$, *there exists the unique* $\varepsilon^2(\tilde{V}) \in \left(0, \frac{\tilde{V}}{\pi^2}\right)$

such that $W(x; \tilde{V}, \varepsilon^2(\tilde{V}))$ *is a solution of (SLP).*

Moreover, $\varepsilon(\tilde{V})$ *is continuous in* $\left(\frac{m-1}{2}, m-1\right)$,

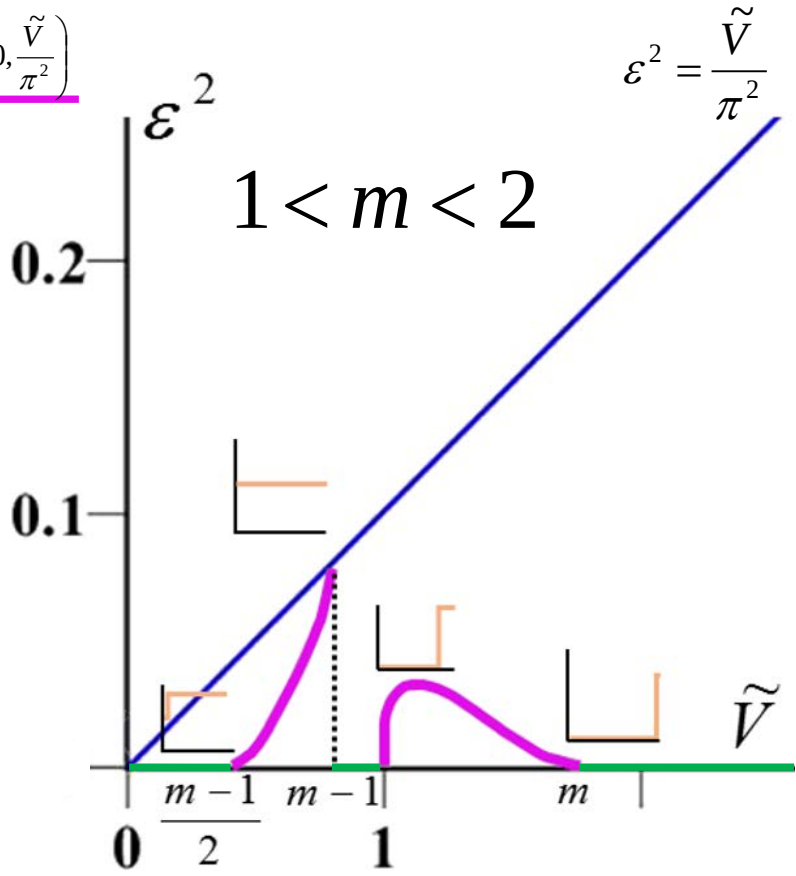
$$\varepsilon^2(\tilde{V}) \to 0 \text{ as } \tilde{V} \downarrow \frac{m-1}{2}, \quad \varepsilon^2(\tilde{V}) \to \frac{m-1}{\pi^2} \text{ as } \tilde{V} \uparrow m-1.$$

(iii) For $\tilde{V} \in (1, m)$, *there exists the unique* $\varepsilon^2(\tilde{V}) \in \left(0, \frac{\tilde{V}}{\pi^2}\right)$

such that $W(x; \tilde{V}, \varepsilon^2(\tilde{V}))$ *is a solution of (SLP).*

Moreover, $\varepsilon(\tilde{V})$ *is continuous in* $(1, m)$,

$$\varepsilon^2(\tilde{V}) \to 0 \text{ as } \tilde{V} \downarrow 1, \quad \varepsilon^2(\tilde{V}) \to 0 \text{ as } \tilde{V} \uparrow m.$$



Theorem 3. *Let* $m = 2$ *be given. The followings hold.*

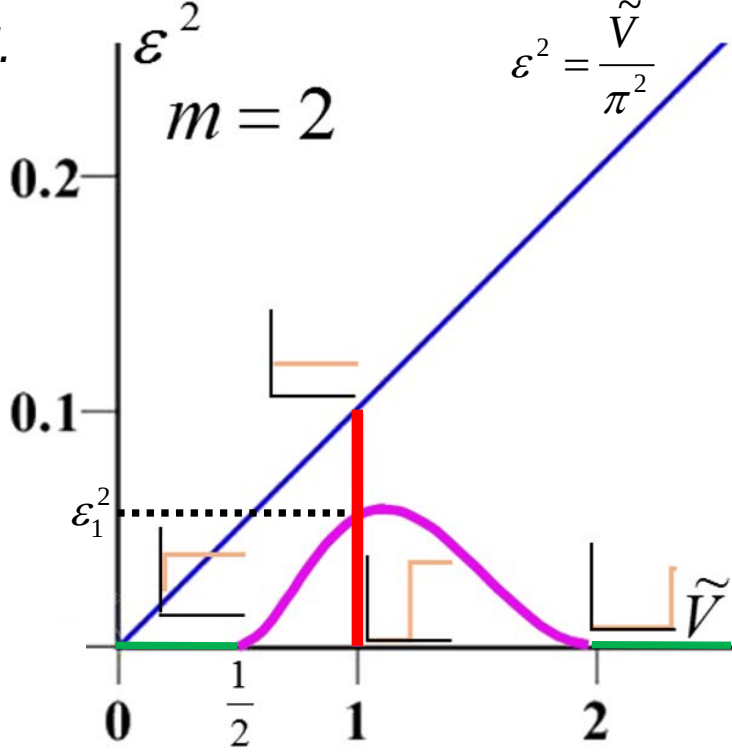
(i) For $\tilde{V} \in \left(0, \frac{1}{2}\right] \cup [2, \infty)$, *there exists no solution of (SLP).*

(ii) For $\tilde{V} \in \left(\frac{1}{2}, 1\right)$, *there exists the unique* $\varepsilon^2(\tilde{V}) \in \left(0, \frac{\tilde{V}}{\pi^2}\right)$

such that $W(x; \tilde{V}, \varepsilon^2(\tilde{V}))$ *is a solution of (SLP).*

Moreover, $\varepsilon(\tilde{V})$ *is continuous in* $\left(\frac{1}{2}, 1\right)$,

$\varepsilon^2(\tilde{V}) \to 0$ as $\tilde{V} \downarrow \frac{1}{2}$, $\varepsilon^2(\tilde{V}) \to \varepsilon_1^2 = 0.05536\cdots$ as $\tilde{V} \uparrow 1$.



(iii For $\tilde{V} = 1$, *there exist infinity many solutions.*
)

More precisely, all solutions are given by $\left\{W(x; 1, \varepsilon^2) : \varepsilon^2 \in \left(0, \frac{1}{\pi^2}\right)\right\}$.

(iv) For $\tilde{V} \in (1, 2)$ *there exists the unique* $\varepsilon^2(\tilde{V}) \in \left(0, \frac{\tilde{V}}{\pi^2}\right)$

such that $W(x; \tilde{V}, \varepsilon^2(\tilde{V}))$ *is a solution of (SLP).*

Moreover, $\varepsilon(\tilde{V})$ *is continuous in* $(1, 2)$,

$\varepsilon^2(\tilde{V}) \to \varepsilon_1^2$ as $\tilde{V} \downarrow 1$, $\varepsilon^2(\tilde{V}) \to 0$ as $\tilde{V} \uparrow 2$.

Theorem 4. *Let $2 < m < 3$ be given. The followings hold.*

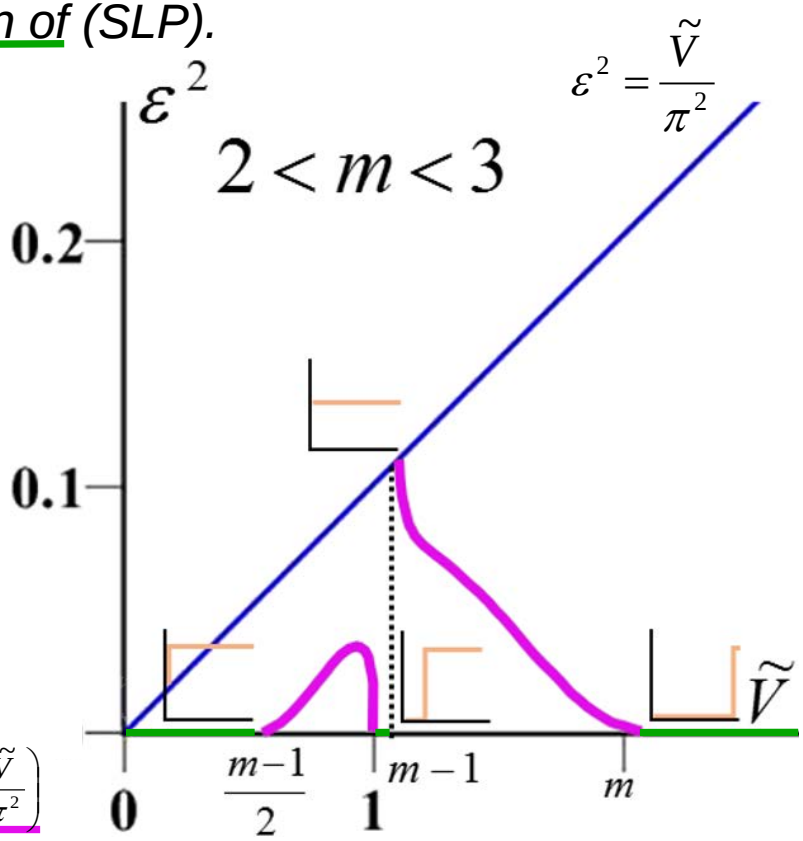
(i) For $\tilde{V} \in \left(0, \frac{m-1}{2}\right] \cup [1, m-1] \cup [m, \infty)$, there exists no solution of (SLP).

(ii) For $\tilde{V} \in \left(\frac{m-1}{2}, 1\right)$, there exists the unique $\varepsilon^2(\tilde{V}) \in \left(0, \frac{\tilde{V}}{\pi^2}\right)$

such that $W(x; \tilde{V}, \varepsilon^2(\tilde{V}))$ is a solution of (SLP).

Moreover, $\varepsilon(\tilde{V})$ is continuous in $\left(\frac{m-1}{2}, 1\right)$,

$$\varepsilon^2(\tilde{V}) \to 0 \text{ as } \tilde{V} \downarrow \frac{m-1}{2}, \quad \varepsilon^2(\tilde{V}) \to 0 \text{ as } \tilde{V} \uparrow 1.$$



(iii) For $\tilde{V} \in (m-1, 1)$, there exists the unique $\varepsilon^2(\tilde{V}) \in \left(0, \frac{\tilde{V}}{\pi^2}\right)$

such that $W(x; \tilde{V}, \varepsilon^2(\tilde{V}))$ is a solution of (SLP).

Moreover, $\varepsilon(\tilde{V})$ is continuous in $(m-1, m)$,

$$\varepsilon^2(\tilde{V}) \to \frac{m-1}{\pi^2} \text{ as } \tilde{V} \downarrow m-1, \quad \varepsilon^2(\tilde{V}) \to 0 \text{ as } \tilde{V} \uparrow m.$$

Theorem 5. *Let* $m \geq 3$ *be given. The followings hold.*

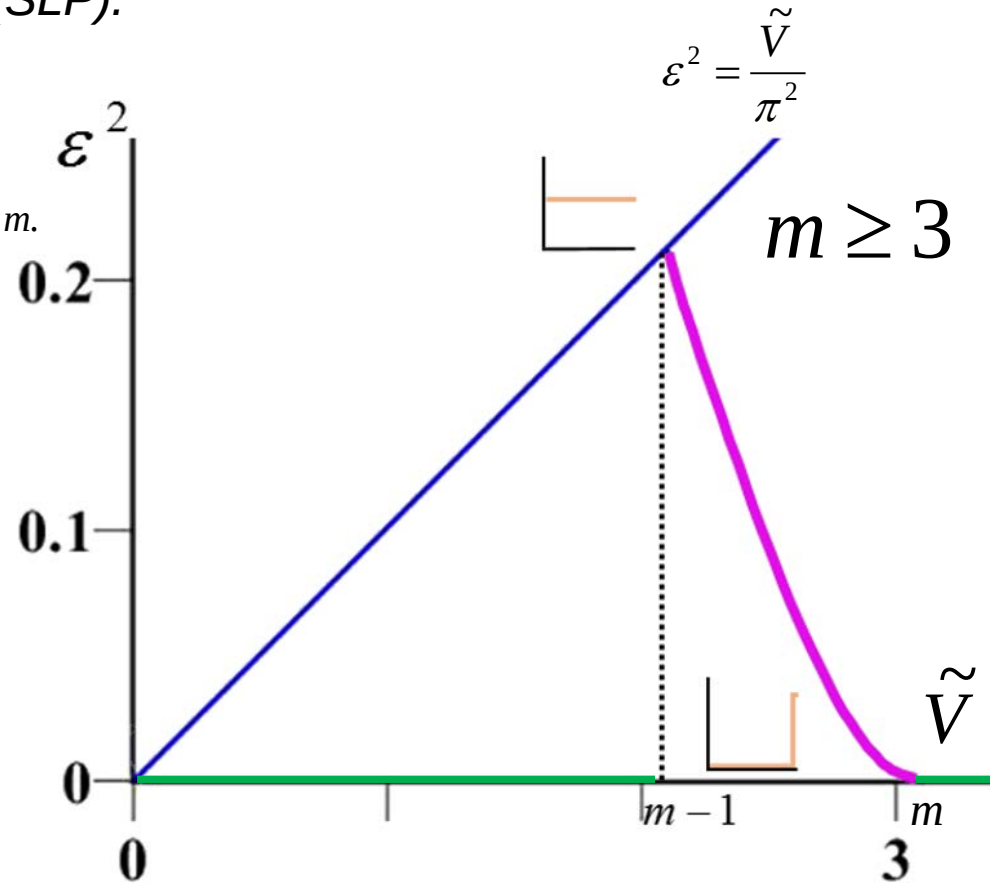
(i) For $\tilde{V} \in (0,\ m-1] \cup [m,\ \infty)$, *there exists no solution of (SLP).*

(ii) For $\tilde{V} \in (m-1,\ m)$, *there exists the unique* $\varepsilon^2(\tilde{V}) \in \left(0, \frac{\tilde{V}}{\pi^2}\right)$

such that $W(x; \tilde{V}, \varepsilon^2(\tilde{V}))$ *is a solution of (SLP).*

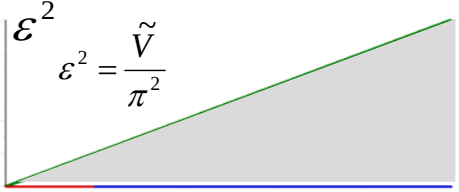
Moreover, $\varepsilon(\tilde{V})$ *is continuous in* $(m-1,\ m)$,

$\varepsilon^2(\tilde{V}) \to \frac{m-1}{\pi^2}$ as $\tilde{V} \downarrow m-1$, $\varepsilon^2(\tilde{V}) \to 0$ as $\tilde{V} \uparrow m$.



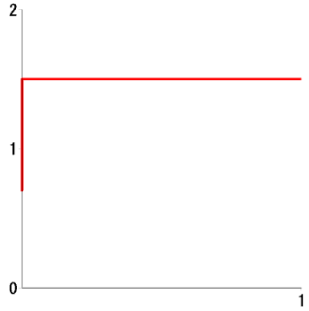
Prop.

$$\lim_{\varepsilon^2 \uparrow \frac{\tilde{V}}{\pi^2}} W(x;\tilde{V},\varepsilon^2) = 1,$$



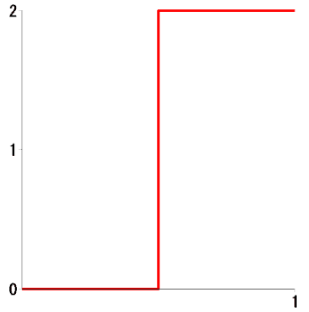
$$\begin{cases} \varepsilon^2 W_{xx} + W(W-1)(\tilde{V}+1-W) = 0, & x \in (0,1), \\ W_x(0) = W_x(1) = 0, \\ W(x) > 0,\ W'(x) > 0, & x \in (0,1) \end{cases}$$

$$\lim_{\varepsilon^2 \downarrow 0} W(x;\tilde{V},\varepsilon^2) = \begin{cases} \dfrac{\tilde{V}+2}{3} - \dfrac{(3s_0-1)\sqrt{\tilde{V}(\tilde{V}+1)}}{\sqrt{8s_0}} & (x=0), \\ V+1 & (0<x\le 1). \end{cases} \quad \text{for} \quad 0<\tilde{V}<1.$$

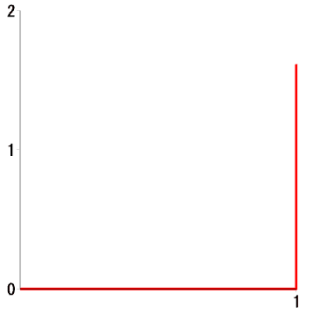


with $s_0 := \dfrac{1}{9} \cdot \dfrac{7\tilde{V}^2 + 7\tilde{V} + 4 - 2\sqrt{2}(2\tilde{V}+1)\sqrt{(\tilde{V}+2)(1-\tilde{V})}}{V(\tilde{V}+1)}.$

$$W\left(\frac{1}{2};1,\varepsilon^2\right) = 1, \qquad \lim_{\varepsilon^2 \downarrow 0} W(x;1,\varepsilon^2) = \begin{cases} 0 & \left(0 \le x < \dfrac{1}{2}\right), \\ 2 & \left(\dfrac{1}{2} < x < 1\right). \end{cases}$$



$$\lim_{\varepsilon^2 \downarrow 0} W(x;\tilde{V},\varepsilon^2) = \begin{cases} 0 & (0 \le x < 1), \\ \dfrac{\tilde{V}+2}{3} + \dfrac{\sqrt{2}}{4} \cdot \dfrac{(3s_1-1)\sqrt{\tilde{V}+1}}{\sqrt{s_1}} & (x=1). \end{cases} \quad \text{for} \quad 1<\tilde{V}.$$



with $s_1 := \dfrac{1}{9} \cdot \dfrac{4\tilde{V}^2 + 7\tilde{V} + 7 - 2\sqrt{2}(\tilde{V}+2)\sqrt{(2\tilde{V}+1)(\tilde{V}-1)}}{\tilde{V}+1}.$

定理のより詳細な説明と証明のアイデア

Now, let us introduce an auxiliary problem to investigate (SLP). Let $\tilde{V} > 0$ be given, let us consider the problem

$$(AP;\tilde{V})\begin{cases} \varepsilon^2 W_{xx} + W(W-1)(\tilde{V}+1-W) = 0 \quad \text{in } (0,1), & (2.1) \\ W_x(0) = W_x(1) = 0, & (2.2) \\ W_x(x) > 0 \text{ in } (0,1). & (2.3) \end{cases}$$

We note that (AP;$\tilde{V}$) is equivalent to

$$\begin{cases} \varepsilon^2 W_{xx} + W(W-1)(\tilde{V}+1-W) = 0 \quad \text{in } (0,1), \\ W_x(0) = W_x(1) = 0, \\ 0 < W(0) < \tilde{V}+1, \quad W_x(x) > 0 \text{ in } (0,1) \end{cases}$$

for given $\tilde{V} > 0$, since it is easy to see that a condition $0 < W(0) < \tilde{V}+1$ holds for any solution of (AP;$\tilde{V}$).

The existence and the uniqueness of the solution $W(x)$ of $(\mathrm{AP};\tilde{V})$ is well-known (see, e.g. Smoller-Wasserman [5] and Smoller [6]). However, we need to know more precise information to investigate (SLP). The following theorem gives the representation formula for all solutions of $(\mathrm{AP};\tilde{V})$.

Theorem 2.1. *Let $\tilde{V} > 0$. There exists a solution of $(AP;\tilde{V})$, if and only if $(\tilde{V},\ \varepsilon^2) \in \mathcal{G}$, where*

$$\mathcal{G} := \left\{ (\tilde{V},\ \varepsilon^2) :\ 0 < \varepsilon^2 < \frac{\tilde{V}}{\pi^2} \right\}. \tag{2.4}$$

Moreover, the solution is unique. The solution $W(x;\tilde{V},\varepsilon^2)$ has properties

$$0 < W(x;\tilde{V},\varepsilon^2) < \tilde{V} + 1, \tag{2.5}$$

$$W(x;\tilde{V},\varepsilon^2) = \tilde{V} + 1 - \tilde{V} \cdot W\left(1 - x; \frac{1}{\tilde{V}}, \frac{\varepsilon^2}{\tilde{V}^2}\right). \tag{2.6}$$

The solution $W(x;\tilde{V},\varepsilon^2)$ *is represented by*

$$W(x,\tilde{V},\varepsilon^2)=\frac{\tilde{V}+2}{3}$$
$$+\frac{1}{\sqrt{3}}\sqrt{\tilde{V}^2+\tilde{V}+1}\cdot\frac{\beta\cdot(1-hs)\mathrm{sn}^2(K(\sqrt{h})x,\sqrt{h})+\alpha\cdot\mathrm{cn}^2(K(\sqrt{h})x,\sqrt{h})}{(1-hs)\mathrm{sn}^2(K(\sqrt{h})x,\sqrt{h})+\mathrm{cn}^2(K(\sqrt{h})x,\sqrt{h})},\tag{2.7}$$

$$\alpha:=\alpha(h,s)=\frac{3hs^2-2(1+h)s+1}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}},\tag{2.8}$$

$$\beta:=\beta(h,s)=\frac{-hs^2-2(1-h)s+1}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}},\tag{2.9}$$

where $(h,s)=\Big(h(\tilde{V},\varepsilon^2),s(\tilde{V},\varepsilon^2)\Big)$ *is the unique solution of the following system of transcendental equations*

$$(\mathrm{E})\begin{cases}\mathcal{E}(h,s)=\sqrt{3}\cdot\dfrac{\varepsilon}{\sqrt{\tilde{V}^2+\tilde{V}+1}}, & (2.10)\\[2ex] \mathcal{A}(h,s)=\dfrac{1}{3\sqrt{3}}\cdot\dfrac{(1-\tilde{V})(2\tilde{V}+1)(\tilde{V}+2)}{\sqrt{\tilde{V}^2+\tilde{V}+1}^{\,3}}, & (2.11)\\[2ex] 0<h<1,\quad 0<s<1, & (2.12)\end{cases}$$

where

$$\mathcal{E}(h,s):=\frac{\sqrt{2s(1-s)(1-sh)}/K(\sqrt{h})}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(1+h)s+3}},\tag{2.13}$$

$$\mathcal{A}(h,s):=\frac{2(hs^2-2sh+1)(hs^2-2s+1)(1-hs^2)}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}^{\,3}}.\tag{2.14}$$

We show the graph of $\mathcal{A}(h,s)$ and $\mathcal{E}(h,s)$ in Figures 1 and 2.

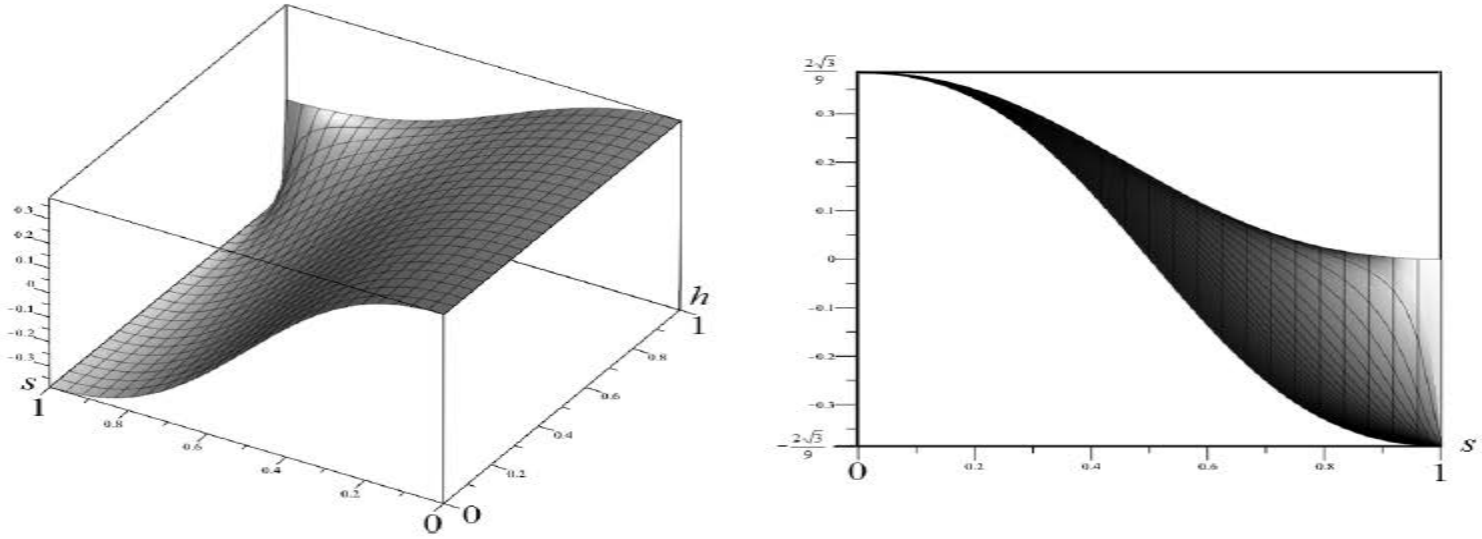


FIGURE 1. Graph of $\mathcal{A}(h,s)$

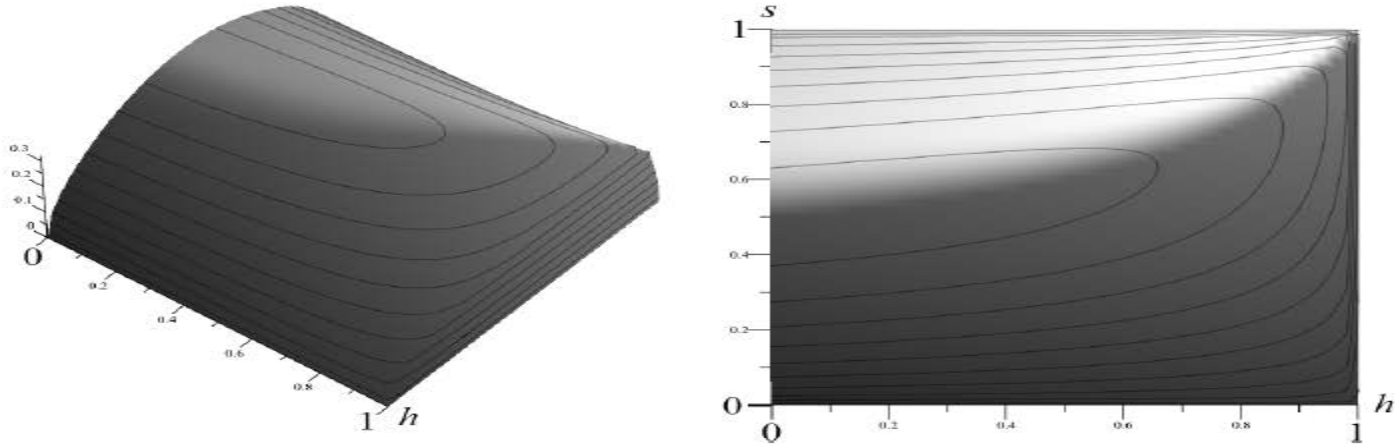


FIGURE 2. Graph of $\mathcal{E}(h,s)$

Theorem 2.2. *Let $W(x;\tilde{V},\varepsilon^2)$ be the unique solution of (AP;$\tilde{V}$), and*

$$m(\tilde{V},\ \varepsilon^2) := \int_0^1 W(x;\tilde{V},\varepsilon^2)dx + \tilde{V}, \tag{2.15}$$

then

$$m(\tilde{V},\varepsilon^2) = 2\tilde{V} + 2 - \tilde{V}m\left(\frac{1}{\tilde{V}}, \frac{\varepsilon^2}{\tilde{V}^2}\right) \quad \text{for any} \quad \tilde{V} > 0, \quad \varepsilon > 0. \tag{2.16}$$

In particular,

$$m(1,\varepsilon^2) = 2 \quad \text{for any} \quad \varepsilon > 0. \tag{2.17}$$

Moreover, it holds that

$$m(\tilde{V},\ \varepsilon^2) = \frac{4\tilde{V}+2}{3} + \frac{1}{\sqrt{3}} \cdot \sqrt{\tilde{V}^2 + \tilde{V} + 1} \cdot \mathcal{M}(h,s), \tag{2.18}$$

$$\mathcal{M}(h,s) := \frac{-(hs^2 - 2(1+h)s + 3) + 4(1-s)(1-sh)\Pi(-sh,\sqrt{h})/K(\sqrt{h})}{\sqrt{3h^2s^4 - 4(h^2+h)s^3 + (4h^2+2h+4)s^2 - 4(1+h)s + 3}}, \tag{2.19}$$

where $h = h(\tilde{V},\varepsilon^2)$, $s = s(\tilde{V},\varepsilon^2)$ are given in Theorem2.1. Here, $K(\cdot)$ is the complete elliptic integral of the 1st kind, and $\Pi(\cdot,\cdot)$ is the complete elliptic integral of the third kind.

Let us define the global bifurcation sheet S by

$$S := \left\{ \left(\tilde{V}, \varepsilon^2, \int_0^1 W dx + \tilde{V} \right) : (\tilde{V},\ \varepsilon^2) \in \mathcal{G} \right\}.$$

We obtain exact representation the global bifurcation sheet S as

$$S = \left\{ \left(\tilde{V}, \varepsilon^2, m(\tilde{V},\ \varepsilon^2) \right) : (\tilde{V},\ \varepsilon^2) \in \mathcal{G} \right\} \tag{2.20}$$

by Theorem 2.2 . For each m, we can obtain the bifurcation diagram by

$$\left\{ (\tilde{V},\ \varepsilon^2) \in \mathcal{G} : m(\tilde{V},\ \varepsilon^2) = m \right\} \tag{2.21}$$

directly from the global bifurcation sheet S.

We will mathematically investigate precise properties of the global bifurcation sheet and bifurcation diagrams in a forth-coming paper. For instance, we see the following facts:

- For each fixed $\tilde{V} \in (0, \infty)$, $W(x; \tilde{V}, \varepsilon^2) \to 1$ as $\varepsilon^2 \to \tilde{V}/\pi^2$ uniformly on $[0, 1]$.
- For each fixed $\tilde{V} \in (0, \infty)$, $m(\tilde{V}, \varepsilon^2) \to \tilde{V} + 1$ as $\varepsilon^2 \to \tilde{V}/\pi^2$.

- For each fixed $\tilde{V} \in (0, 1)$, $W(x; \tilde{V}, \varepsilon^2) \to \tilde{V} + 1$ as $\varepsilon^2 \to 0$ in $(0, 1]$.
- For each fixed $\tilde{V} \in (1, \infty)$, $W(x; \tilde{V}, \varepsilon^2) \to 0$ as $\varepsilon^2 \to 0$ in $[0, 1)$.

- For each fixed $\tilde{V} \in (0, 1)$, $m(\tilde{V}, \varepsilon^2) \to 2\tilde{V} + 1$ as $\varepsilon^2 \to 0$.
- For each fixed $\tilde{V} \in (1, \infty)$, $m(\tilde{V}, \varepsilon^2) \to \tilde{V}$ as $\varepsilon^2 \to 0$.

- For $m \in (0, 1]$, bifurcation diagrams are the empty set.
- For $m \in (1, \infty)$, bifurcation diagram given by (2.21) are graphs with $\tilde{V}$ axis (smooth single-valued function in $\tilde{V}$) except $m = 2$ with $\tilde{V} = 1$.

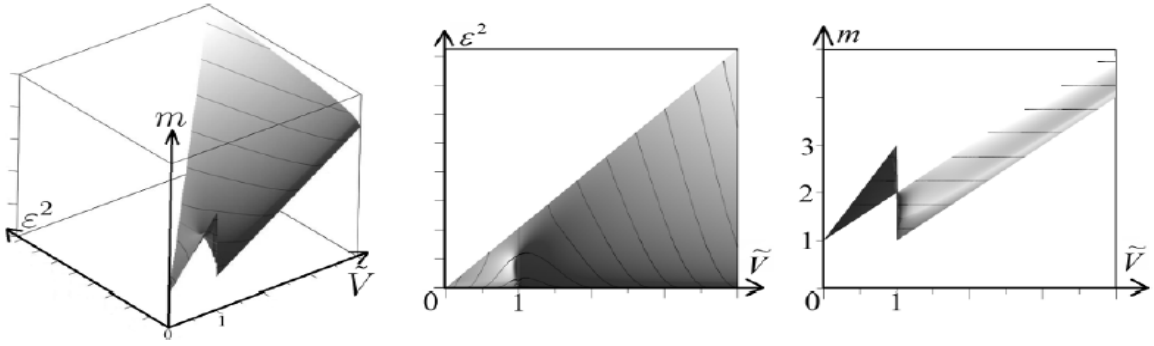
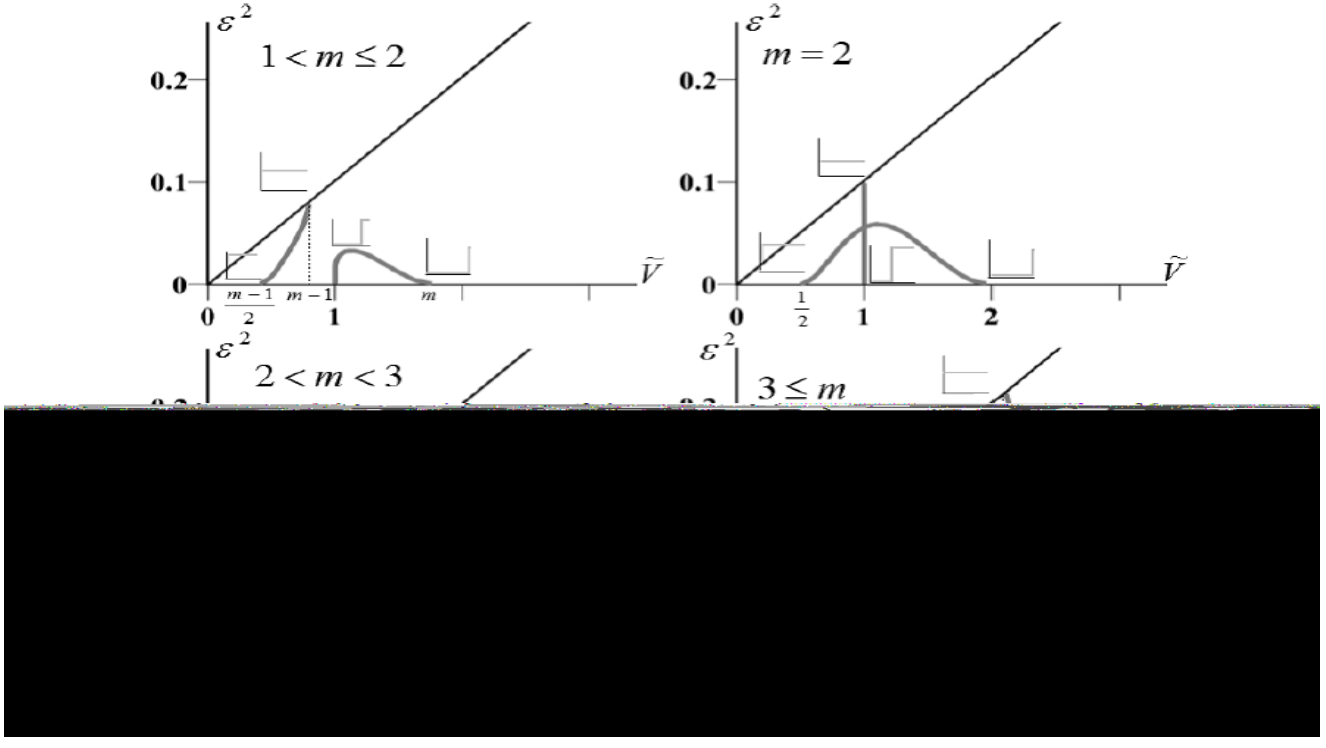


FIGURE 3. Global bifurcation sheet

The Figure 4 show the bifurcation diagrams for various m with the profiles of solutions of (SLP).



Proposition 3.1. *Let $W(x)$ be a solution of $(AP;\tilde{V})$, and*

$$u(x) := \frac{\sqrt{3}}{\sqrt{\lambda^2-\lambda+1}}\left(\lambda\, W(x) - \left(\frac{1}{3}+\frac{\lambda}{3}\right)\right), \tag{3.1}$$

where

$$\lambda := \frac{1}{\tilde{V}+1}. \tag{3.2}$$

Then $u(x)$ satisfies

$$\begin{cases} \left(\dfrac{\sqrt{3}\varepsilon}{\sqrt{\tilde{V}^2+\tilde{V}+1}}\right)^2 u_{xx} - u^3 + u \\ \qquad + \dfrac{1}{3\sqrt{3}} \cdot \dfrac{(1-\tilde{V})(2\tilde{V}+1)(\tilde{V}+2)}{\sqrt{\tilde{V}^2+\tilde{V}+1}^3} \quad \text{in } (0,1), & (3.3) \\ u_x(0) = u_x(1) = 0, & (3.4) \\ u_x(x) > 0 \quad \text{in } (0,1), & (3.5) \end{cases}$$

Proposition 3.2. *Let $\tilde{V} > 0$. There exists a solution $W(x)$ of $(AP;\tilde{V})$, if and only if (E) has a solution (h, s). For the solution (h, s) of (E), $(AP;\tilde{V})$ has a solution in the form (2.7) with (2.8) and (2.9).*

Proposition 3.3. *Let $\tilde{V} > 0$. There exists a solution $(h, s) = (h(\tilde{V}, \varepsilon^2), s(\tilde{V}, \varepsilon^2))$ of (E), if and only if $(\tilde{V}, \varepsilon^2) \in \mathcal{G}$, where $\mathcal{G}$ is defined by (2.4). Moreover, the solution is unique.*

Proposition 3.4. *Let $\tilde{V} > 0$, $\varepsilon > 0$, $(h, s) = (h(\tilde{V}, \varepsilon^2), s(\tilde{V}, \varepsilon^2))$ be the unique solution of (E), $W(x; \tilde{V}, \varepsilon^2)$ be the unique solution of (E) in the form (2.7) with (2.8) and (2.9), and $u(x)$ be defined by (3.1) and (3.2) with $W(x) = W(x; \tilde{V}, \varepsilon^2)$. Then*

$$\int_0^1 u(x)dx = \mathcal{M}(h(\tilde{V}, \varepsilon^2), s(\tilde{V}, \varepsilon^2)) \tag{3.7}$$

where $\mathcal{M}(h, s)$ is defined by (2.19).

Proof of Theorem 2.1. We see from Proposition 3.2 and Proposition 3.3 that conclusions hold except (2.6).

We see that

$$\tilde{V} + 1 - \tilde{V} \cdot W\left(1 - x; \frac{1}{\tilde{V}}, \frac{\varepsilon^2}{\tilde{V}^2}\right).$$

is a solution of (AP; $\tilde{V}$). Thus, we obtain (2.6) by the uniqueness of solutions of $(AP; \tilde{V})$.

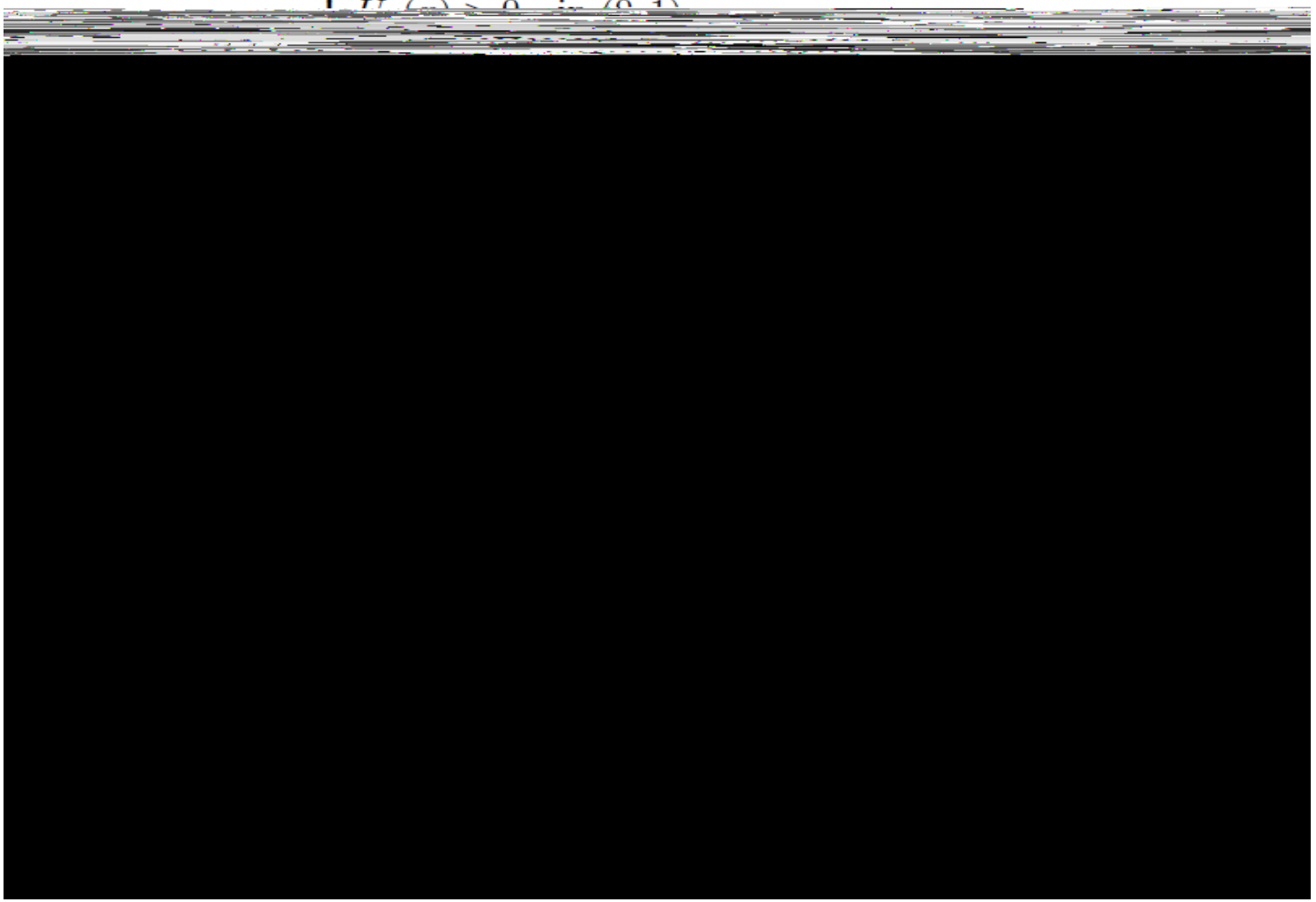
Proof of Theorem 2.2. We obtain conclusions by (2.16), Proposition 3.1, and Proposition 3.4.

Proof of Proposition 3.1. Let us put

$$U(x) := \frac{W(x)}{\tilde{V}+1}.$$

We get

$$\begin{cases} (\varepsilon\lambda)^2 U_{xx} + U(1-U)(U-\lambda) = 0 \quad \text{in } (0,1), \\ U_x(0) = U_x(1) = 0, \\ U(x) > 0 \quad \text{in } (0,1) \end{cases}$$



5. **Proof of Proposition 3.2 and 3.4.** In this section we give a proof of Proposition 3.2 and 3.4. We employ representation formulas obtained in Proposition 1.1 and its proof in Kosugi-Morita-Yotsutani[3].

Lemma 5.1. *Let $E > 0$ and A be constants. Then all the solution of*

$$\begin{cases} E^2 u_{xx} - u^3 + u - A = 0 \quad \text{in } (0,1), \\ u_x(0) = u_x(1) = 0, \\ u_x(x) > 0 \quad \text{in } (0,1). \end{cases}$$

are represented by two parameters (h,s) with $0 < h < 1$ and $0 < s < 1$ as follows.

$$u(x;h,s) = \frac{\beta \cdot (1-hs)\mathrm{sn}^2(K(\sqrt{h})x, \sqrt{h}) + \alpha \cdot \mathrm{cn}^2(K(\sqrt{h})x, \sqrt{h})}{(1-hs)\mathrm{sn}^2(K(\sqrt{h})x, \sqrt{h}) + \mathrm{cn}^2(K(\sqrt{h})x, \sqrt{h})}, \tag{5.1}$$

$$\alpha := \alpha(h,s), \quad \beta := \beta(h,s), \tag{5.2}$$

where $\alpha(h,s)$ and $\beta(h,s)$ are defined by (2.8) and (2.9), and (h,s) is a solution of the following system of transcendental equations

$$(\mathrm{E}) \begin{cases} \mathcal{E}(h,s) = E & (5.3) \\ \mathcal{A}(h,s) = A & (5.4) \\ 0 < h < 1, \quad 0 < s < 1, & (5.5) \end{cases}$$

where $\mathcal{E}(h,s)$ and $\mathcal{A}(h,s)$ are defined by (2.13) and (2.14) respectively.

Moreover,

$$\int_0^1 u(x)dx = \mathcal{M}(h,s),$$

where $\mathcal{M}(h,s)$ is defined (2.19).

Proof of Proposition 3.2. We obtain conclusions by Proposition3.1 and Lemma5.1.

Proof of Proposition 3.4. We obtain conclusions by Proposition3.2 and Lemma5.1.

6. **Proof of Proposition 3.3.** In this section we give a proof of Proposition 3.3. First we note that

$$3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3$$
$$= s^2(3\,s^2-4\,s+4)h^2-2\,s(2\,s^2-s+2)h+4\,s^2-4\,s+3>0$$

and $\mathcal{A}(h,s)$ and $\mathcal{E}(h,s)$ are well-defined in $(h,s)\in(0,1)\times(0,1)$, since

$$s^2\left(2\,s^2-\,s+2\right)^2-\,s^2\left(3\,s^4-4\,s^3+4\,s^2\right)\left(4\,s^2-4\,s+3\right)$$
$$=-8\,s^2\left(s^2-s+1\right)(s-1)^2<0.$$

there. We prepare several lemmas.

We see from Lemma 3.2 and the proof of Lemma 3.4 of [3] that the following lemma holds.

We show the graph of $\mathcal{A}(h,s)$ and $\mathcal{E}(h,s)$ in Figures 1 and 2.

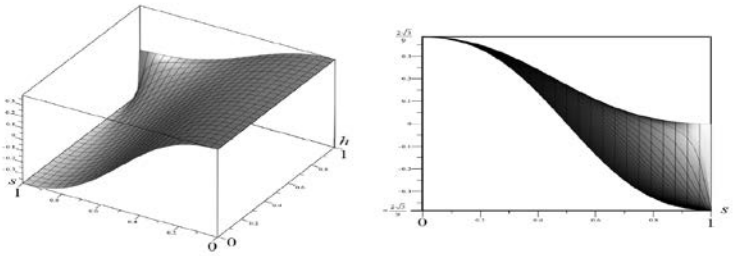


FIGURE 1. Graph of $\mathcal{A}(h,s)$

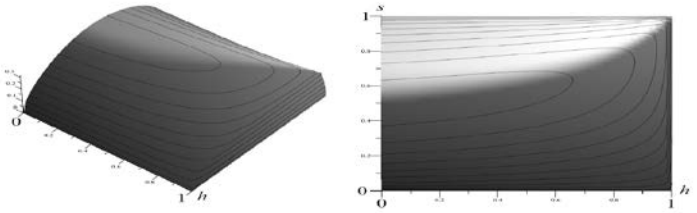


FIGURE 2. Graph of $\mathcal{E}(h,s)$

Lemma 6.1. *Let $\mathcal{E}(h,s)$ be defined by (2.13). The derivative of $\mathcal{E}(h,s)$ with respect to s satisfies*

$$\frac{\partial}{\partial s}\mathcal{E}(h,s)\begin{cases} >0, & s\in(0,\sigma(h)), \quad h\in[0,1),\\ =0, & s=\sigma(h), \quad h\in[0,1),\\ <0, & s\in(\sigma(h),1), \quad h\in[0,1), \end{cases} \tag{6.1}$$

and

$$\mathcal{E}(h,s)<\mathcal{E}(h,\sigma(h)) \quad \text{for all } (h,s)\in[0,1)\times[0,1]\setminus\{(h,s): s=\sigma(h)\}, \tag{6.2}$$

where $\sigma(h):=1/(1+\sqrt{1-h})$.

Moreover,

$$\mathcal{E}(h,\sigma(h))=\frac{1}{\sqrt{2(2-h)}\,K(\sqrt{h})}, \tag{6.3}$$

$$\frac{d}{dh}\mathcal{E}(h,\sigma(h))<0 \quad \text{for } h\in[0,1), \tag{6.4}$$

and

$$\mathcal{E}(0,\sigma(0))=\frac{1}{\pi}, \quad \mathcal{E}(h,\sigma(h))\to 0 \text{ as } h\ \to 1. \tag{6.5}$$

In addition,

$$\mathcal{E}(0,s)=\frac{2\sqrt{2\,s\,(1-s)}}{\pi\sqrt{4\,s^2-4\,s+3}}. \tag{6.6}$$

Lemma 6.2. *Let*

$$r(v) := \frac{\sqrt{3}}{9} \cdot \frac{(1-v)(2v+1)(v+2)}{\sqrt{v^2+v+1}^3}. \tag{6.7}$$

Then, $r(v)$ is monotone decreasing in $(0,\infty)$ and

$$r(0) = \frac{2\sqrt{3}}{9}, \quad r(v) \to -\frac{2\sqrt{3}}{9} \text{ as } v \to \infty. \tag{6.8}$$

Lemma 6.3. *Let $\mathcal{A}(h,s)$ be defined by (2.14). Then*

$$\mathcal{A}(h,0) = \frac{2\sqrt{3}}{9}, \quad \mathcal{A}(h,1) = -\frac{2\sqrt{3}}{9} \quad \text{for all } h \in [0,1), \tag{6.9}$$

$$\mathcal{A}_s(h,s) < 0 \quad \text{for all } (h,s) \in (0,1)\times(0,1). \tag{6.10}$$

Lemma 6.4. *Let $\tilde{V} > 0$ be fixed. There exists a unique curve*

$$s(h; \tilde{V}) \in C^{\infty}[0, 1) \tag{6.15}$$

such that

$$\mathcal{A}(h, s(h; \tilde{V})) = \frac{1}{3\sqrt{3}} \cdot \frac{(1 - \tilde{V})(2\tilde{V} + 1)(\tilde{V} + 2)}{\sqrt{\tilde{V}^2 + \tilde{V} + 1}^3}, \quad 0 < s(h; \tilde{V}) < 1. \tag{6.16}$$

Moreover

$$s(0; \tilde{V}) = \frac{1}{2} - \frac{1 - \tilde{V}}{\sqrt{2}\sqrt{(\tilde{V} + 2)(2\tilde{V} + 1)}}, \tag{6.17}$$

$$\mathcal{E}(0, s(0; \tilde{V})) = \frac{\sqrt{3}\sqrt{\tilde{V}}}{\pi \sqrt{\tilde{V}^2 + \tilde{V} + 1}}, \tag{6.18}$$

and

$$\mathcal{E}(h, s(h; \tilde{V})) \to 0 \text{ as } h \to 1. \tag{6.19}$$

Lemma 6.5. *Let $\mathcal{E}(h, s)$ be defined by (2.13), and $s(h; \tilde{V})$ defined in Lemma 6.4, then for each fixed $\tilde{V} > 0$*

$$\frac{d\mathcal{E}(h, s(h; \tilde{V}))}{dh} < 0 \quad \text{in}(0, 1). \tag{6.20}$$

Proof of Proposition 3.3 First, we note that

$$0 < \frac{\sqrt{3}\varepsilon}{\sqrt{\tilde{V}^2 + \tilde{V} + 1}} < \mathcal{E}(0, s(0; \tilde{V}))$$

is equivalent to

$$0 < \frac{\sqrt{3}\varepsilon}{\sqrt{\tilde{V}^2 + \tilde{V} + 1}} < \frac{\sqrt{3}\sqrt{\tilde{V}}}{\pi\sqrt{\tilde{V}^2 + \tilde{V} + 1}},$$

that is,

$$0 < \varepsilon < \frac{\sqrt{\tilde{V}}}{\pi}.$$

Thus, we complete the proof by Lemmas 6.4 and 6.5.

We show the graph of $\mathcal{A}(h, s)$ and $\mathcal{E}(h, s)$ in Figures 1 and 2.

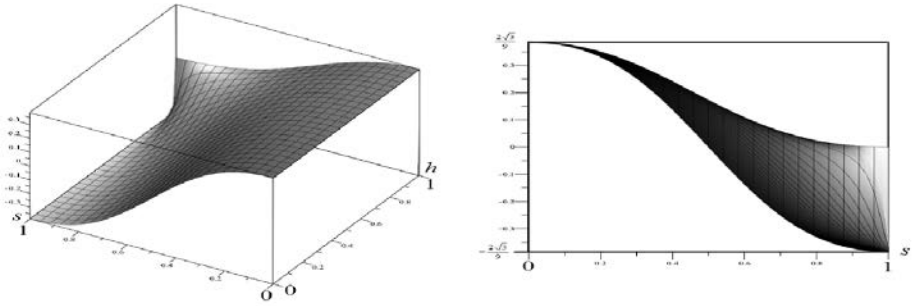


FIGURE 1. Graph of $\mathcal{A}(h, s)$

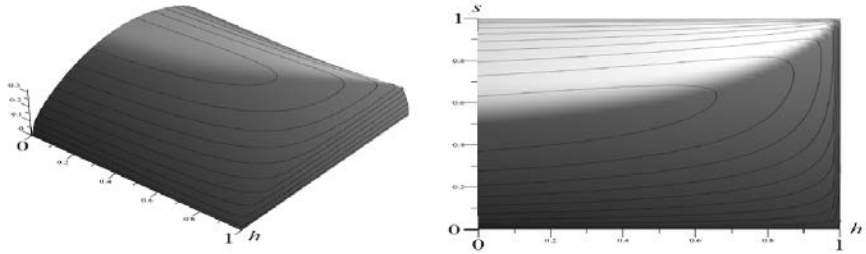


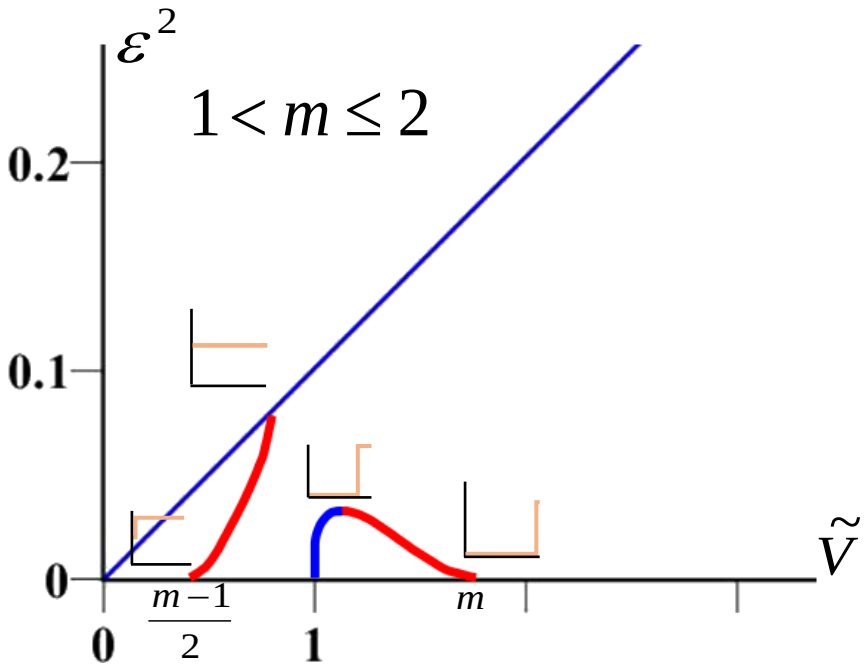
FIGURE 2. Graph of $\mathcal{E}(h, s)$

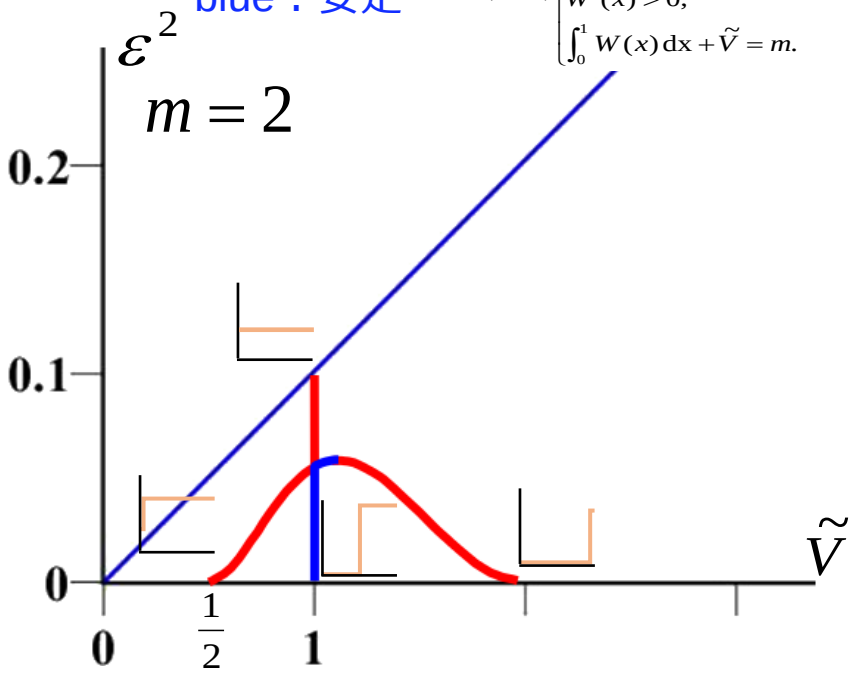
(SLP)の定常解の安定性

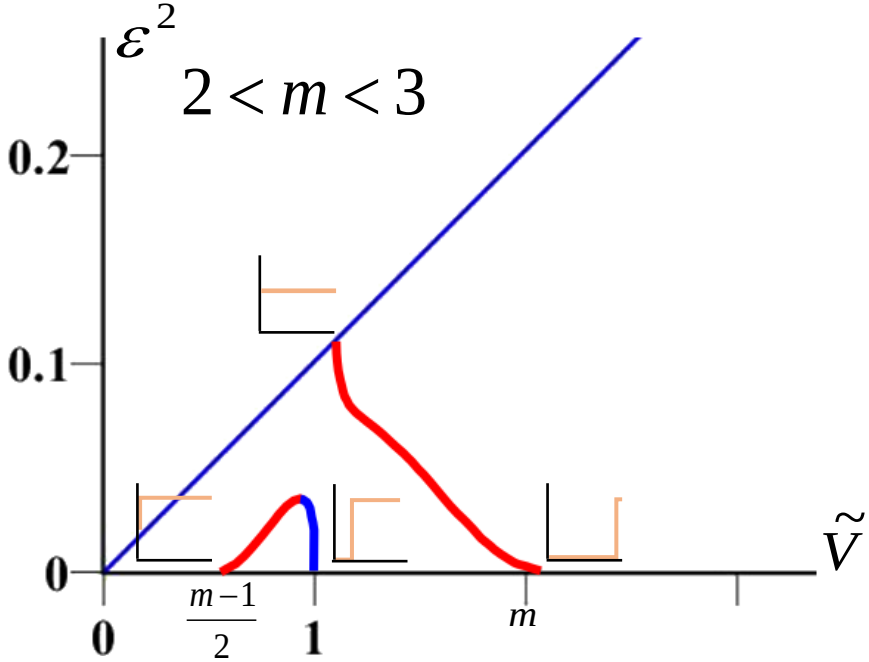
数値計算

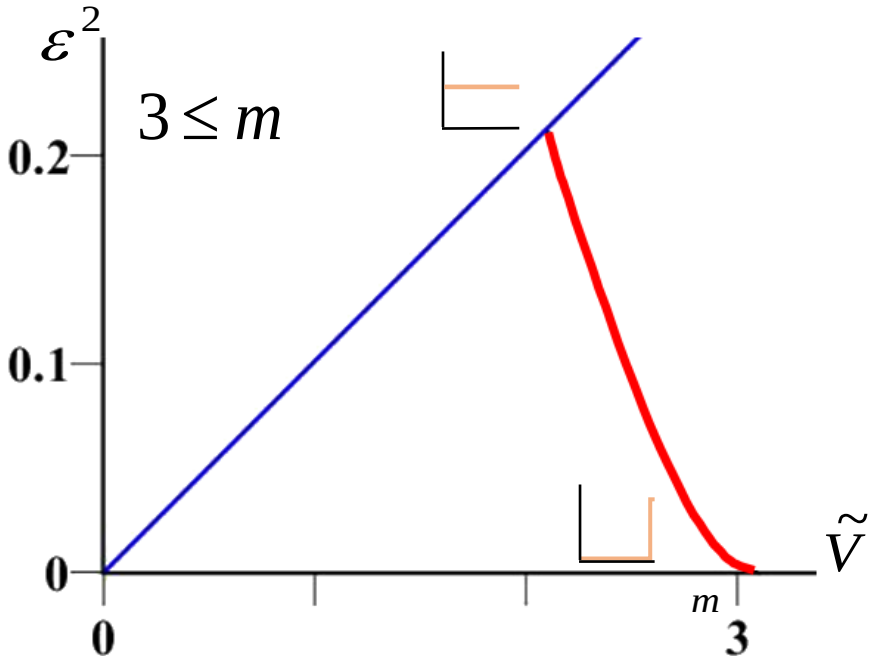
red：不安定
blue：安定

$$(SLP)\begin{cases}\varepsilon^2 W_{xx}+W(W-1)(\tilde{V}+1-W)=0,\\ W_x(0)=W_x(1)=0,\\ W'(x)>0,\\ \int_0^1 W(x)\,dx+\tilde{V}=m.\end{cases}$$





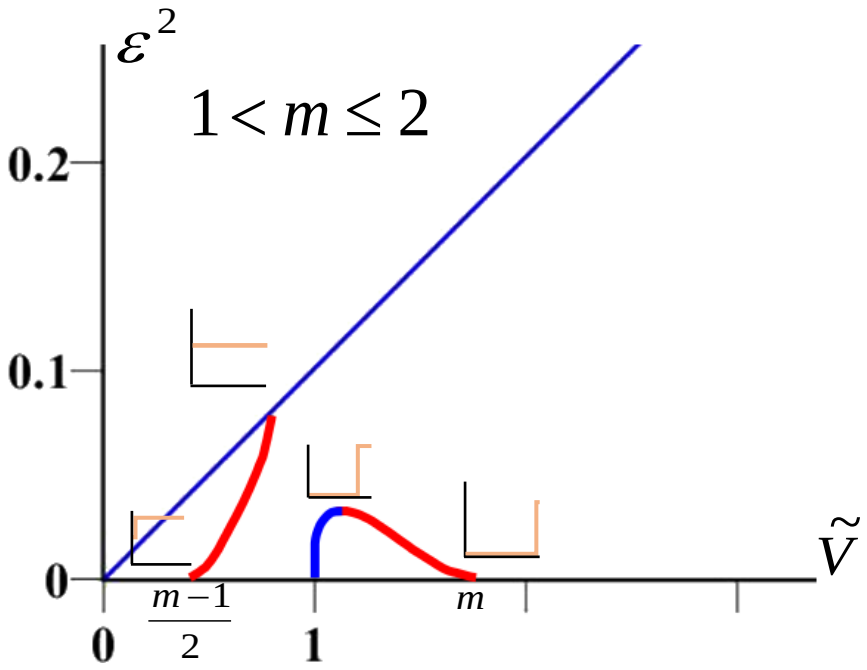


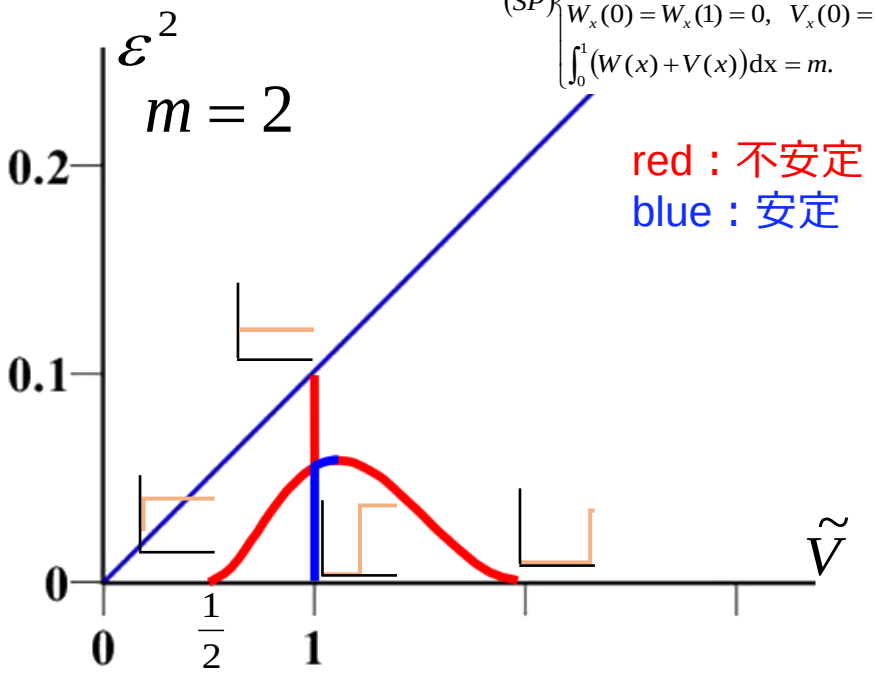


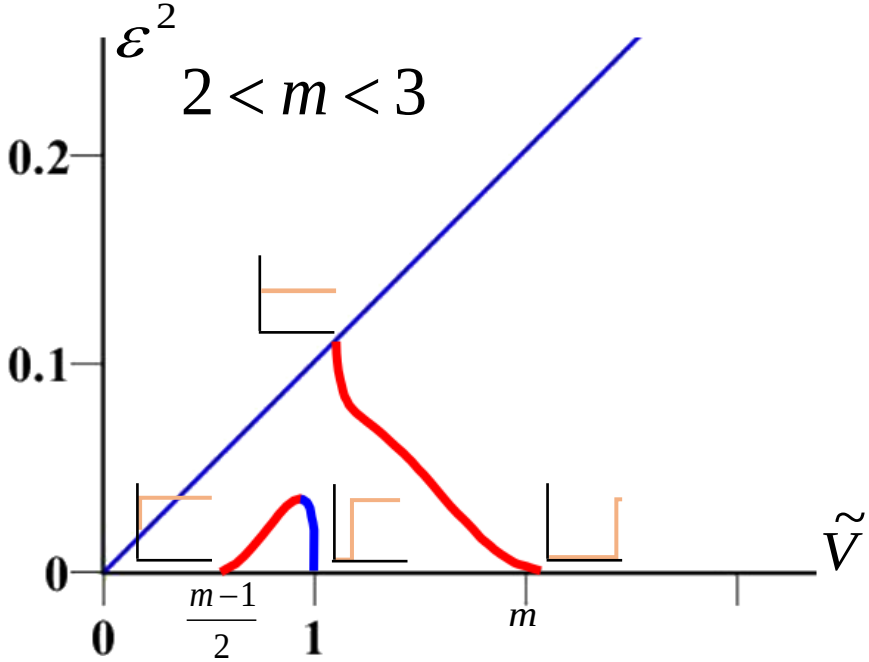
(SP)の定常解の安定性 $D=10000$ (SLP)と同様

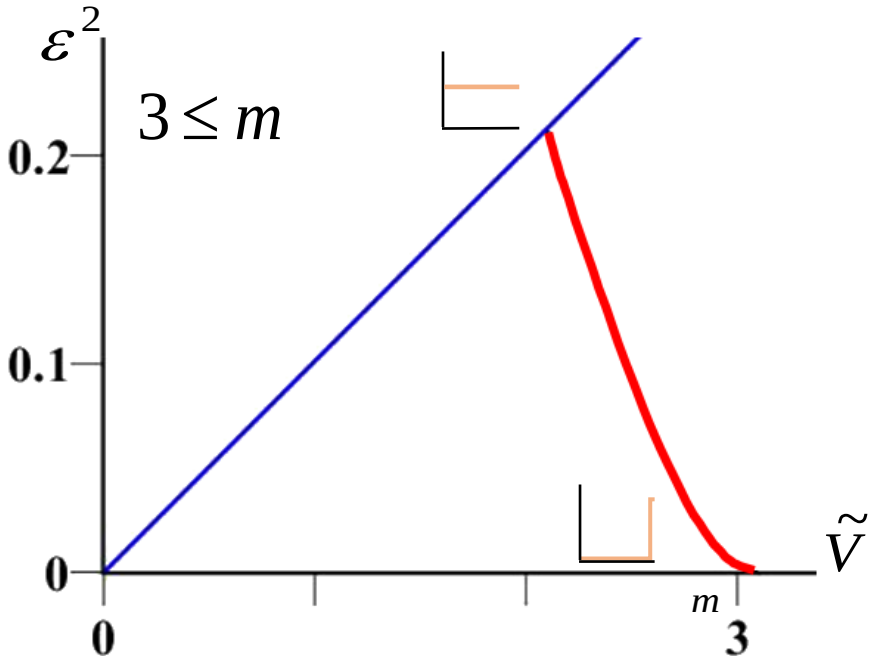
$$(SP)\begin{cases}\varepsilon^2 W_{xx}+W(W-1)(V+1-W)=0,\\ DV_{xx}-W(W-1)(V+1-W)=0,\\ W_x(0)=W_x(1)=0,\quad V_x(0)=V_x(1)=0,\\ \int_0^1 (W(x)+V(x))\mathrm{dx}=m.\end{cases}$$

red：不安定
blue：安定









Idea of proof of Theorem C

$$\frac{\partial m(\tilde{V},\varepsilon^2)}{\partial \varepsilon} < 0 \text{ for } \quad 0<\tilde{V}<1,\ 0<\varepsilon^2<\frac{\tilde{V}}{\pi^2}.$$

$$m(\tilde{V},\ \varepsilon^2) := \int_0^1 W(x;\tilde{V},\varepsilon^2)\,dx + \tilde{V}$$

Idea of proof of Theorem C

$\frac{\partial m(\tilde{V},\varepsilon^2)}{\partial \varepsilon} < 0$ *for* $0 < \tilde{V} < 1,\ 0 < \varepsilon^2 < \frac{\tilde{V}}{\pi^2}.$

$m(\tilde{V},\varepsilon^2)$ *is represented by*

$$m(\tilde{V},\varepsilon^2) := \frac{4\tilde{V}+2}{3} + \frac{1}{\sqrt{3}} \cdot \sqrt{\tilde{V}^2 + \tilde{V} + 1} \cdot M(h,s),$$

$$m(\tilde{V},\ \varepsilon^2) := \int_0^1 W(x;\tilde{V},\varepsilon^2)\,dx + \tilde{V}$$

$$M(h,\ s) := \frac{-(hs^2 - 2(1+h)s + 3) + 4(1-s)(1-sh)\Pi(-sh,\sqrt{h})/K(\sqrt{h})}{\sqrt{3h^2s^4 - 4(h^2+h)s^3 + (4h^2+2h+4)s^2 - 4(h+1)s + 3}},$$

Idea of proof of Theorem C

$\frac{\partial m(\tilde{V},\varepsilon^2)}{\partial \varepsilon} < 0$ *for* $0<\tilde{V}<1,\ 0<\varepsilon^2<\frac{\tilde{V}}{\pi^2}.$

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$$M(h,s) := \frac{-(hs^2-2(1+h)s+3)+4(1-s)(1-sh)\Pi(-sh,\sqrt{h})/K(\sqrt{h})}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}},$$

where $(h,s)=(h(\tilde{V},\varepsilon^2),s(\tilde{V},\varepsilon^2))$ *is the unique solution of the following system of transcendental equations*

$$\begin{cases} E(h,s) := \dfrac{\sqrt{2s(1-s)(1-sh)}/K(\sqrt{h})}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}} = \sqrt{3}\cdot\dfrac{\varepsilon}{\sqrt{\tilde{V}^2+\tilde{V}+1}}, \\ A(h,s) := \dfrac{2(hs^2-sh+1)(hs^2-2s+1)(1-hs^2)}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}^3} = \dfrac{1}{3\sqrt{3}}\cdot\dfrac{(1-\tilde{V})(2\tilde{V}+1)(\tilde{V}+2)}{\sqrt{\tilde{V}^2+\tilde{V}+1}^3}, \\ 0<h<1,\quad 0<s<\dfrac{1}{1+\sqrt{1-h}}. \end{cases}$$

Idea of proof of Theorem C

$$\frac{\partial m(\tilde{V},\varepsilon^2)}{\partial \varepsilon} < 0 \quad \text{for} \quad 0<\tilde{V}<1,\ 0<\varepsilon^2<\frac{\tilde{V}}{\pi^2}.$$

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$$m(\tilde{V},\varepsilon^2) := \int_0^1 W(x;\tilde{V},\varepsilon^2)\,dx + \tilde{V}$$

$$M(h,s) := \frac{-(hs^2-2(1+h)s+3)+4(1-s)(1-sh)\Pi(-sh,\sqrt{h})/K(\sqrt{h})}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}},$$

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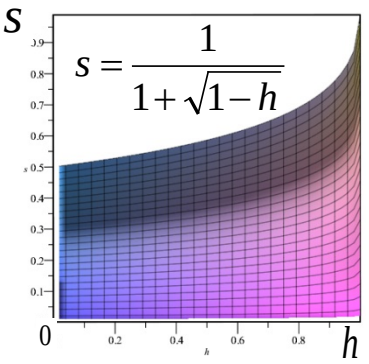


Idea of proof of Theorem C

$$\frac{\partial m(\tilde{V},\varepsilon^2)}{\partial \varepsilon} < 0 \quad \text{for} \quad 0<\tilde{V}<1,\ 0<\varepsilon^2<\frac{\tilde{V}}{\pi^2}.$$

$m(\tilde{V},\varepsilon^2)$ *is represented by*

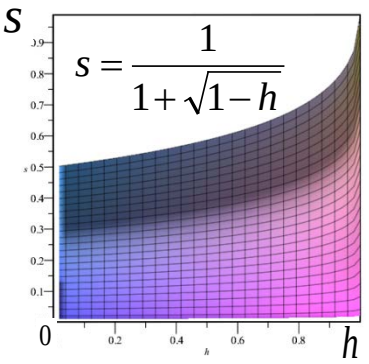
$$m(\tilde{V},\varepsilon^2) := \frac{4\tilde{V}+2}{3}+\frac{1}{\sqrt{3}}\cdot\sqrt{\tilde{V}^2+\tilde{V}+1}\cdot M(h,s),$$

$$m(\tilde{V},\varepsilon^2) := \int_0^1 W(x;\tilde{V},\varepsilon^2)\,dx+\tilde{V}$$

$$M(h,s) := \frac{-(hs^2-2(1+h)s+3)+4(1-s)(1-sh)\Pi(-sh,\sqrt{h})\big/K(\sqrt{h})}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}},$$

where $(h,s)=(h(\tilde{V},\varepsilon^2),\ s(\tilde{V},\varepsilon^2))$ *is the unique solution of the following system of transcendental equations*

$$\begin{cases} E(h,s) := \dfrac{\sqrt{2s(1-s)(1-sh)}\big/K(\sqrt{h})}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}} = \sqrt{3}\cdot\dfrac{\varepsilon}{\sqrt{\tilde{V}^2+\tilde{V}+1}}, \\ A(h,s) := \dfrac{2(hs^2-sh+1)(hs^2-2s+1)(1-hs^2)}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}^3} = \dfrac{1}{3\sqrt{3}}\cdot\dfrac{(1-\tilde{V})(2\tilde{V}+1)(\tilde{V}+2)}{\sqrt{\tilde{V}^2+\tilde{V}+1}^3}, \\ 0<h<1,\quad 0<s<\dfrac{1}{1+\sqrt{1-h}}. \end{cases}$$



Here, $K(\cdot)$ *is the complete elliptic integral of the first kind,*

$\Pi(\cdot,\cdot)$ *is the complete elliptic integral of the third kind.*

Idea of proof of Theorem C

$\dfrac{\partial m(\tilde{V},\varepsilon^2)}{\partial \varepsilon} < 0$ *for* $0<\tilde{V}<1,\ 0<\varepsilon^2<\dfrac{\tilde{V}}{\pi^2}.$

$m(\tilde{V},\varepsilon^2)$ *is represented by*

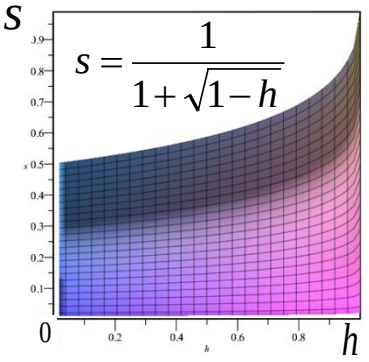
$$m(\tilde{V},\varepsilon^2) := \frac{4\tilde{V}+2}{3} + \frac{1}{\sqrt{3}}\cdot\sqrt{\tilde{V}^2+\tilde{V}+1}\cdot M(h,s),$$

$$m(\tilde{V},\varepsilon^2) := \int_0^1 W(x;\tilde{V},\varepsilon^2)\,dx + \tilde{V}$$

$$M(h,s) := \frac{-(hs^2-2(1+h)s+3)+4(1-s)(1-sh)\Pi(-sh,\sqrt{h})\big/K(\sqrt{h})}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}},$$

where $(h,s)=(h(\tilde{V},\varepsilon^2),\ s(\tilde{V},\varepsilon^2))$ *is the unique solution of the following system of transcendental equations*

$$\begin{cases} E(h,s) := \dfrac{\sqrt{2s(1-s)(1-sh)}\big/K(\sqrt{h})}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}} = \sqrt{3}\cdot\dfrac{\varepsilon}{\sqrt{\tilde{V}^2+\tilde{V}+1}}, \\ A(h,s) := \dfrac{2(hs^2-sh+1)(hs^2-2s+1)(1-hs^2)}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}^3} = \dfrac{1}{3\sqrt{3}}\cdot\dfrac{(1-\tilde{V})(2\tilde{V}+1)(\tilde{V}+2)}{\sqrt{\tilde{V}^2+\tilde{V}+1}^3}, \\ 0<h<1,\quad 0<s<\dfrac{1}{1+\sqrt{1-h}}. \end{cases}$$

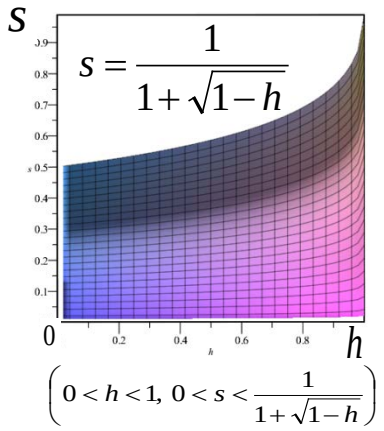


Here, $K(\cdot)$ *is the complete elliptic integral of the first kind,*

$\Pi(\cdot,\cdot)$ *is the complete elliptic integral of the third kind.*

$$K(k) := \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}},\qquad \Pi(\mu,k) := \int_0^{\frac{\pi}{2}} \frac{d\varphi}{(1+\mu\sin^2\varphi)\sqrt{1-k^2\sin^2\varphi}}.$$

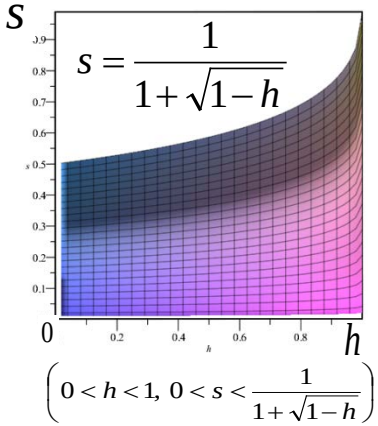
$$\begin{cases} E(h,s) := \dfrac{\sqrt{2s(1-s)(1-sh)}\,/\,K(\sqrt{h})}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}} = \sqrt{3}\cdot\dfrac{\varepsilon}{\sqrt{\tilde{V}^2+\tilde{V}+1}}, \\ A(h,s) := \dfrac{2(hs^2-sh+1)(hs^2-2s+1)(1-hs^2)}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}^3} = \dfrac{1}{3\sqrt{3}}\cdot\dfrac{(1-\tilde{V})(2\tilde{V}+1)(\tilde{V}+2)}{\sqrt{\tilde{V}^2+\tilde{V}+1}^3}. \end{cases}$$



$$\left(0 < h < 1,\ 0 < s < \frac{1}{1+\sqrt{1-h}}\right)$$

$$\begin{cases} E(h,s) := \dfrac{\sqrt{2s(1-s)(1-sh)}\,/\,K(\sqrt{h})}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}} = \sqrt{3}\cdot\dfrac{\varepsilon}{\sqrt{\tilde{V}^2+\tilde{V}+1}}, \\ A(h,s) := \dfrac{2(hs^2-sh+1)(hs^2-2s+1)(1-hs^2)}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}^3} = \dfrac{1}{3\sqrt{3}}\cdot\dfrac{(1-\tilde{V})(2\tilde{V}+1)(\tilde{V}+2)}{\sqrt{\tilde{V}^2+\tilde{V}+1}^3}. \end{cases}$$

$$m(\tilde{V},\varepsilon^2) := \frac{4\tilde{V}+2}{3} + \frac{1}{\sqrt{3}}\cdot\sqrt{\tilde{V}^2+\tilde{V}+1}\cdot M(h,s),$$

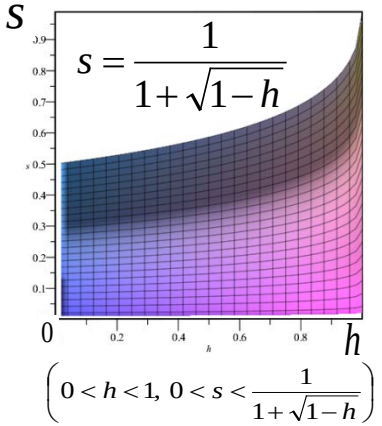


$$\left(0 < h < 1,\ 0 < s < \frac{1}{1+\sqrt{1-h}}\right)$$

$$\begin{cases} E(h,s) := \dfrac{\sqrt{2s(1-s)(1-sh)}\,/\,K(\sqrt{h})}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}} = \sqrt{3}\cdot\dfrac{\varepsilon}{\sqrt{\tilde{V}^2+\tilde{V}+1}}, \\ A(h,s) := \dfrac{2(hs^2-sh+1)(hs^2-2s+1)(1-hs^2)}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}^3} = \dfrac{1}{3\sqrt{3}}\cdot\dfrac{(1-\tilde{V})(2\tilde{V}+1)(\tilde{V}+2)}{\sqrt{\tilde{V}^2+\tilde{V}+1}^3}. \end{cases}$$

$$m(\tilde{V},\varepsilon^2) := \frac{4\tilde{V}+2}{3}+\frac{1}{\sqrt{3}}\cdot\sqrt{\tilde{V}^2+\tilde{V}+1}\cdot M(h,s),$$

$$\frac{\partial m(\tilde{V},\varepsilon^2)}{\partial\varepsilon} = \frac{\partial\left(\dfrac{4\tilde{V}+2}{3}+\sqrt{\dfrac{\tilde{V}^2+\tilde{V}+1}{3}}\cdot M\right)}{\partial\varepsilon} = \sqrt{\frac{\tilde{V}^2+\tilde{V}+1}{3}}\,\frac{\partial M}{\partial\varepsilon}$$



$$\left(0<h<1,\ 0<s<\frac{1}{1+\sqrt{1-h}}\right)$$

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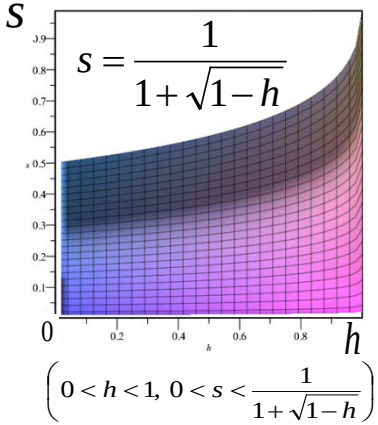

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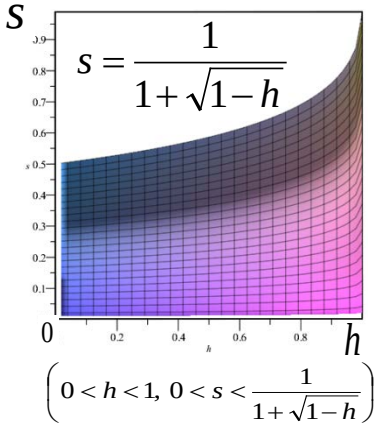

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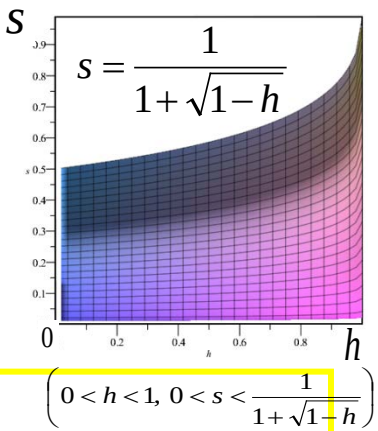

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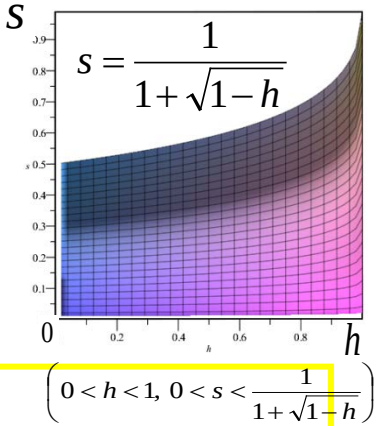

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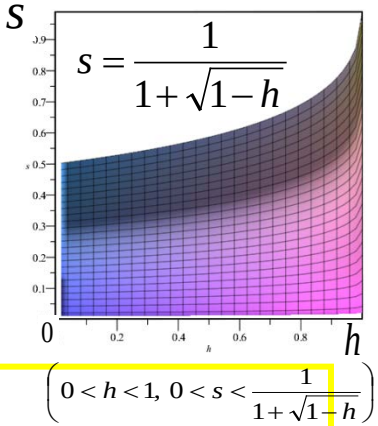

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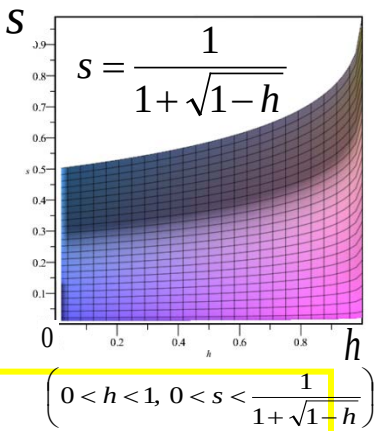

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Plan of proof(continued)

$$\frac{\partial m(\tilde{V},\varepsilon^2)}{\partial \varepsilon} = \sqrt{\frac{\tilde{V}^2+\tilde{V}+1}{3}} \cdot \frac{M_h \cdot A_s - M_s \cdot A_h}{E_h \cdot A_s - E_s \cdot A_h}.$$

Plan of proof(continued)

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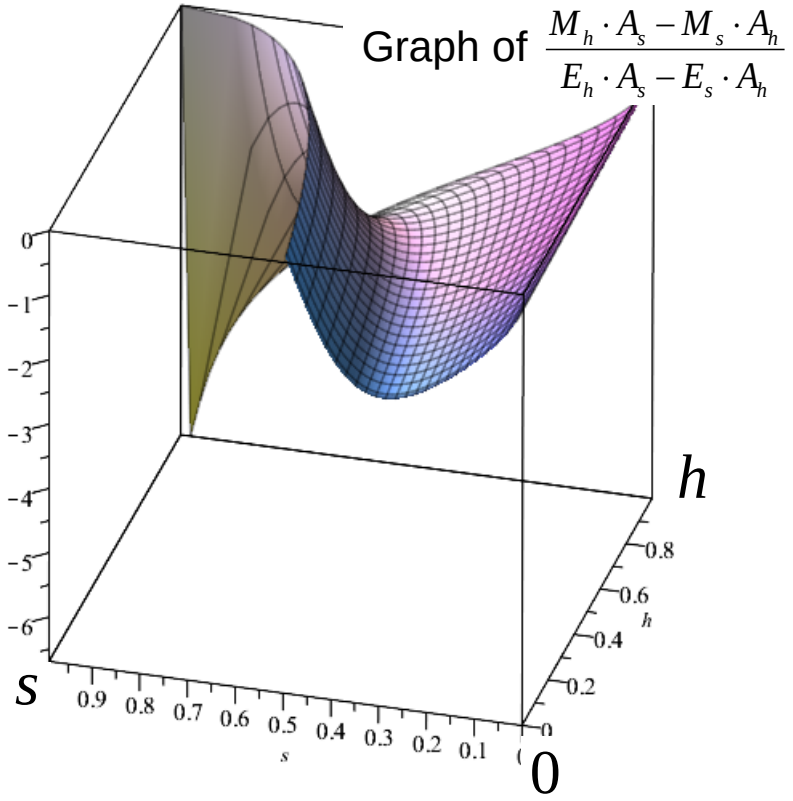
Graph of $\frac{M_h\cdot A_s-M_s\cdot A_h}{E_h\cdot A_s-E_s\cdot A_h}$



Plan of proof(continued)

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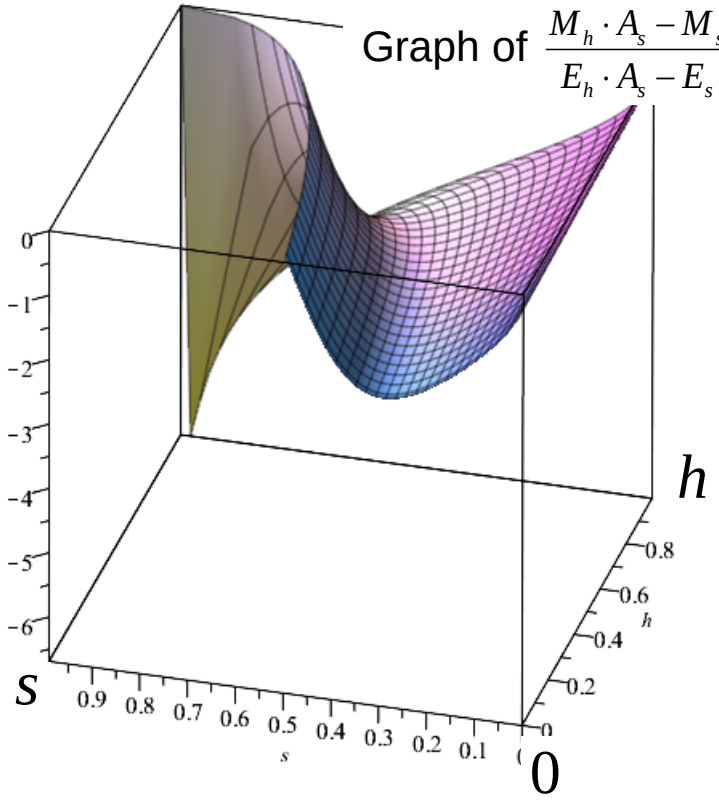

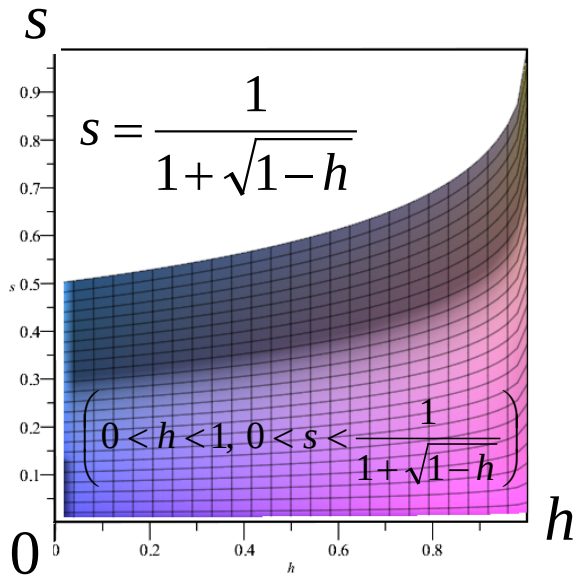

Plan of proof(continued)

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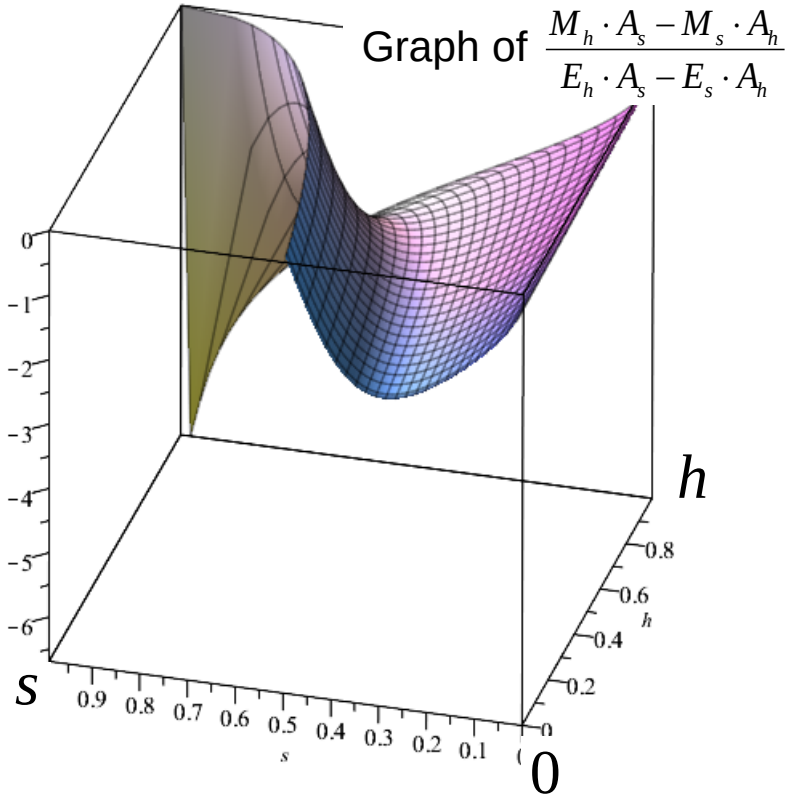
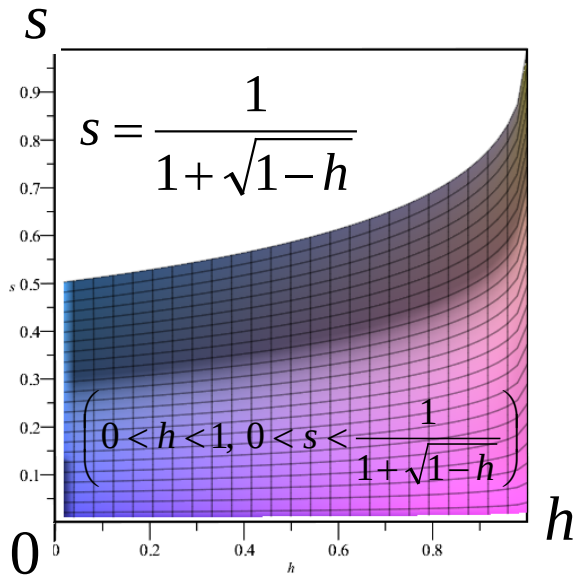
We may show the following inequality equations.

$$M_h \cdot A_s - M_s \cdot A_h < 0, \quad E_h \cdot A_s - E_s \cdot A_h > 0.$$





Plan of proof(continued)

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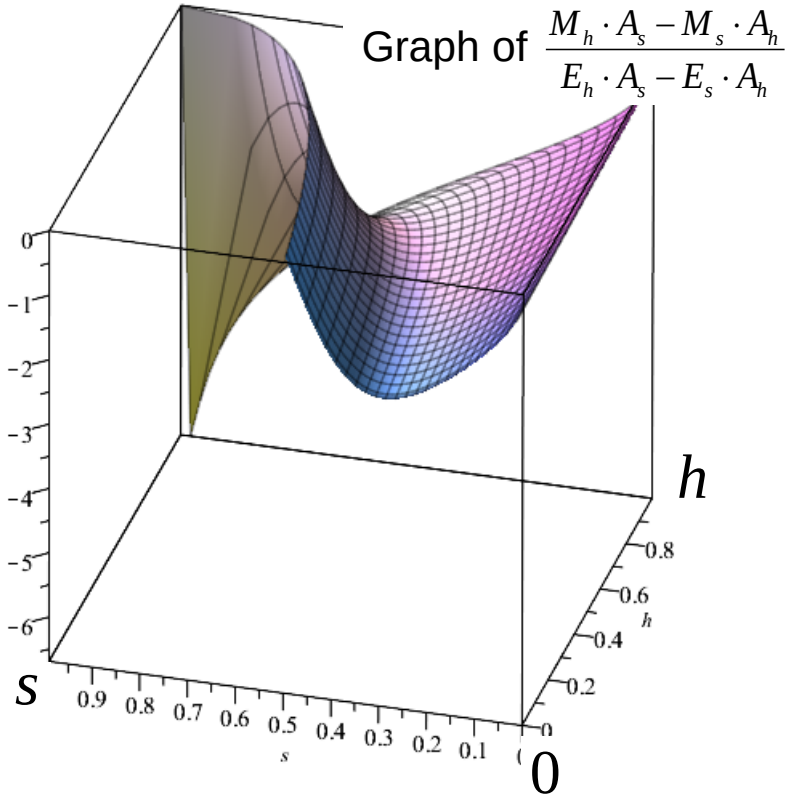

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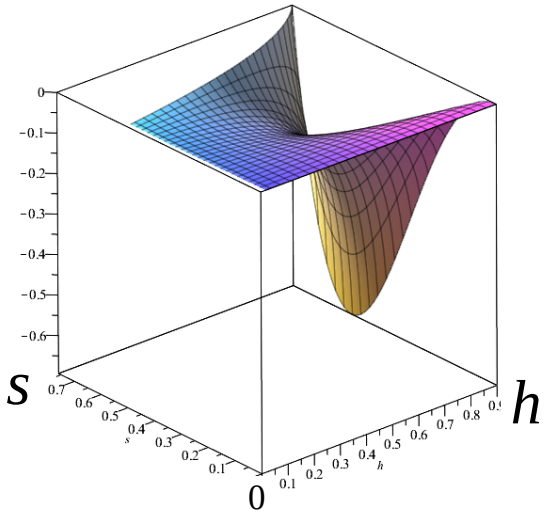
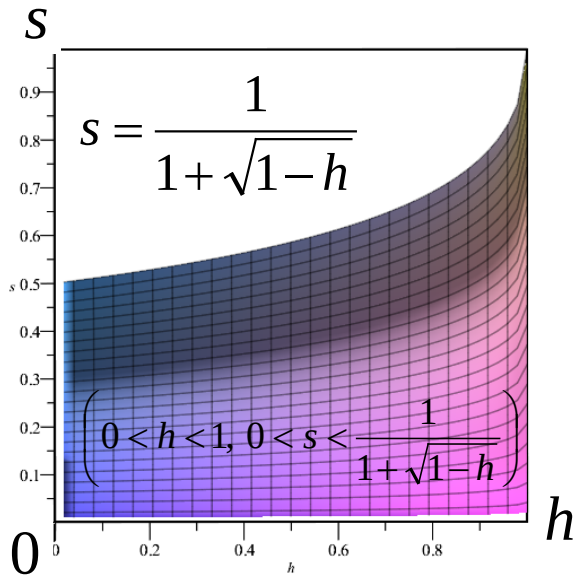




Plan of proof(continued)

$$\frac{\partial m(\tilde{V},\varepsilon^2)}{\partial \varepsilon}=\sqrt{\frac{\tilde{V}^2+\tilde{V}+1}{3}}\cdot\frac{M_h\cdot A_s-M_s\cdot A_h}{E_h\cdot A_s-E_s\cdot A_h}.$$

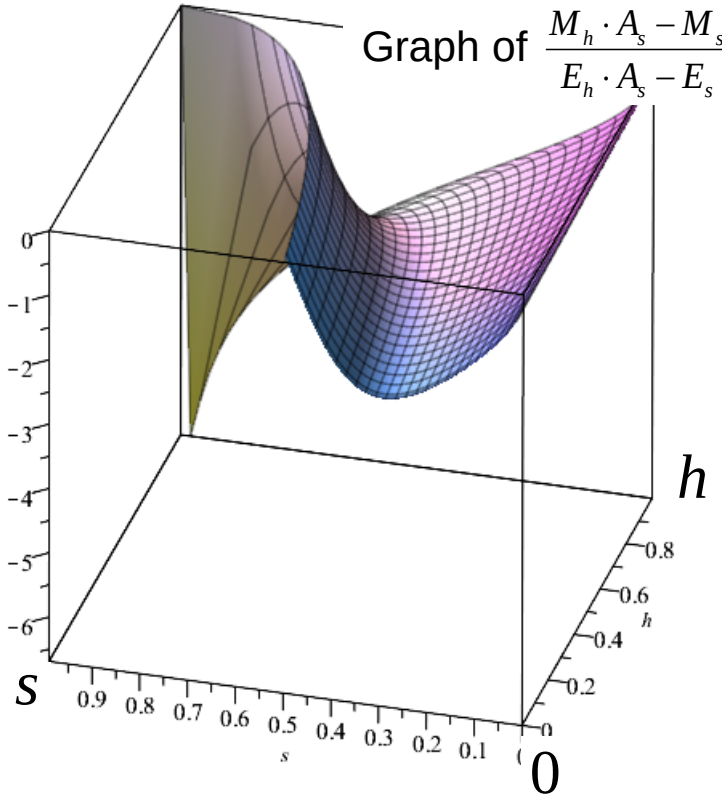
Here,

$$M(h,\ s):=\frac{-(hs^2-2(1+h)s+3)+4(1-s)(1-sh)\Pi(-sh,\sqrt{h})/K(\sqrt{h})}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}}$$

$$A(h,s):=\frac{2(hs^2-sh+1)(hs^2-2s+1)(1-hs^2)}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}^3}$$

$$E(h,s):=\frac{\sqrt{2s(1-s)(1-sh)}/K(\sqrt{h})}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}}$$

Graph of $\frac{M_h\cdot A_s-M_s\cdot A_h}{E_h\cdot A_s-E_s\cdot A_h}$



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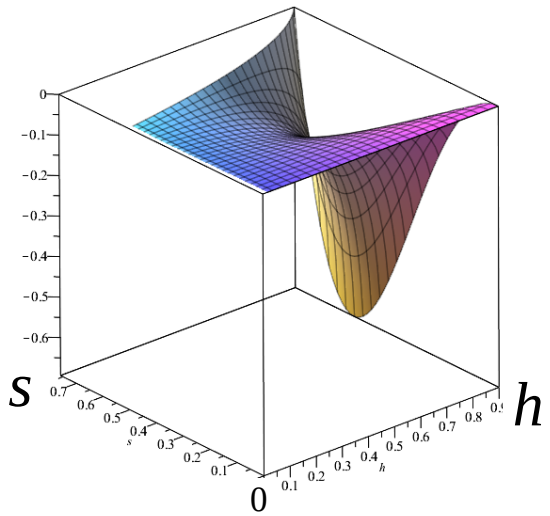

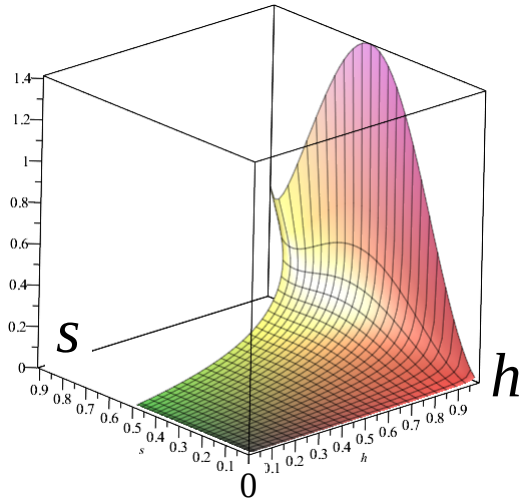

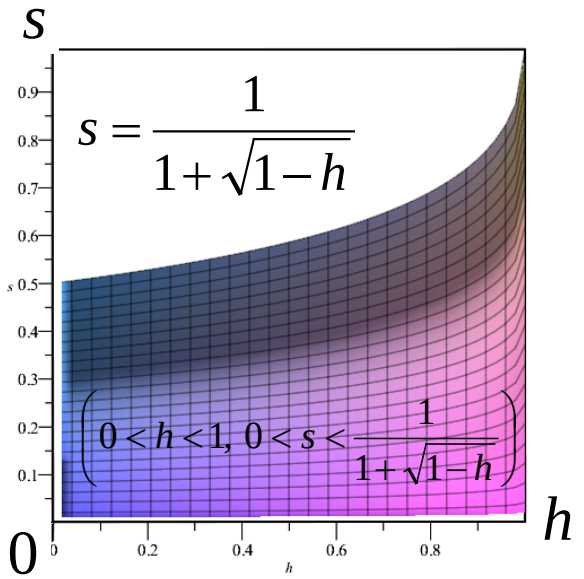

Plan of proof(continued)

$$\frac{\partial m(\tilde{V},\varepsilon^2)}{\partial \varepsilon}=\sqrt{\frac{\tilde{V}^2+\tilde{V}+1}{3}}\cdot\frac{M_h\cdot A_s-M_s\cdot A_h}{E_h\cdot A_s-E_s\cdot A_h}.$$

Here,

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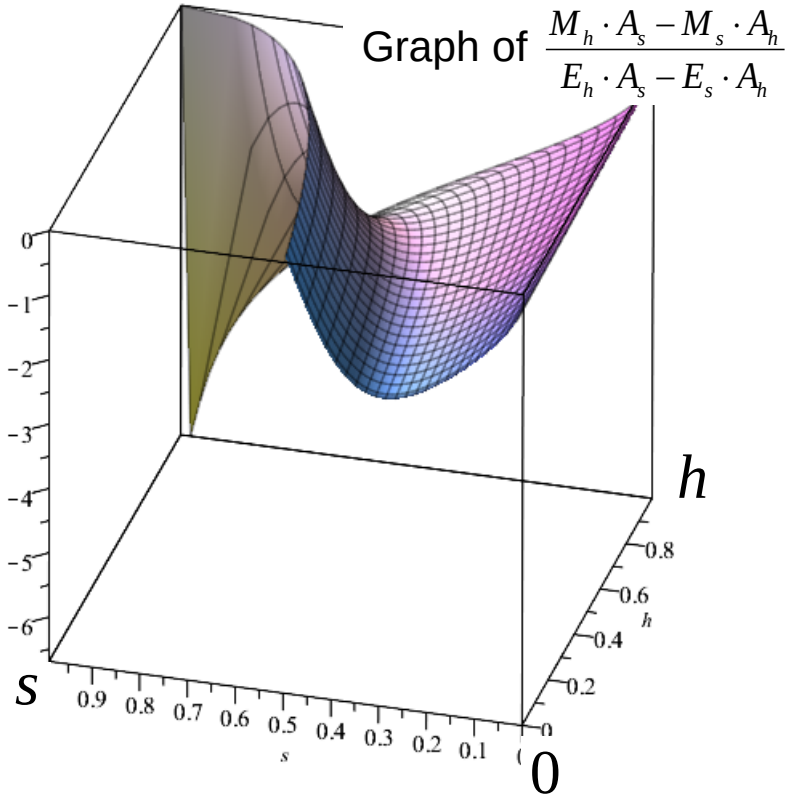
$$A(h,s):=\frac{2(hs^2-sh+1)(hs^2-2s+1)(1-hs^2)}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}^3}$$

$$E(h,s):=\frac{\sqrt{2s(1-s)(1-sh)}/K(\sqrt{h})}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}}$$

Graph of $\frac{M_h\cdot A_s-M_s\cdot A_h}{E_h\cdot A_s-E_s\cdot A_h}$



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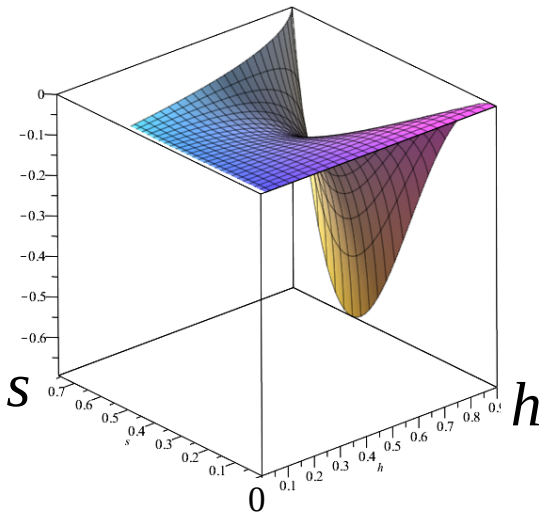
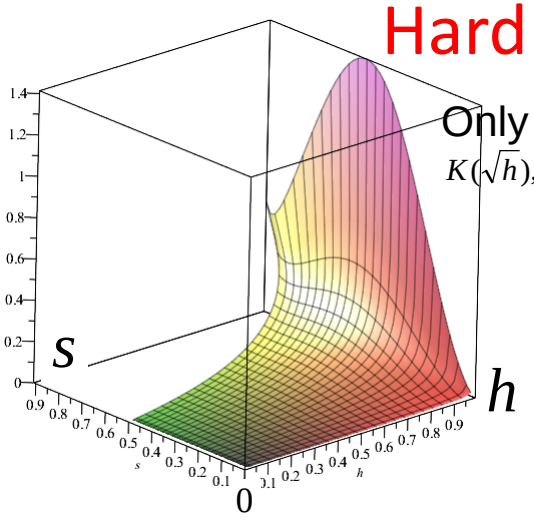
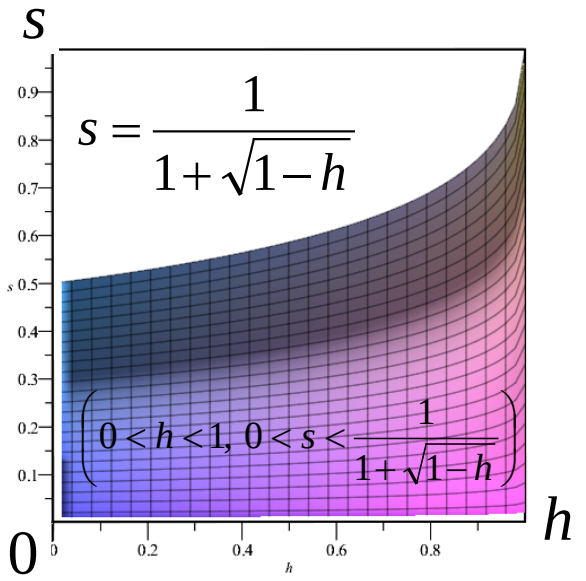


Hard

Only use $K(\sqrt{h}), E(\sqrt{h})$





Plan of proof(continued)

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Here,

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$$A(h,s):=\frac{2(hs^2-sh+1)(hs^2-2s+1)(1-hs^2)}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}^3}$$

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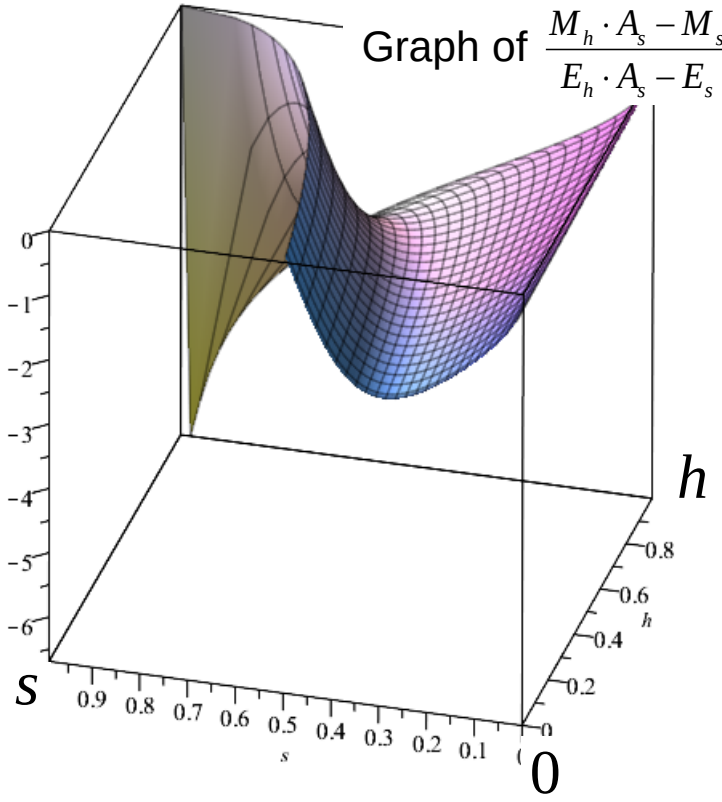
Graph of $\frac{M_h\cdot A_s-M_s\cdot A_h}{E_h\cdot A_s-E_s\cdot A_h}$



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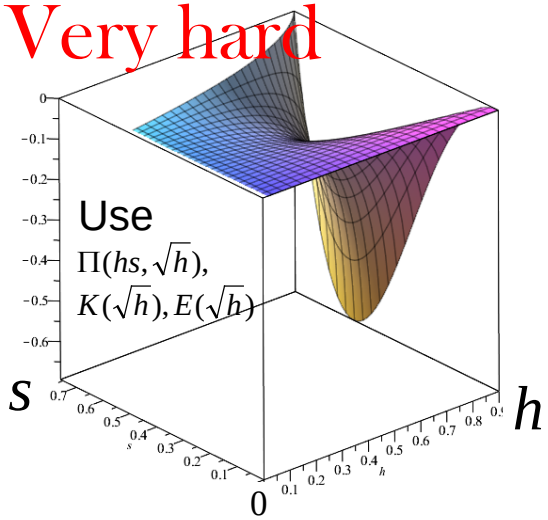
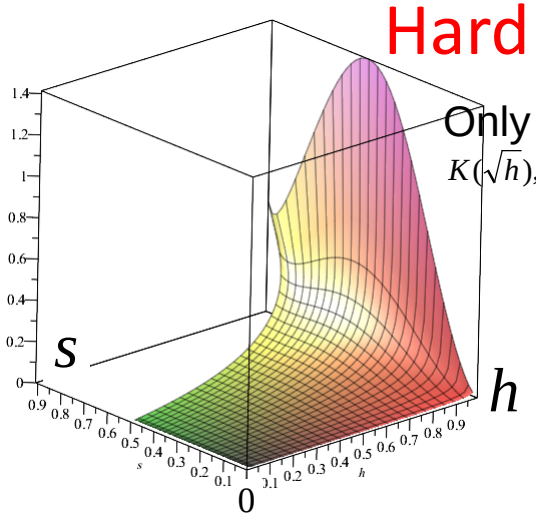
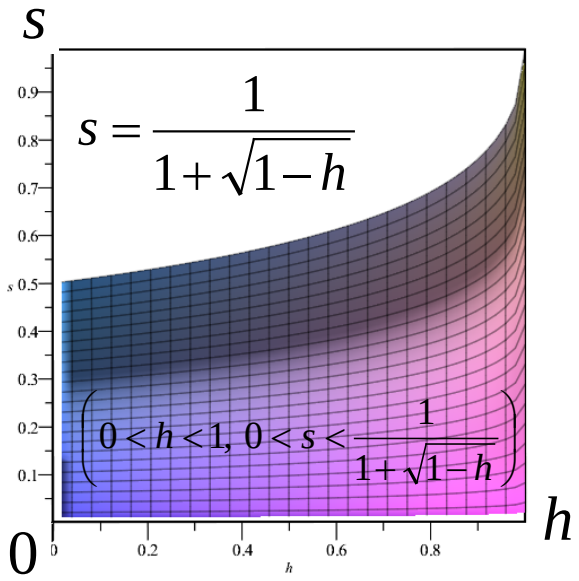
Very hard



Hard





Plan of proof(continued)

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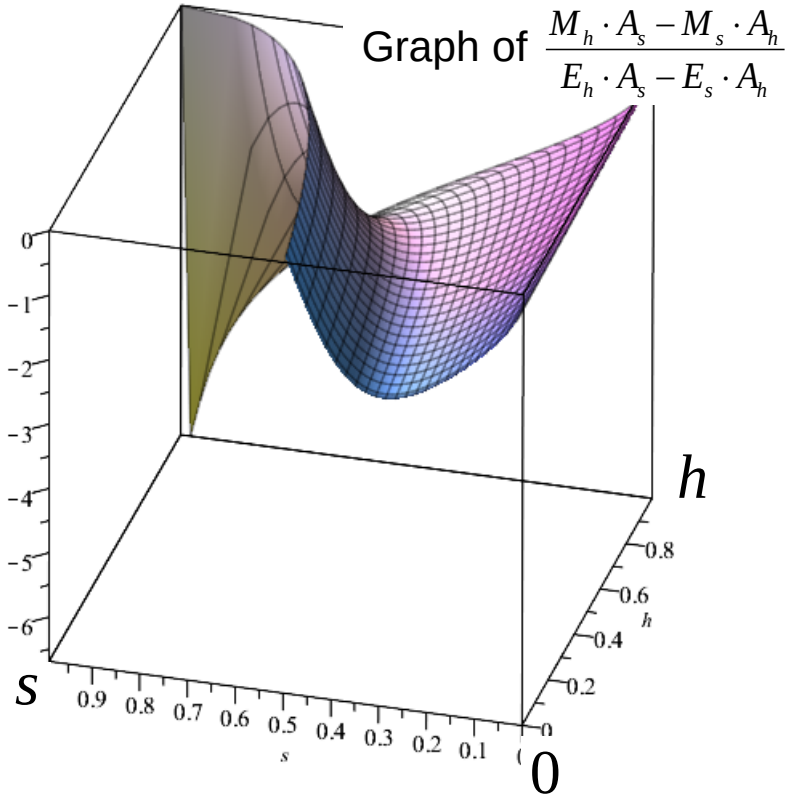
$$A(h,s):=\frac{2(hs^2-sh+1)(hs^2-2s+1)(1-hs^2)}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}^3}$$

$$E(h,s):=\frac{\sqrt{2s(1-s)(1-sh)}/K(\sqrt{h})}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}}$$

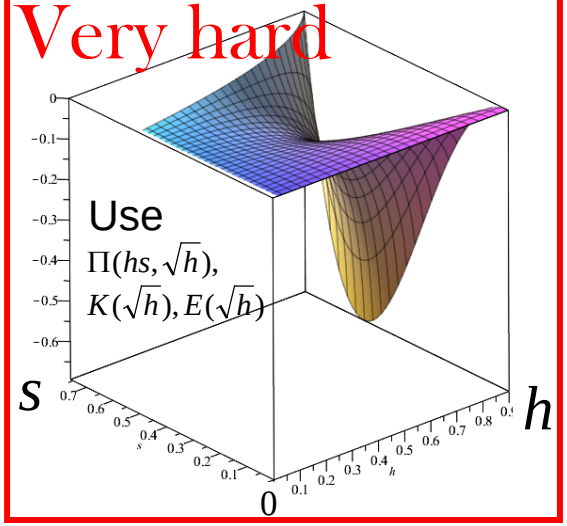
Graph of $\frac{M_h\cdot A_s-M_s\cdot A_h}{E_h\cdot A_s-E_s\cdot A_h}$



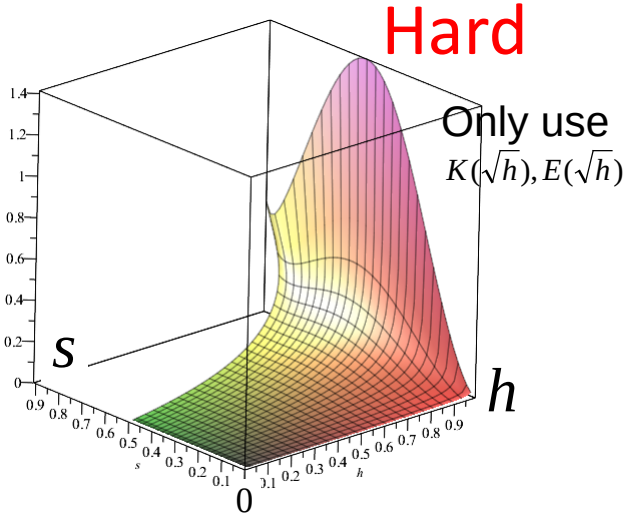
We may show the following inequality equations.

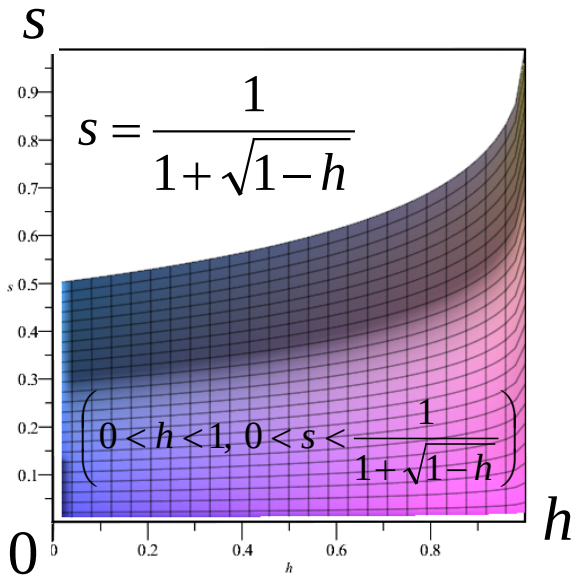
$M_h\cdot A_s-M_s\cdot A_h<0,$ $E_h\cdot A_s-E_s\cdot A_h>0.$

Very hard



Hard





We show $M_h \cdot A_s - M_s \cdot A_h < 0$

$\mathcal{M}_h \cdot \mathcal{A}_s - \mathcal{M}_s \cdot \mathcal{A}_h =$

$$M(h,\ s) := \frac{-(hs^2 - 2(1+h)s + 3) + 4(1-s)(1-sh)\Pi(-sh, \sqrt{h}) / K(\sqrt{h})}{\sqrt{3h^2s^4 - 4(h^2+h)s^3 + (4h^2+2h+4)s^2 - 4(h+1)s + 3}}$$

$$A(h,s) := \frac{2(hs^2 - sh + 1)(hs^2 - 2s + 1)(1 - hs^2)}{\sqrt{3h^2s^4 - 4(h^2+h)s^3 + (4h^2+2h+4)s^2 - 4(h+1)s + 3}^3}$$

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$$\mathcal{M}_h \cdot \mathcal{A}_s - \mathcal{M}_s \cdot \mathcal{A}_h =$$

$$32s(1-hs)^2(1-s)^2 \Bigg(\Big(\big(-2h^4s^4 + 2h^3s^4 + 4h^3s^3 - 2h^2s^4 + 4h^2s^3 - 12h^2s^2 + 4h^2s - 2h^2 + 4hs + 2h - 2 \big) E(\sqrt{h}) + \big(h^4s^4 - 3h^3s^4 + 4h^3s^3 + 2h^2s^4 - 6h^3s^2 - 4h^2s^3 + 6h^2s^2 + 4h^2s + h^2 - 4hs - 3h + 2 \big) K(\sqrt{h}) \Big) \Pi(hs, \sqrt{h}) + \big(-h^3s^3 - h^2s^3 + 6h^2s^2 - 3h^2s + 2h^2 - 3hs - 2h + 2 \big) K(\sqrt{h}) E(\sqrt{h}) + \big(-h^3s^3 + 3h^3s^2 + h^2s^3 - 3h^2s^2 - 3h^2s - h^2 + 3hs + 3h - 2 \big) K(\sqrt{h})^2 \Bigg)$$

$$\Big/ K(\sqrt{h})^2 (-1+h)\, h \left(3h^2s^4 - 4h^2s^3 + 4h^2s^2 - 4hs^3 + 2hs^2 - 4hs + 4s^2 - 4s + 3 \right)^3$$

We show $M_h \cdot A_s - M_s \cdot A_h < 0$

$$M(h,s) := \frac{-(hs^2 - 2(1+h)s + 3) + 4(1-s)(1-sh)\Pi(-sh, \sqrt{h}) / K(\sqrt{h})}{\sqrt{3h^2s^4 - 4(h^2+h)s^3 + (4h^2+2h+4)s^2 - 4(h+1)s + 3}}$$

$$A(h,s) := \frac{2(hs^2 - sh + 1)(hs^2 - 2s + 1)(1 - hs^2)}{\sqrt{3h^2s^4 - 4(h^2+h)s^3 + (4h^2+2h+4)s^2 - 4(h+1)s + 3}^3}$$

$$\mathcal{M}_h \cdot \mathcal{A}_s - \mathcal{M}_s \cdot \mathcal{A}_h =$$

$$32s(1-hs)^2(1-s)^2 \Big(\Big((-2h^4s^4 + 2h^3s^4 + 4h^3s^3 - 2h^2s^4 + 4h^2s^3 - 12h^2s^2$$

$$+4h^2s - 2h^2 + 4hs + 2h - 2)\, E(\sqrt{h}) + (h^4s^4 - 3h^3s^4 + 4h^3s^3 + 2h^2s^4 - 6h^3s^2$$

$$-4h^2s^3 + 6h^2s^2 + 4h^2s + h^2 - 4hs - 3h + 2)\, K(\sqrt{h}) \Big) \Pi(hs, \sqrt{h})$$

$$+ (-h^3s^3 - h^2s^3 + 6h^2s^2 - 3h^2s + 2h^2 - 3hs - 2h + 2)\, K(\sqrt{h}) E(\sqrt{h})$$

$$+ (-h^3s^3 + 3h^3s^2 + h^2s^3 - 3h^2s^2 - 3h^2s - h^2 + 3hs + 3h - 2)\, K(\sqrt{h})^2 \Big)$$

$$\Big/ K(\sqrt{h})^2 (-1+h)\, h \left(3h^2s^4 - 4h^2s^3 + 4h^2s^2 - 4hs^3 + 2hs^2 - 4hs + 4s^2 - 4s + 3\right)^3$$

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$$M(h,s) := \frac{-(hs^2 - 2(1+h)s + 3) + 4(1-s)(1-sh)\Pi(-sh, \sqrt{h})/K(\sqrt{h})}{\sqrt{3h^2s^4 - 4(h^2+h)s^3 + (4h^2+2h+4)s^2 - 4(h+1)s + 3}}$$

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$$+4h^2s - 2h^2 + 4hs + 2h - 2\big)E(\sqrt{h}) + \big(h^4s^4 - 3h^3s^4 + 4h^3s^3 + 2h^2s^4 - 6h^3s^2$$

$$-4h^2s^3 + 6h^2s^2 + 4h^2s + h^2 - 4hs - 3h + 2\big)K(\sqrt{h})\Big)\Pi(hs, \sqrt{h})$$

$$+\big(-h^3s^3 - h^2s^3 + 6h^2s^2 - 3h^2s + 2h^2 - 3hs - 2h + 2\big)K(\sqrt{h})E(\sqrt{h})$$

$$+\big(-h^3s^3 + 3h^3s^2 + h^2s^3 - 3h^2s^2 - 3h^2s - h^2 + 3hs + 3h - 2\big)K(\sqrt{h})^2\Big)$$

$$\Big/ K(\sqrt{h})^2 \underbrace{(-1+h)}_{\text{negative}} \underbrace{h\left(3h^2s^4 - 4h^2s^3 + 4h^2s^2 - 4hs^3 + 2hs^2 - 4hs + 4s^2 - 4s + 3\right)^3}_{\text{positive}}$$

We show $M_h \cdot A_s - M_s \cdot A_h < 0$

$$M(h,s) := \frac{-(hs^2 - 2(1+h)s + 3) + 4(1-s)(1-sh)\Pi(-sh, \sqrt{h})/K(\sqrt{h})}{\sqrt{3h^2s^4 - 4(h^2+h)s^3 + (4h^2+2h+4)s^2 - 4(h+1)s + 3}}$$

$$A(h,s) := \frac{2(hs^2 - sh + 1)(hs^2 - 2s + 1)(1 - hs^2)}{\sqrt{3h^2s^4 - 4(h^2+h)s^3 + (4h^2+2h+4)s^2 - 4(h+1)s + 3}^3}$$

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$$-4h^2s^3 + 6h^2s^2 + 4h^2s + h^2 - 4hs - 3h + 2\big)K(\sqrt{h})\Big)\Pi(-hs, \sqrt{h})$$

$$+\big(-h^3s^3 - h^2s^3 + 6h^2s^2 - 3h^2s + 2h^2 - 3hs - 2h + 2\big)K(\sqrt{h})E(\sqrt{h})$$

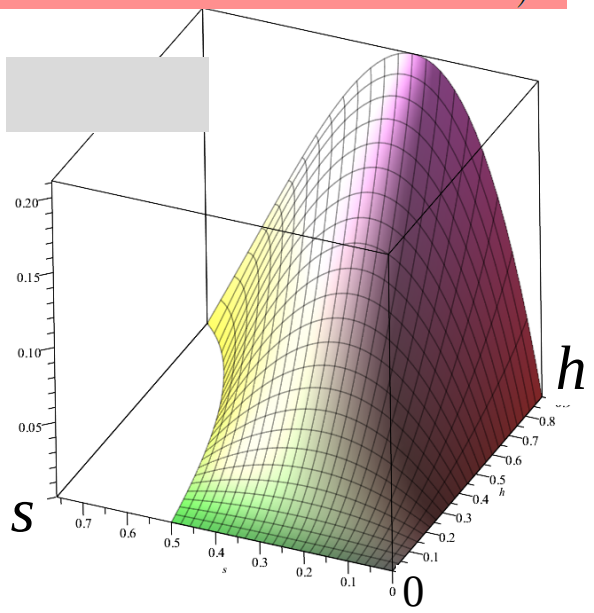
$$+\big(-h^3s^3 + 3h^3s^2 + h^2s^3 - 3h^2s^2 - 3h^2s - h^2 + 3hs + 3h - 2\big)K(\sqrt{h})^2\Big)$$

$$\Big/ K(\sqrt{h})^2 (-1+h)\, h \left(3h^2s^4 - 4h^2s^3 + 4h^2s^2 - 4hs^3 + 2hs^2 - 4hs + 4s^2 - 4s + 3\right)^3$$

negative positive

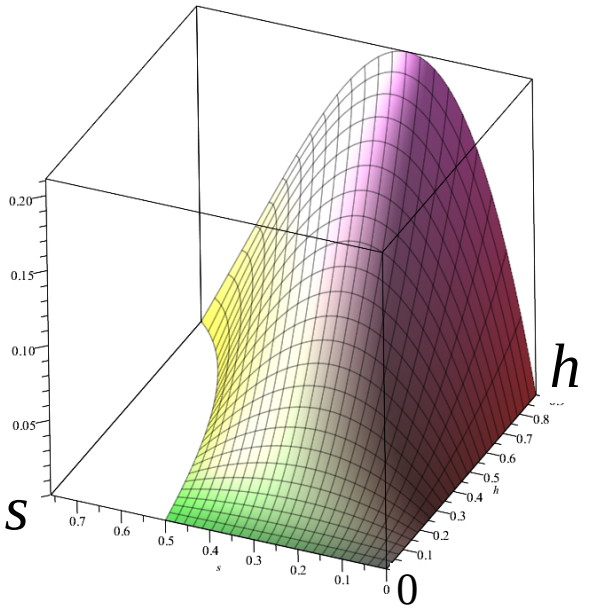
We may show that part of is positive.

Graph of



$$\Big(\Big((-2h^4s^4+2h^3s^4+4h^3s^3-2h^2s^4+4h^2s^3-12h^2s^2$$
$$+4h^2s-2h^2+4hs+2h-2\big)\,E(\sqrt{h})+\big(h^4s^4-3h^3s^4+4h^3s^3+2h^2s^4-6h^3s^2$$
$$-4h^2s^3+6h^2s^2+4h^2s+h^2-4hs-3h+2\big)\,K(\sqrt{h})\Big)\Pi(hs,\sqrt{h})$$
$$+\big(-h^3s^3-h^2s^3+6h^2s^2-3h^2s+2h^2-3hs-2h+2\big)\,K(\sqrt{h})E(\sqrt{h})$$
$$+\big(-h^3s^3+3h^3s^2+h^2s^3-3h^2s^2-3h^2s-h^2+3hs+3h-2\big)\,K(\sqrt{h})^2\Big)$$

We may show that part of [redacted] is positive.



$$\Big(\Big((-2h^4s^4+2h^3s^4+4h^3s^3-2h^2s^4+4h^2s^3-12h^2s^2$$
$$+4h^2s-2h^2+4hs+2h-2\big)\,E(\sqrt{h})+\big(h^4s^4-3h^3s^4+4h^3s^3+2h^2s^4-6h^3s^2$$
$$-4h^2s^3+6h^2s^2+4h^2s+h^2-4hs-3h+2\big)\,K(\sqrt{h})\Big)\Pi(hs,\sqrt{h})$$
$$+\big(-h^3s^3-h^2s^3+6h^2s^2-3h^2s+2h^2-3hs-2h+2\big)\,K(\sqrt{h})E(\sqrt{h})$$
$$+\big(-h^3s^3+3h^3s^2+h^2s^3-3h^2s^2-3h^2s-h^2+3hs+3h-2\big)\,K(\sqrt{h})^2\Big)$$

We may show that part of is positive.

- We can prove to use only $K(\sqrt{h}), E(\sqrt{h})$.

(∵ Take the derivative of $K(\sqrt{h}), E(\sqrt{h})$ with respect to h

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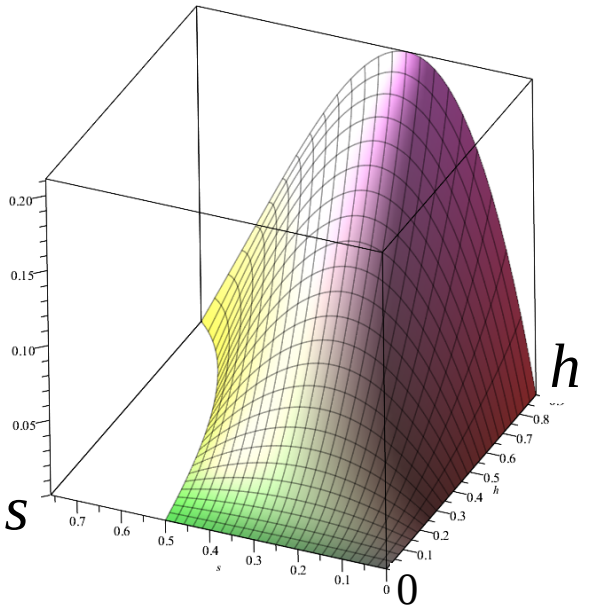

$$\Big(\Big((-2h^4s^4+2h^3s^4+4h^3s^3-2h^2s^4+4h^2s^3-12h^2s^2$$
$$+4h^2s-2h^2+4hs+2h-2\big)\,E(\sqrt{h})+\big(h^4s^4-3h^3s^4+4h^3s^3+2h^2s^4-6h^3s^2$$
$$-4h^2s^3+6h^2s^2+4h^2s+h^2-4hs-3h+2\big)\,K(\sqrt{h})\Big)\Pi(hs,\sqrt{h})$$
$$+\big(-h^3s^3-h^2s^3+6h^2s^2-3h^2s+2h^2-3hs-2h+2\big)\,K(\sqrt{h})E(\sqrt{h})$$
$$+\big(-h^3s^3+3h^3s^2+h^2s^3-3h^2s^2-3h^2s-h^2+3hs+3h-2\big)\,K(\sqrt{h})^2\Big)$$

We may show that part of [ ] is positive.

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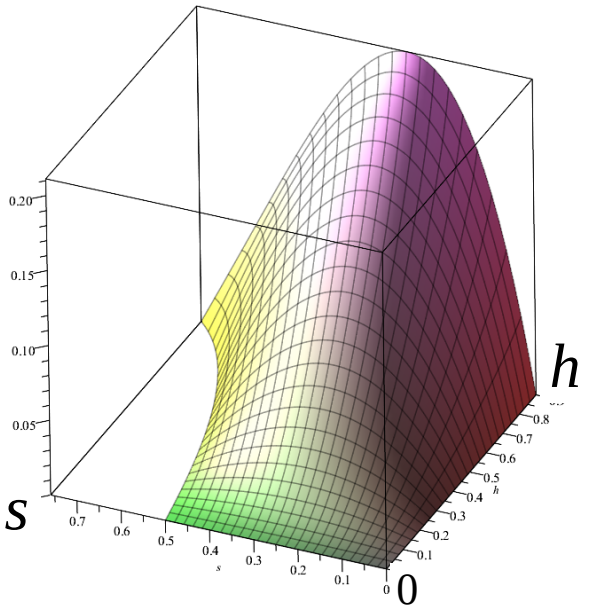

$$\Big(\Big((-2h^4s^4+2h^3s^4+4h^3s^3-2h^2s^4+4h^2s^3-12h^2s^2$$
$$+4h^2s-2h^2+4hs+2h-2\Big)E(\sqrt{h})+\big(h^4s^4-3h^3s^4+4h^3s^3+2h^2s^4-6h^3s^2$$
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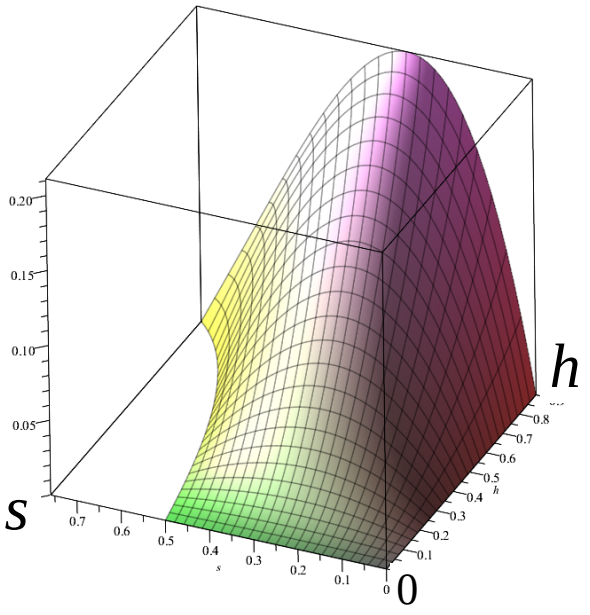
Only $K(\sqrt{h}), E(\sqrt{h})$

- Using $\Pi(hs, \sqrt{h})$ case

We can prove but hard calculation.



Differential of a $\Pi(-hs, \sqrt{h})$

$\Pi(-hs, \sqrt{h})$ は微分すると $\Pi(-hs,\sqrt{h}), K(\sqrt{h}), E(\sqrt{h})$ がでる.

$$\frac{\partial}{\partial s}\Pi(-hs,\sqrt{h}) = \frac{1}{2}\frac{(1-hs^2)\Pi(-hs,\sqrt{h}) - sE(\sqrt{h}) - (1-s)K(\sqrt{h})}{s(1-s)(1-sh)}$$

$$\frac{\partial}{\partial h}\Pi(-hs,\sqrt{h}) = \frac{1}{2}\frac{sh(1-h)\Pi(-hs,\sqrt{h}) + E(\sqrt{h}) - (1-h)K(\sqrt{h})}{h(1-h)(1-sh)}$$

Idea of except $\Pi(-hs, \sqrt{h})$

K.Kosugi, Y. Morita and Y. Yotsutani, DCDS 19(2007),

Take the derivative of $\dfrac{2\sqrt{(1-s)(1-sh)}}{\sqrt{s}} \cdot \Pi(-hs,\sqrt{h})$ with respect to h

Only $K(\sqrt{h}), E(\sqrt{h})$

$$\frac{\partial}{\partial s}\left(\frac{2\sqrt{(1-s)(1-sh)}}{\sqrt{s}} \cdot \Pi(-hs,\sqrt{h})\right) = -\frac{sE(\sqrt{h}) + (1-s)K(\sqrt{h})}{s\sqrt{s(1-s)(1-sh)}}$$

$$\frac{\partial}{\partial h}\left(\frac{2\sqrt{(1-s)(1-sh)}}{\sqrt{s}} \cdot \Pi(-hs,\sqrt{h})\right) = \frac{(1-s)\left(E(\sqrt{h}) - (1-h)K(\sqrt{h})\right)}{h(1-h)\sqrt{s(1-s)(1-sh)}}$$

$$\Big(\left(-2h^4s^4+2h^3s^4+4h^3s^3-2h^2s^4+4h^2s^3-12h^2s^2+4h^2s-2h^2+4hs+2h-2\right)E(\sqrt{h})$$
$$+\left(h^4s^4-3h^3s^4+4h^3s^3+2h^2s^4-6h^3s^2\ -4h^2s^3+6h^2s^2+4h^2s+h^2-4hs-3h+2\right)K(\sqrt{h})\Big)\Pi(hs,\sqrt{h})$$
$$+\left(-h^3s^3-h^2s^3+6h^2s^2-3h^2s+2h^2-3hs-2h+2\right)K(\sqrt{h})E(\sqrt{h})$$
$$+\left(-h^3s^3+3h^3s^2+h^2s^3-3h^2s^2-3h^2s-h^2+3hs+3h-2\right)K(\sqrt{h})^2>0$$

を示す.

$$\Big(\left(-2h^4s^4+2h^3s^4+4h^3s^3-2h^2s^4+4h^2s^3-12h^2s^2+4h^2s-2h^2+4hs+2h-2\right)E(\sqrt{h})$$
$$+\left(h^4s^4-3h^3s^4+4h^3s^3+2h^2s^4-6h^3s^2-4h^2s^3+6h^2s^2+4h^2s+h^2-4hs-3h+2\right)K(\sqrt{h})\Big)\Pi(hs,\sqrt{h})$$
$$+\left(-h^3s^3-h^2s^3+6h^2s^2-3h^2s+2h^2-3hs-2h+2\right)K(\sqrt{h})E(\sqrt{h})$$
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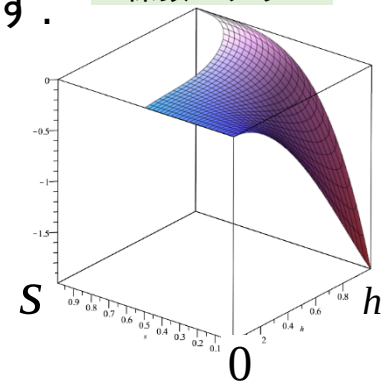
$\Pi(hs,\sqrt{h})$ の係数は負であることを確認する.

$\Pi(hs,\sqrt{h})$ の係数のグラフ を示す.



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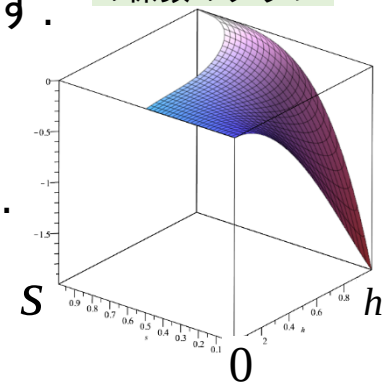

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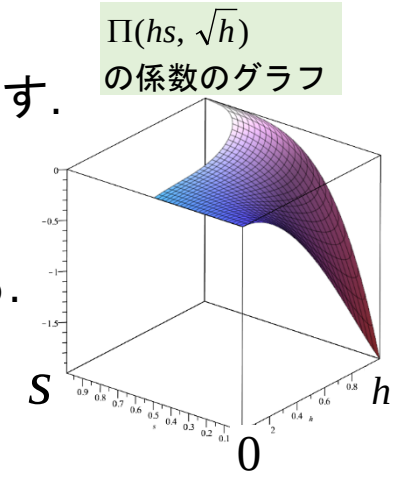

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$$+\left(h^4s^4-3h^3s^4+4h^3s^3+2h^2s^4-6h^3s^2-4h^2s^3+6h^2s^2+4h^2s+h^2-4hs-3h+2\right)K(\sqrt{h})\Big)\Pi(hs,\sqrt{h})$$
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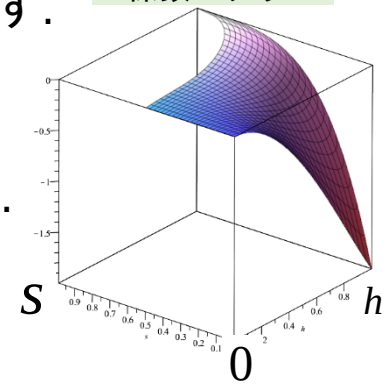
$\Pi(hs,\sqrt{h})$ の係数のグラフ



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$$>0$$

s で微分したものが

$K(\sqrt{h}),E(\sqrt{h})$ のみで書けるように変形した.

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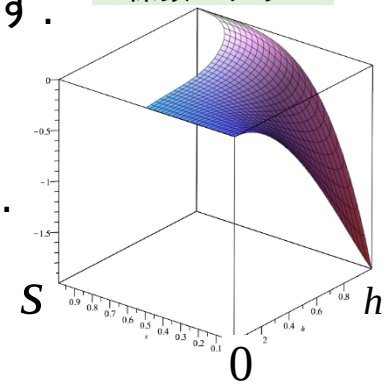

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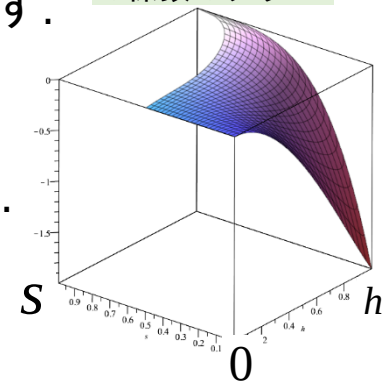

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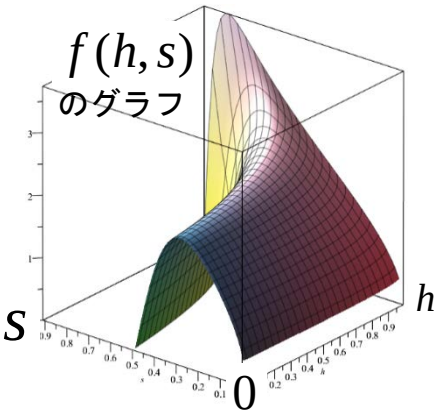

$$\left(0<h<1,\quad 0<s<\frac{1}{1+\sqrt{1-h}}\right)$$

$$\Big(\big(-2h^4s^4+2h^3s^4+4h^3s^3-2h^2s^4+4h^2s^3-12h^2s^2+4h^2s-2h^2+4hs+2h-2\big)E(\sqrt{h})$$
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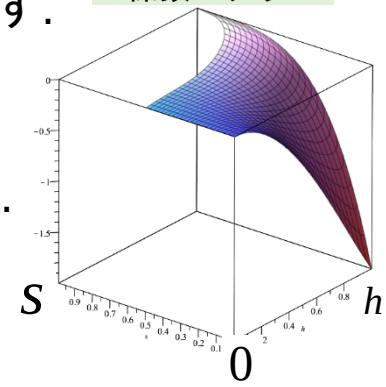

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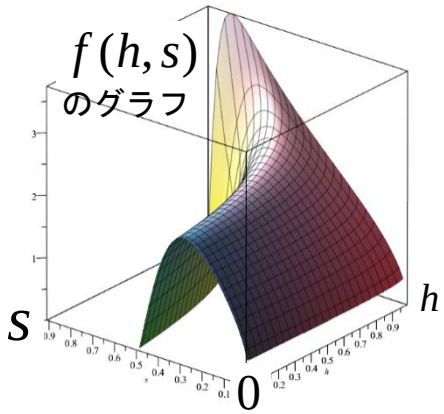
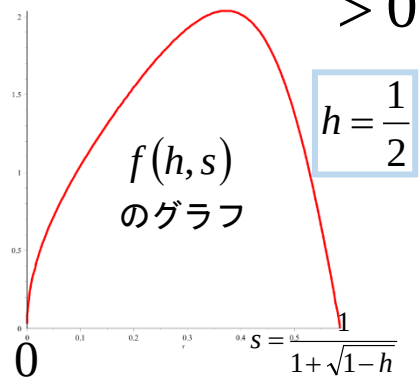
>0

以下, h を固定して s の関数と思う.

$$\lim_{s\to0}f(h,0)>0,\quad f\left(h,\frac{1}{1+\sqrt{1-h}}\right)=0$$

$f(h,s)$ のグラフ





hを固定して, s で偏微分する.

$f_s(h, s)$

hを固定して, s で偏微分する.

$$
\begin{aligned}
f_s(h,s) = 2\sqrt{2}\cdot\Big\{ & 2\Big(h^2\left(h^2-h+1\right)s^4-2h^2\left(h+1\right)s^3+6h^2s^2-2h\left(h+1\right)s+h^2-h+1\Big)^2E(\sqrt{h})^3 \\
& +\left(1-h\right)\Big(-3h^4\left(2-h\right)\left(h^2-h+1\right)s^8+6h^4\left(3h^2-3h+4\right)s^7-2h^4\left(23h^2-6h+25\right)s^6 \\
& +2h^3\left(16h^3+31h^2+41h-4\right)s^5-2h^2\left(8h^4+17h^3+87h^2-5h-2\right)s^4 \\
& +2h^2\left(16h^3+31h^2+41h-4\right)s^3-2h^2\left(23h^2-6h+25\right)s^2+6h\left(3h^2-3h+4\right)s \\
& -3\left(2-h\right)\left(h^2-h+1\right)\Big)K(\sqrt{h})E(\sqrt{h})^2 \\
& +\left(1-h\right)^2\Big(3h^4\left(h^2-2h+2\right)s^8-12h^4\left(h^2-2h+2\right)s^7+4h^4\left(3h^2-4h+4\right)s^6 \\
& -4h^3\left(h^2+6h-10\right)s^5-2h^2\left(8h^3-33h^2+30h+10\right)s^4-4h^2\left(h^2+6h-10\right)s^3 \\
& +4h^2\left(3h^2-4h+4\right)s^2-12h\left(h^2-2h+2\right)s+3h^2-6h+6\Big)K(\sqrt{h})^2E(\sqrt{h}) \\
& +\left(1-h\right)^3\Big(h^4\left(h-2\right)s^8-2h^4\left(3h-4\right)s^7+2h^4\left(3h+1\right)s^6-2h^3\left(5h+12\right)s^5 \\
& -2h^2\left(2h^2-21h-6\right)s^4-2h^2\left(5h+12\right)s^3+2h^2\left(3h+1\right)s^2 \\
& -2h\left(3h-4\right)s+h-2\Big)K(\sqrt{h})^3\Big\} \\
& \Big/\sqrt{s}\sqrt{\left(s-1\right)\left(hs-1\right)}\Big(\left(2h^4s^4-2h^3s^4-4h^3s^3+2h^2s^4-4h^2s^3+12h^2s^2-4h^2s\right. \\
& \left.+2h^2-4hs-2h+2\right)E(\sqrt{h})+\left(-h^4s^4+3h^3s^4-4h^3s^3-2h^2s^4+6h^3s^2\right. \\
& \left.+4h^2s^3-6h^2s^2-4h^2s-h^2+4hs+3h-2\right)K(\sqrt{h})\Big)^2
\end{aligned}
$$

hを固定して, s で偏微分する.

$$
\begin{aligned}
f_s(h,s) = {} & 2\sqrt{2}\cdot\Big\{2\Big(h^2\left(h^2-h+1\right)s^4-2h^2(h+1)s^3+6h^2s^2-2h(h+1)s+h^2-h+1\Big)^2E(\sqrt{h})^3 \\
& +(1-h)\Big(-3h^4(2-h)\left(h^2-h+1\right)s^8+6h^4\left(3h^2-3h+4\right)s^7-2h^4\left(23h^2-6h+25\right)s^6 \\
& +2h^3\left(16h^3+31h^2+41h-4\right)s^5-2h^2\left(8h^4+17h^3+87h^2-5h-2\right)s^4 \\
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& -3(2-h)\left(h^2-h+1\right)\Big)K(\sqrt{h})E(\sqrt{h})^2 \\
& +(1-h)^2\Big(3h^4\left(h^2-2h+2\right)s^8-12h^4\left(h^2-2h+2\right)s^7+4h^4\left(3h^2-4h+4\right)s^6 \\
& -4h^3\left(h^2+6h-10\right)s^5-2h^2\left(8h^3-33h^2+30h+10\right)s^4-4h^2\left(h^2+6h-10\right)s^3 \\
& +4h^2\left(3h^2-4h+4\right)s^2-12h\left(h^2-2h+2\right)s+3h^2-6h+6\Big)K(\sqrt{h})^2E(\sqrt{h}) \\
& +(1-h)^3\Big(h^4(h-2)s^8-2h^4(3h-4)s^7+2h^4(3h+1)s^6-2h^3(5h+12)s^5 \\
& -2h^2\left(2h^2-21h-6\right)s^4-2h^2(5h+12)s^3+2h^2(3h+1)s^2 \\
& -2h(3h-4)s+h-2\Big)K(\sqrt{h})^3\Big\} \\
& \Big/\sqrt{s}\sqrt{(s-1)(hs-1)}\Big(\left(2h^4s^4-2h^3s^4-4h^3s^3+2h^2s^4-4h^2s^3+12h^2s^2-4h^2s\right. \\
& \left.+2h^2-4hs-2h+2\right)E(\sqrt{h})+\left(-h^4s^4+3h^3s^4-4h^3s^3-2h^2s^4+6h^3s^2\right. \\
& \left.+4h^2s^3-6h^2s^2-4h^2s-h^2+4hs+3h-2\right)K(\sqrt{h})\Big)^2
\end{aligned}
$$

正

hを固定して, s で偏微分する.

$$f_s(h,s) = 2\sqrt{2}\cdot\Big\{2\Big(h^2\left(h^2-h+1\right)s^4-2h^2\left(h+1\right)s^3+6h^2s^2-2h\left(h+1\right)s+h^2-h+1\Big)^2E(\sqrt{h})^3$$
$$+\left(1-h\right)\Big(-3h^4\left(2-h\right)\left(h^2-h+1\right)s^8+6h^4\left(3h^2-3h+4\right)s^7-2h^4\left(23h^2-6h+25\right)s^6$$
$$+2h^3\left(16h^3+31h^2+41h-4\right)s^5-2h^2\left(8h^4+17h^3+87h^2-5h-2\right)s^4$$
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$$-3\left(2-h\right)\left(h^2-h+1\right)\Big)K(\sqrt{h})E(\sqrt{h})^2$$
$$+\left(1-h\right)^2\Big(3h^4\left(h^2-2h+2\right)s^8-12h^4\left(h^2-2h+2\right)s^7+4h^4\left(3h^2-4h+4\right)s^6$$
$$-4h^3\left(h^2+6h-10\right)s^5-2h^2\left(8h^3-33h^2+30h+10\right)s^4-4h^2\left(h^2+6h-10\right)s^3$$
$$+4h^2\left(3h^2-4h+4\right)s^2-12h\left(h^2-2h+2\right)s+3h^2-6h+6\Big)K(\sqrt{h})^2E(\sqrt{h})$$
$$+\left(1-h\right)^3\Big(h^4\left(h-2\right)s^8-2h^4\left(3h-4\right)s^7+2h^4\left(3h+1\right)s^6-2h^3\left(5h+12\right)s^5$$
$$-2h^2\left(2h^2-21h-6\right)s^4-2h^2\left(5h+12\right)s^3+2h^2\left(3h+1\right)s^2$$
$$-2h\left(3h-4\right)s+h-2\Big)K(\sqrt{h})^3\Big\}$$

この分子が正から負に1回だけ符号変化することを示せばよい

$$\Big/\sqrt{s}\sqrt{\left(s-1\right)\left(hs-1\right)}\Big(\left(2h^4s^4-2h^3s^4-4h^3s^3+2h^2s^4-4h^2s^3+12h^2s^2-4h^2s\right.$$
$$\left.+2h^2-4hs-2h+2\right)E(\sqrt{h})+\left(-h^4s^4+3h^3s^4-4h^3s^3-2h^2s^4+6h^3s^2\right.$$
$$\left.+4h^2s^3-6h^2s^2-4h^2s-h^2+4hs+3h-2\right)K(\sqrt{h})\Big)^2$$

正

$$
\begin{aligned}
g(h,s) := \Big\{ & 2\left(h^2\left(h^2-h+1\right)s^4-2h^2(h+1)s^3+6h^2s^2-2h(h+1)s+h^2-h+1\right)^2 E(\sqrt{h})^3 \\
& +(1-h)\left(-3h^4(2-h)\left(h^2-h+1\right)s^8+6h^4\left(3h^2-3h+4\right)s^7-2h^4\left(23h^2-6h+25\right)s^6\right. \\
& +2h^3\left(16h^3+31h^2+41h-4\right)s^5-2h^2\left(8h^4+17h^3+87h^2-5h-2\right)s^4 \\
& +2h^2\left(16h^3+31h^2+41h-4\right)s^3-2h^2\left(23h^2-6h+25\right)s^2+6h\left(3h^2-3h+4\right)s \\
& \left.-3(2-h)\left(h^2-h+1\right)\right)K(\sqrt{h})E(\sqrt{h})^2 \\
& +(1-h)^2\left(3h^4\left(h^2-2h+2\right)s^8-12h^4\left(h^2-2h+2\right)s^7+4h^4\left(3h^2-4h+4\right)s^6\right. \\
& -4h^3\left(h^2+6h-10\right)s^5-2h^2\left(8h^3-33h^2+30h+10\right)s^4-4h^2\left(h^2+6h-10\right)s^3 \\
& \left.+4h^2\left(3h^2-4h+4\right)s^2-12h\left(h^2-2h+2\right)s+3h^2-6h+6\right)K(\sqrt{h})^2E(\sqrt{h}) \\
& +(1-h)^3\left(h^4(h-2)s^8-2h^4(3h-4)s^7+2h^4(3h+1)s^6-2h^3(5h+12)s^5\right. \\
& -2h^2\left(2h^2-21h-6\right)s^4-2h^2(5h+12)s^3+2h^2(3h+1)s^2 \\
& \left.-2h(3h-4)s+h-2\right)K(\sqrt{h})^3\Big\}
\end{aligned}
$$

これが正から負に1回だけ符号変化することを示せばよい

$$
\begin{aligned}
g(h,s) := \Big\{ & 2\left(h^2\left(h^2-h+1\right)s^4-2h^2(h+1)s^3+6h^2s^2-2h(h+1)s+h^2-h+1\right)^2 E(\sqrt{h})^3 \\
& +(1-h)\left(-3h^4(2-h)\left(h^2-h+1\right)s^8+6h^4\left(3h^2-3h+4\right)s^7-2h^4\left(23h^2-6h+25\right)s^6\right. \\
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& +(1-h)^2\left(3h^4\left(h^2-2h+2\right)s^8-12h^4\left(h^2-2h+2\right)s^7+4h^4\left(3h^2-4h+4\right)s^6\right. \\
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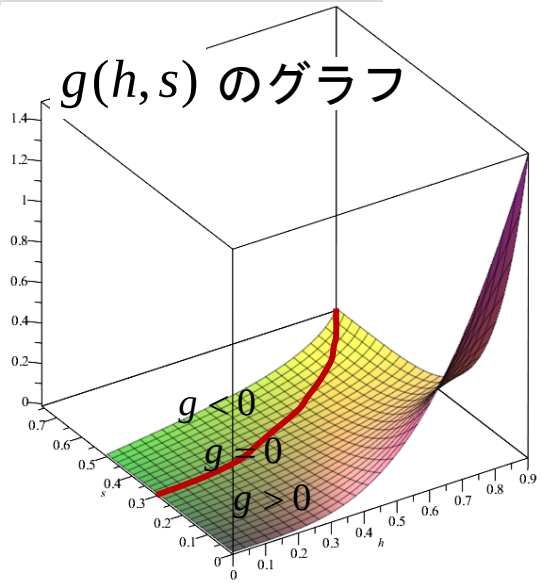
\end{aligned}
$$

これが正から負に1回だけ符号変化することを示せばよい



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g(h,s) := \Big\{ & 2\left(h^2\left(h^2-h+1\right)s^4-2h^2(h+1)s^3+6h^2s^2-2h(h+1)s+h^2-h+1\right)^2 E(\sqrt{h})^3 \\
& +(1-h)\left(-3h^4(2-h)\left(h^2-h+1\right)s^8+6h^4\left(3h^2-3h+4\right)s^7-2h^4\left(23h^2-6h+25\right)s^6\right. \\
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& +(1-h)^2\left(3h^4\left(h^2-2h+2\right)s^8-12h^4\left(h^2-2h+2\right)s^7+4h^4\left(3h^2-4h+4\right)s^6\right. \\
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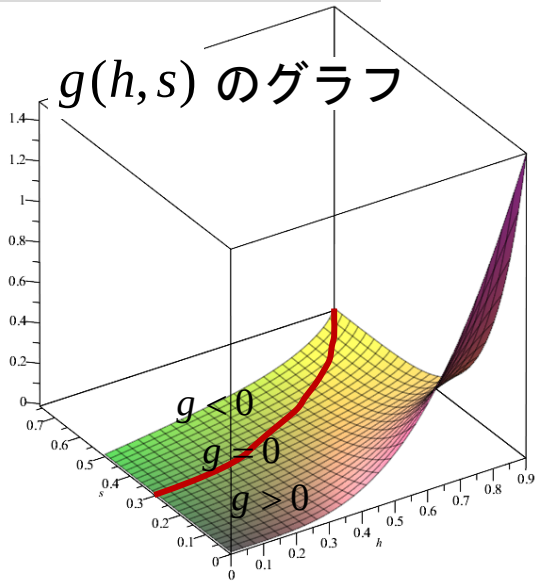
& \left.+4h^2\left(3h^2-4h+4\right)s^2-12h\left(h^2-2h+2\right)s+3h^2-6h+6\right)K(\sqrt{h})^2E(\sqrt{h}) \\
& +(1-h)^3\left(h^4(h-2)s^8-2h^4(3h-4)s^7+2h^4(3h+1)s^6-2h^3(5h+12)s^5\right. \\
& -2h^2\left(2h^2-21h-6\right)s^4-2h^2(5h+12)s^3+2h^2(3h+1)s^2 \\
& \left.-2h(3h-4)s+h-2\right)K(\sqrt{h})^3\Big\}
\end{aligned}
$$

これが正から負に1回だけ符号変化することを示せばよい

$h \in (0,1)$ をとめるごとに

$$g(h,s)=0 \quad \left(0<s<\frac{1}{1+\sqrt{1-h}}\right)$$

となる s がただ一つ存在することを示す.



フーリエの定理　$p(s)$ を実係数の n 次の多項式とする. $p(s)$ の導関数の列

$$p(s),\ p^{(1)}(s),\ p^{(2)}(s),\ \cdots,\ p^{(n)}(s)$$

の s における符号変化数を $v(s)$ とおく. $p(a)\neq 0,\ p(b)\neq 0$ ならば,

区間 (a,b) における $p(s)=0$ の実根の個数は $v(a)-v(b)$ に等しいか,

またそれより偶数だけ少ない.

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系　$v(a)-v(b)=1$ のとき,

区間 (a,b) における $p(s)=0$ の実根の個数は1である.

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各 $h\in(0,1)$ を固定するごとに, s についての8次多項式 $g(h,s)$ に対して,

区間 $\left(0,\ \dfrac{1}{1+\sqrt{1-h}}\right)$ において上記の系を適用する. したがって,

フーリエの定理　$p(s)$ を実係数の n 次の多項式とする. $p(s)$ の導関数の列

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$v(0):=$　$g(h,0),\ g_s^{(1)}(h,0),\cdots,g_s^{(8)}(h,0)$の符号変化数

$v\left(\dfrac{1}{1+\sqrt{1-h}}\right):=$　$g\left(h,\ \dfrac{1}{1+\sqrt{1-h}}\right),\ g_s^{(1)}\left(h,\ \dfrac{1}{1+\sqrt{1-h}}\right),\cdots,\ g_s^{(8)}\left(h,\ \dfrac{1}{1+\sqrt{1-h}}\right)$の符号変化数

とするとき

$$v(0)-v\left(\frac{1}{1+\sqrt{1-h}}\right)=1$$

を示せばよい.

各 $h \in (0,1)$ を固定するごとに, s についての 8次多項式 $g(h,s)$ に対する,

符号変化数 $v(0)$, $v\left(\frac{1}{1+\sqrt{1-h}}\right)$ を調べ, $v(0)-v\left(\frac{1}{1+\sqrt{1-h}}\right)=1$ を確認する.

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	$s=0$
	$0<h<1$
g	+
$g_s^{(1)}$	−
$g_s^{(2)}$	+
$g_s^{(3)}$	−
$g_s^{(4)}$	+
$g_s^{(5)}$	−
$g_s^{(6)}$	+
$g_s^{(7)}$	−
$g_s^{(8)}$	+
符号変化数	8

$s=\dfrac{1}{1+\sqrt{1-h}}$								
0	$\cdots$	h_1	$\cdots$	h_2	$\cdots$	h_3	$\cdots$	1
	−	−	−	−	−	−	−	
	+	+	+	+	+	+	+	
	+	0	−	−	−	−	−	
	−	−	−	0	+	+	+	
	+	+	+	+	+	0	−	
	−	−	−	−	−	−	−	
	+	+	+	+	+	+	+	
	−	−	−	−	−	−	−	
	+	+	+	+	+	+	+	
	7	7	7	7	7	7	7	

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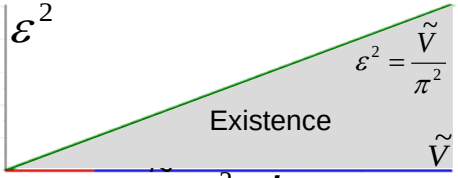
	$s=0$
	$0<h<1$
g	+
$g_s^{(1)}$	−
$g_s^{(2)}$	+
$g_s^{(3)}$	−
$g_s^{(4)}$	+
$g_s^{(5)}$	−
$g_s^{(6)}$	+
$g_s^{(7)}$	−
$g_s^{(8)}$	+
符号変化数	8

$s=\dfrac{1}{1+\sqrt{1-h}}$								
0	$\cdots$	h_1	$\cdots$	h_2	$\cdots$	h_3	$\cdots$	1
	−	−	−	−	−	−	−	
	+	+	+	+	+	+	+	
	+	0	−	−	−	−	−	
	−	−	−	0	+	+	+	
	+	+	+	+	+	0	−	
	−	−	−	−	−	−	−	
	+	+	+	+	+	+	+	
	−	−	−	−	−	−	−	
	+	+	+	+	+	+	+	
	7	7	7	7	7	7	7	

$$v(0)-v\left(\frac{1}{1+\sqrt{1-h}}\right)=8-7=1$$

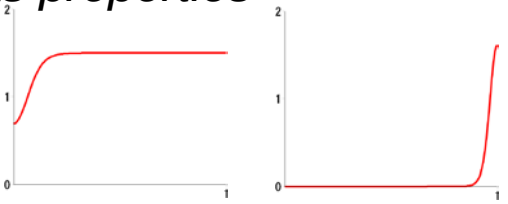
Theorem 1 *Let* $\tilde{V} > 0$. *There exists a solution of* $(AP;\tilde{V})$, *if and only if* $(\tilde{V},\ \varepsilon^2) \in G$, *where*

$$G := \left\{ (\tilde{V},\ \varepsilon^2) : 0 < \varepsilon^2 < \frac{\tilde{V}}{\pi^2} \right\}.$$



Moreover, the solution is unique. The solution $W(x;\tilde{V},\varepsilon^2)$ *has properties*

$$0 < W(x;\tilde{V},\varepsilon^2) < \tilde{V} + 1, \qquad W(x;\tilde{V},\varepsilon^2) = \tilde{V} + 1 - \tilde{V} \cdot W\left(1 - x\ ;\ \frac{1}{\tilde{V}},\ \frac{\varepsilon^2}{\tilde{V}^2}\right).$$



The solution $W(x;\tilde{V},\varepsilon^2)$ *is represented by*

$$W(x;\tilde{V},\varepsilon^2) = \frac{\tilde{V}+2}{3} + \frac{1}{\sqrt{3}} \cdot \sqrt{\tilde{V}^2 + \tilde{V} + 1} \cdot \frac{\beta \cdot (1-hs)\,\mathrm{sn}^2(K(\sqrt{h})x, \sqrt{h}) + \alpha \cdot \mathrm{cn}^2(K(\sqrt{h})x, \sqrt{h})}{(1-hs)\,\mathrm{sn}^2(K(\sqrt{h})x, \sqrt{h}) + \mathrm{cn}^2(K(\sqrt{h})x, \sqrt{h})},$$

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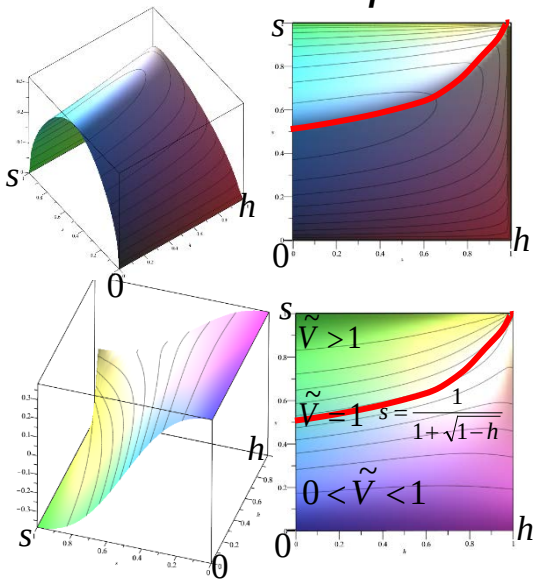
where $(h,\ s) = (h(\tilde{V},\ \varepsilon^2),\ s(\tilde{V},\ \varepsilon^2))$ *is the unique solution of the following system of transcendental equations*



$$\begin{cases} \dfrac{\sqrt{2s(1-s)(1-sh)}\,/\,K(\sqrt{h})}{\sqrt{3h^2s^4 - 4(h^2+h)s^3 + (4h^2+2h+4)s^2 - 4(h+1)s + 3}} = \sqrt{3} \cdot \dfrac{\varepsilon}{\sqrt{\tilde{V}^2 + \tilde{V} + 1}}, \\ \dfrac{2(hs^2 - sh + 1)(hs^2 - 2s + 1)(1 - hs^2)}{\sqrt{3h^2s^4 - 4(h^2+h)s^3 + (4h^2+2h+4)s^2 - 4(h+1)s + 3}^{\,3}} = \dfrac{1}{3\sqrt{3}} \cdot \dfrac{(1-\tilde{V})(2\tilde{V}+1)(\tilde{V}+2)}{\sqrt{\tilde{V}^2 + \tilde{V} + 1}^{\,3}}, \\ 0 < h < 1, \quad 0 < s < 1. \end{cases}$$

Here, $sn(\cdot,\cdot), cn(\cdot,\cdot)$ *are Jacobi's elliptic function,*

$K(\cdot)$ *is complete elliptic integral of the 1st kind.*

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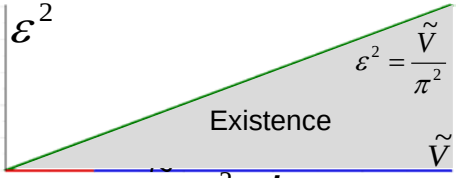

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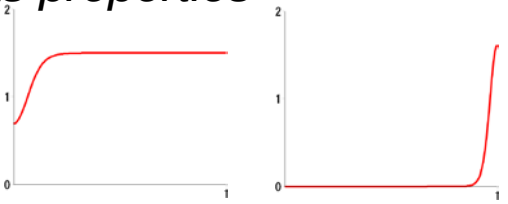

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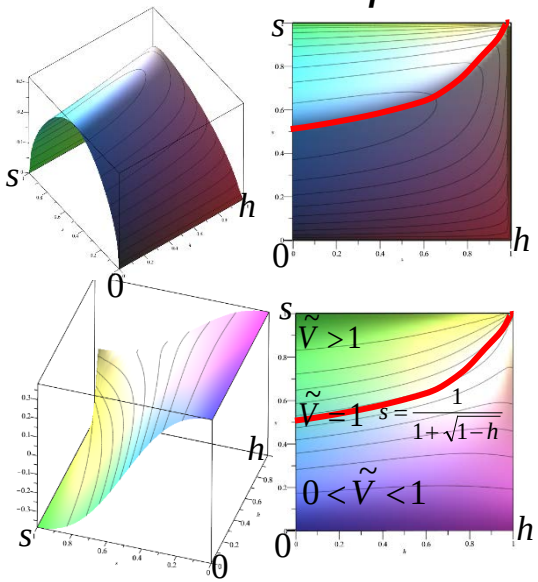

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POINT : Parameterization for all W in $(h,s) \in (0,1) \times (0,1)$

Here, $sn(\cdot,\cdot), cn(\cdot,\cdot)$ are Jacobi's elliptic function, $K(\cdot)$ is complete elliptic integral of the 1st kind.

Theorem 2 *Let* $W(x;\tilde{V},\varepsilon^2)$ *be the unique solution of* $(AP;\tilde{V})$, *and*

$$m(\tilde{V},\varepsilon^2) := \int_0^1 W(x;\tilde{V},\varepsilon^2)\,dx + \tilde{V},$$

then

$$m(\tilde{V},\varepsilon^2) = 2\tilde{V} + 2 - \tilde{V}\cdot m\left(\frac{1}{\tilde{V}}, \frac{\varepsilon^2}{\tilde{V}^2}\right) \qquad \text{for any} \quad \tilde{V} > 0, \quad \varepsilon > 0.$$

In particular, $m(1,\varepsilon^2) = 2$ *for any* $\varepsilon > 0$.

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In particular, $m(1,\varepsilon^2) = 2$ *for any* $\varepsilon > 0$.

Moreover, it holds that

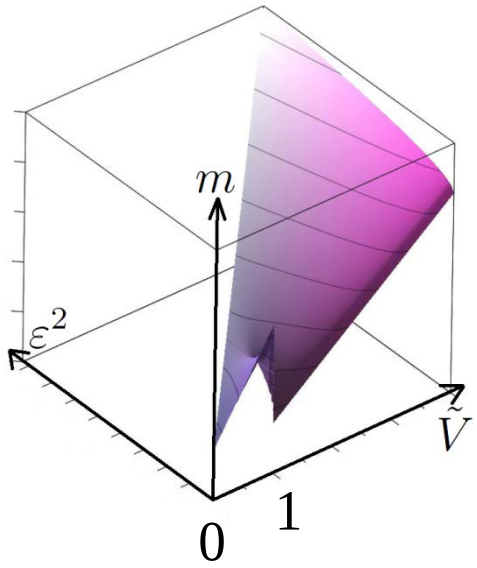
$$m(\tilde{V},\varepsilon^2) := \frac{4\tilde{V}+2}{3} + \frac{1}{\sqrt{3}}\cdot\sqrt{\tilde{V}^2+\tilde{V}+1}\cdot M(h,s)$$

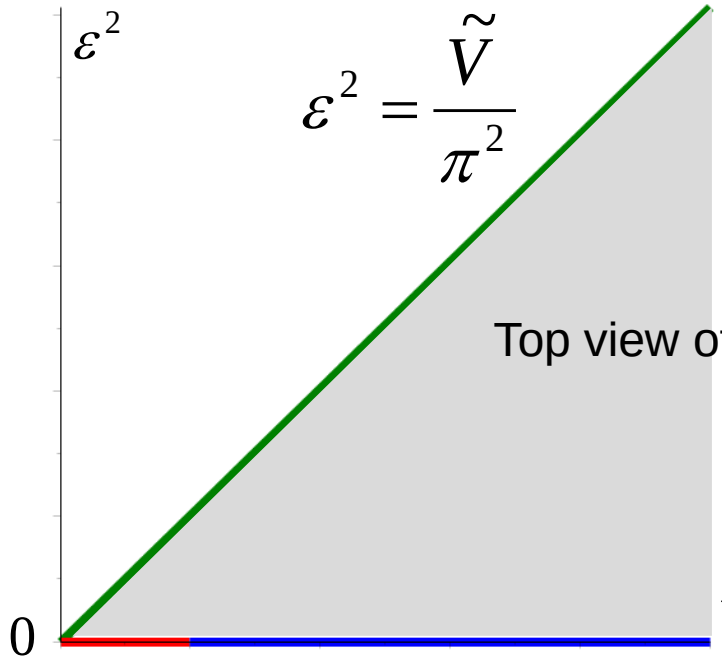
$$M(h,s) := \frac{-(hs^2-2(1+h)s+3)+4(1-s)(1-sh)\Pi(-sh,\sqrt{h})/K(\sqrt{h})}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}}.$$

where $h = h(\tilde{V},\varepsilon^2)$, $s = s(\tilde{V},\varepsilon^2)$ *are given in Theorem 1.*

Here, $K(\cdot)$ *is complete elliptic integral of the 1st kind,*

$\Pi(\cdot,\cdot)$ *is complete elliptic integral of the 3rd kind.*





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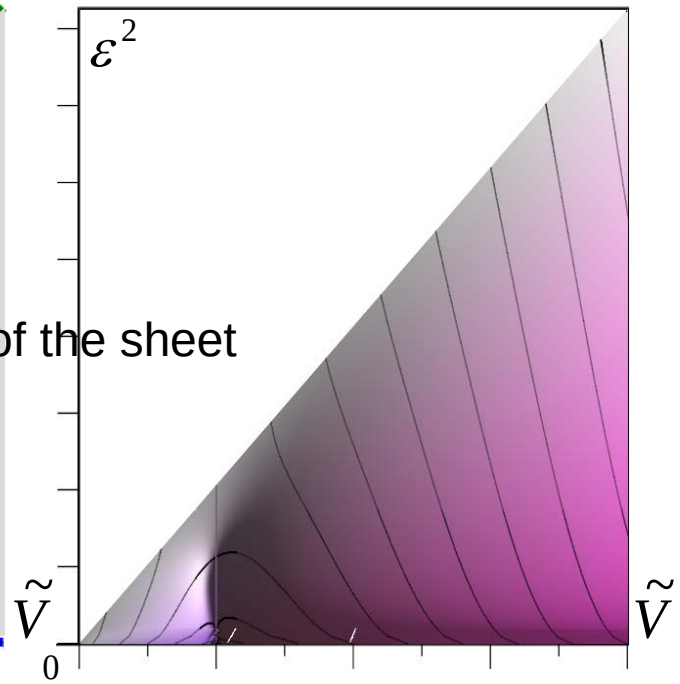
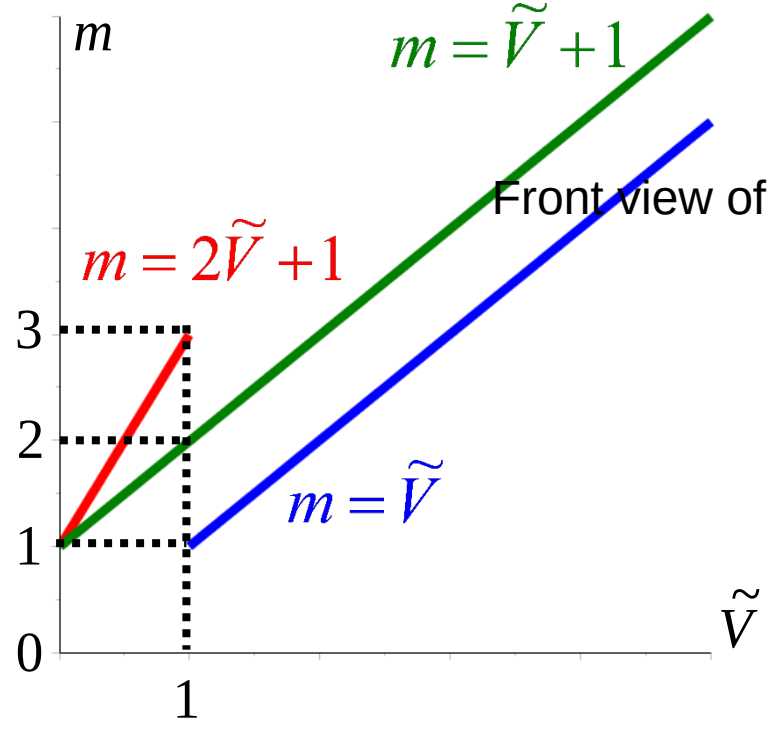
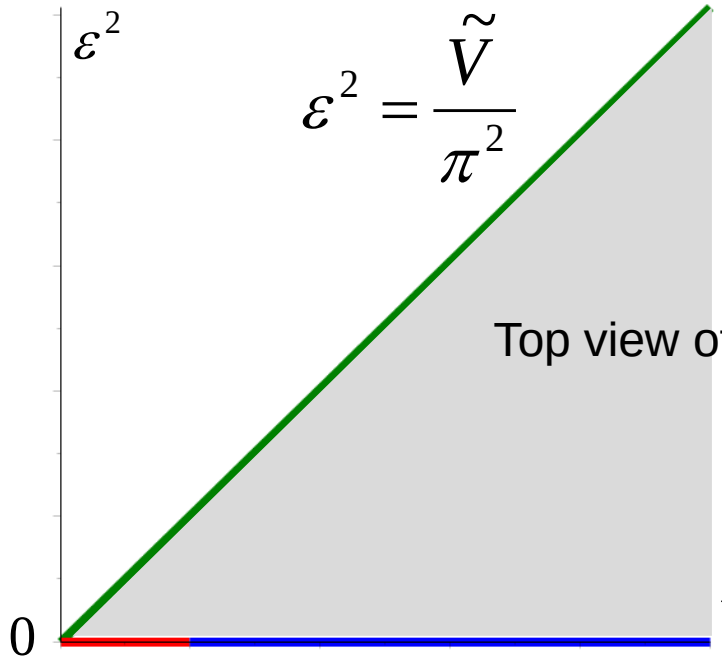




Front view of the sheet



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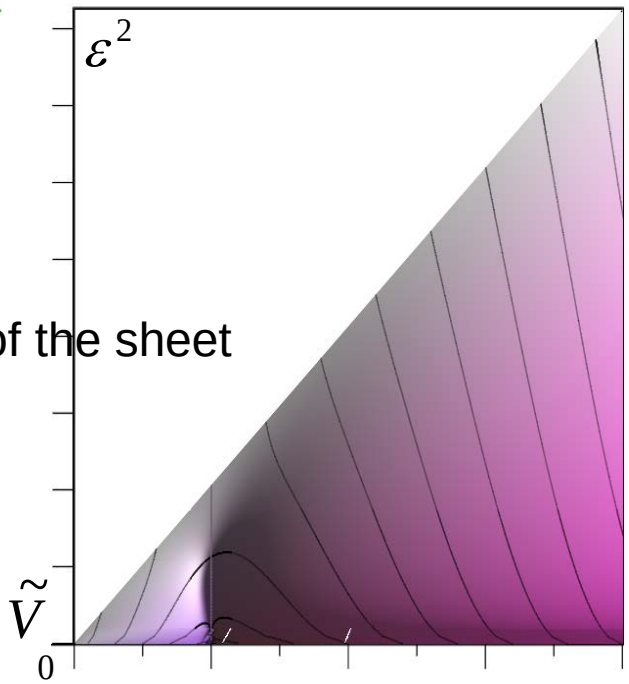
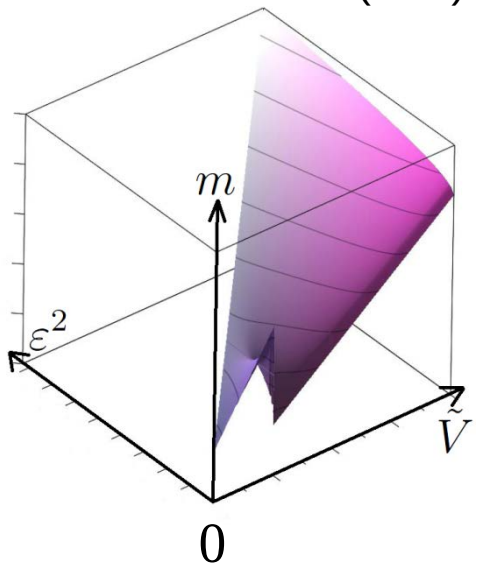


Bifurcation sheet for (SLP)



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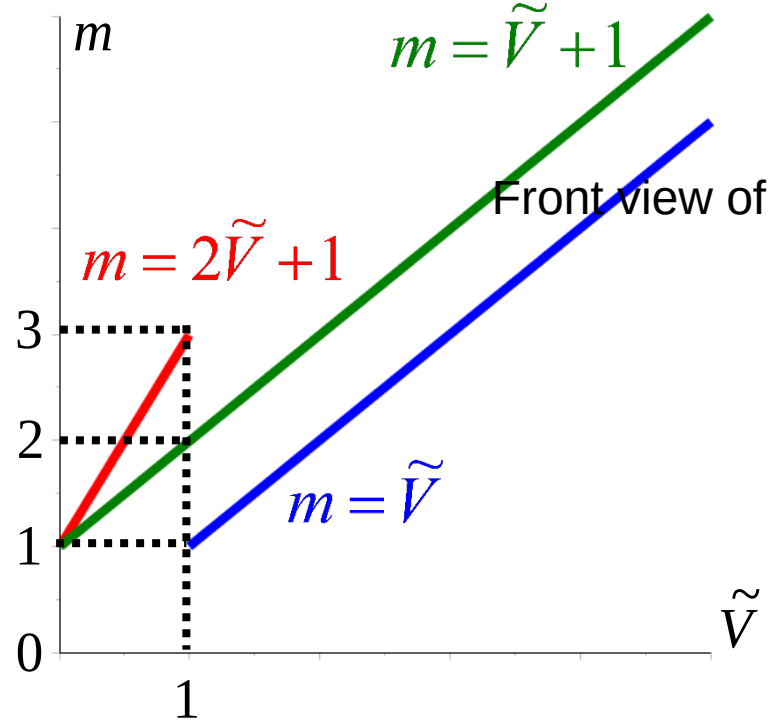

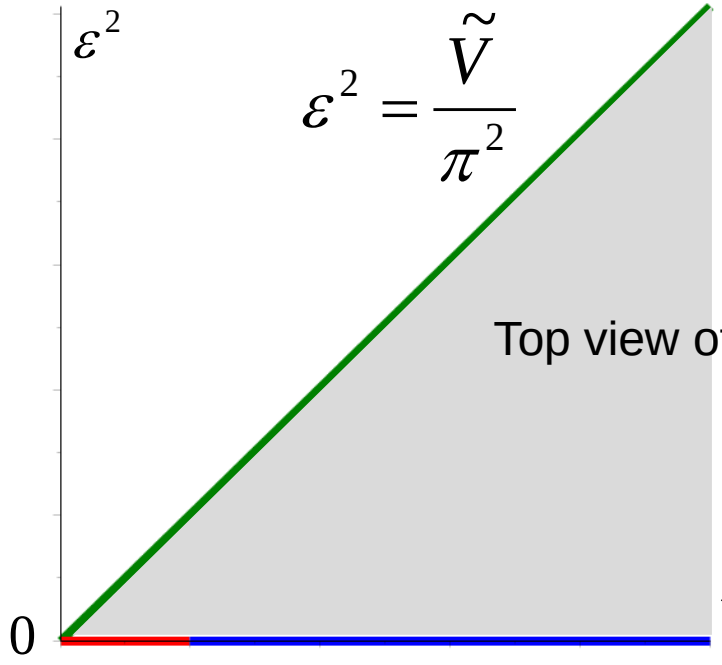

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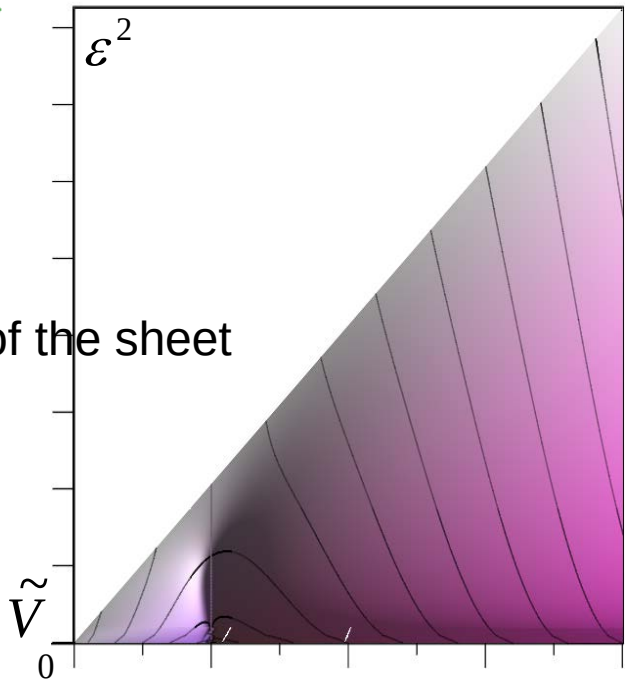

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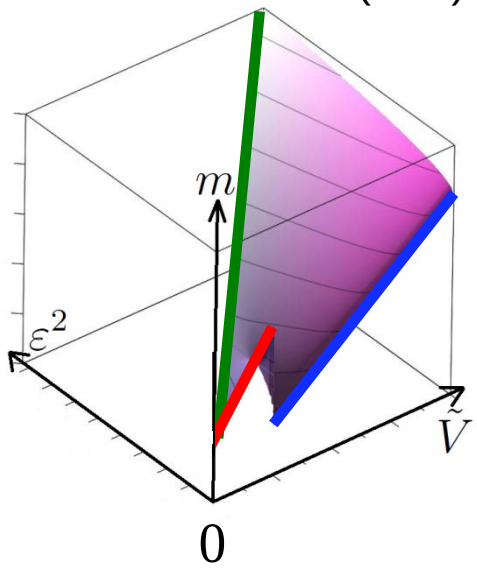

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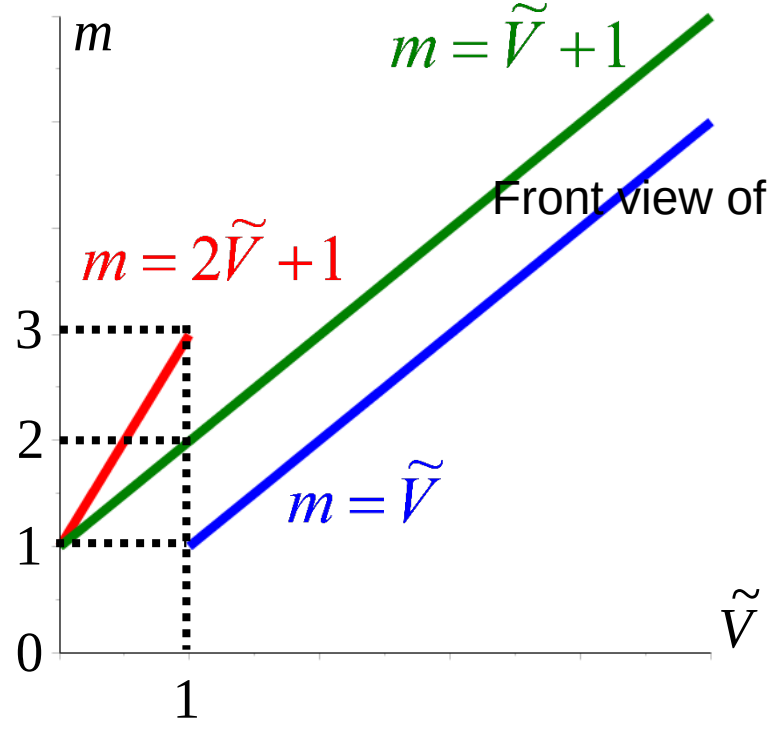

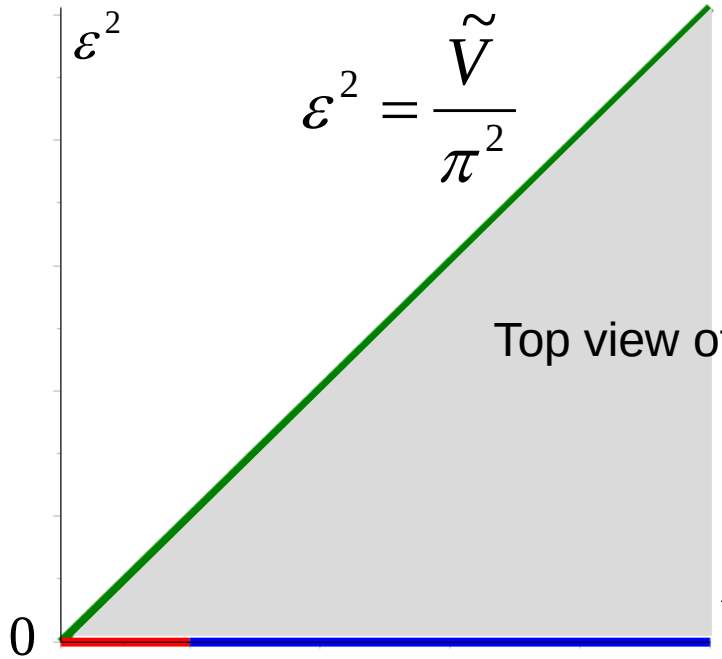

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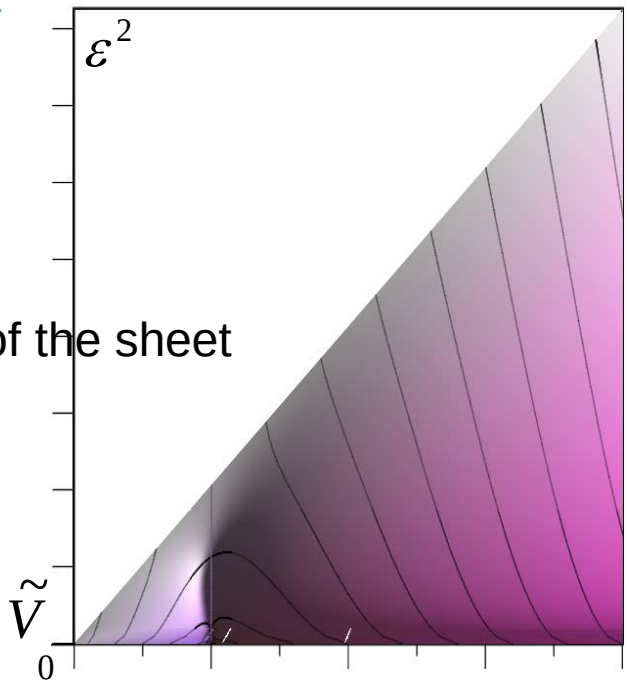

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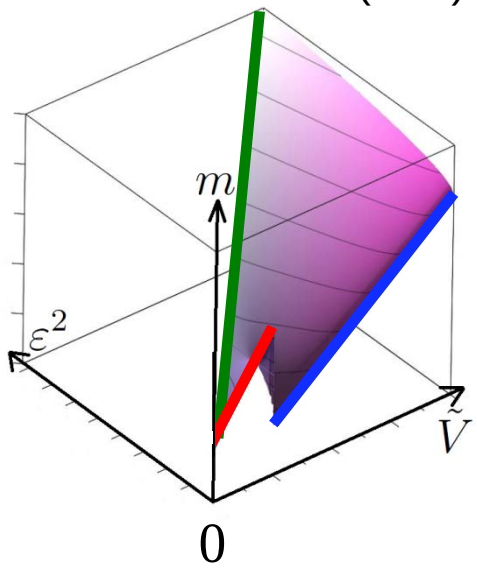

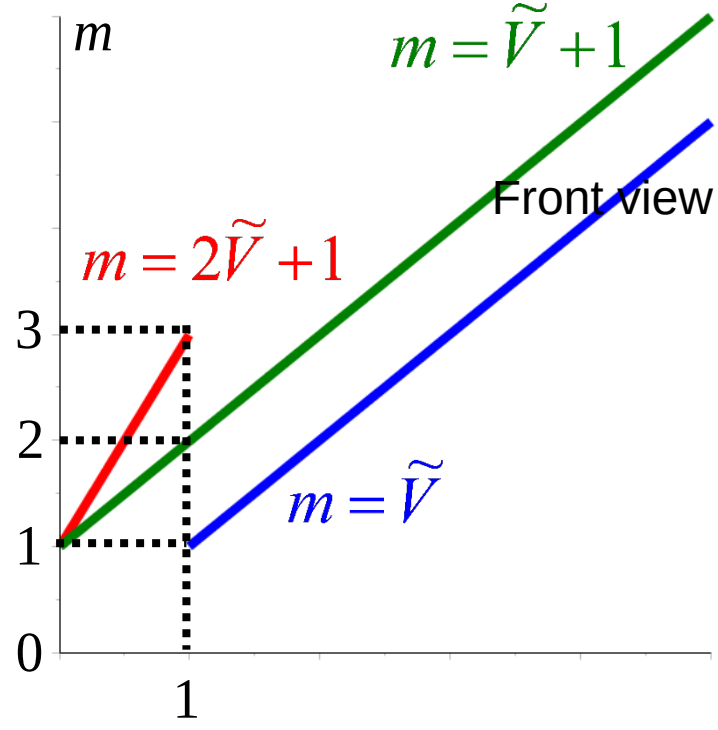

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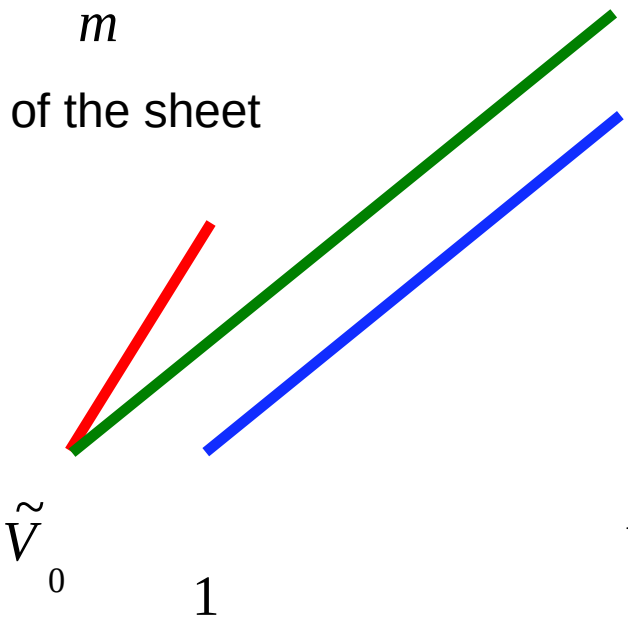
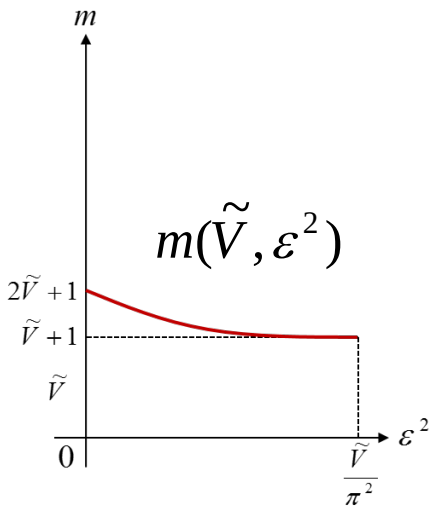


Theorem3 *Let* $\tilde{V} > 0$ *be fixed. It holds that*

$$\frac{\partial m(\tilde{V}, \varepsilon^2)}{\partial \varepsilon} < 0 \quad \textit{for} \quad \varepsilon^2 \in \left(0, \frac{\tilde{V}}{\pi^2}\right) \textit{ with } \tilde{V} \in (0, 1),$$



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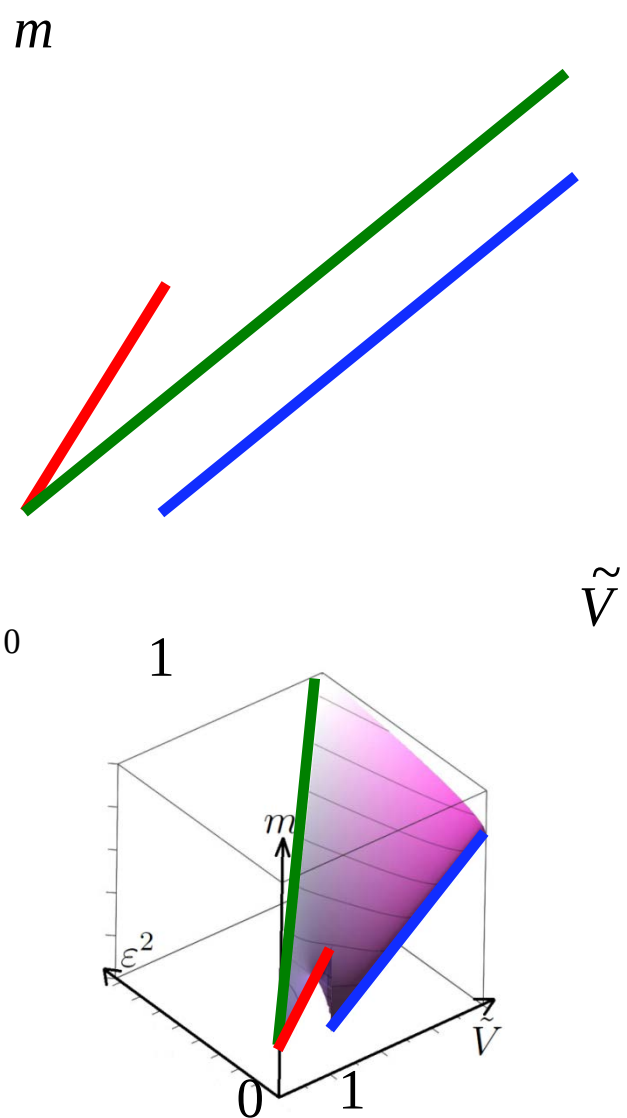

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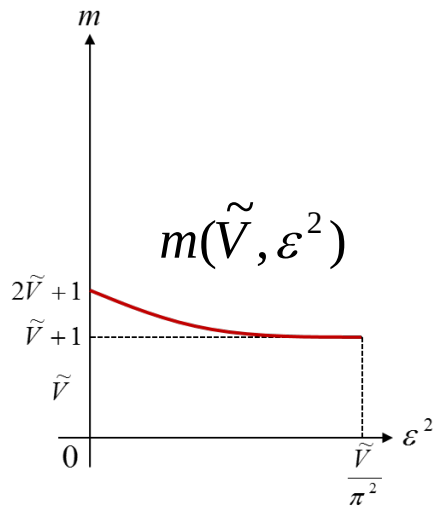

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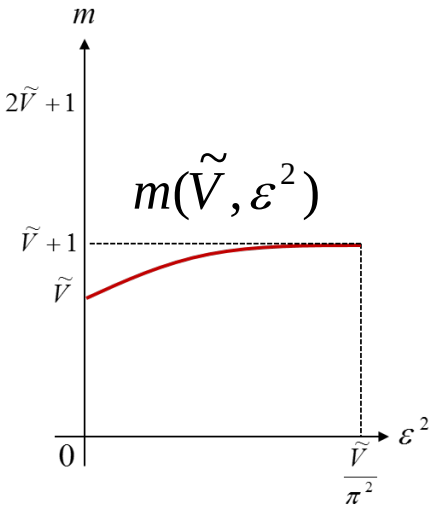
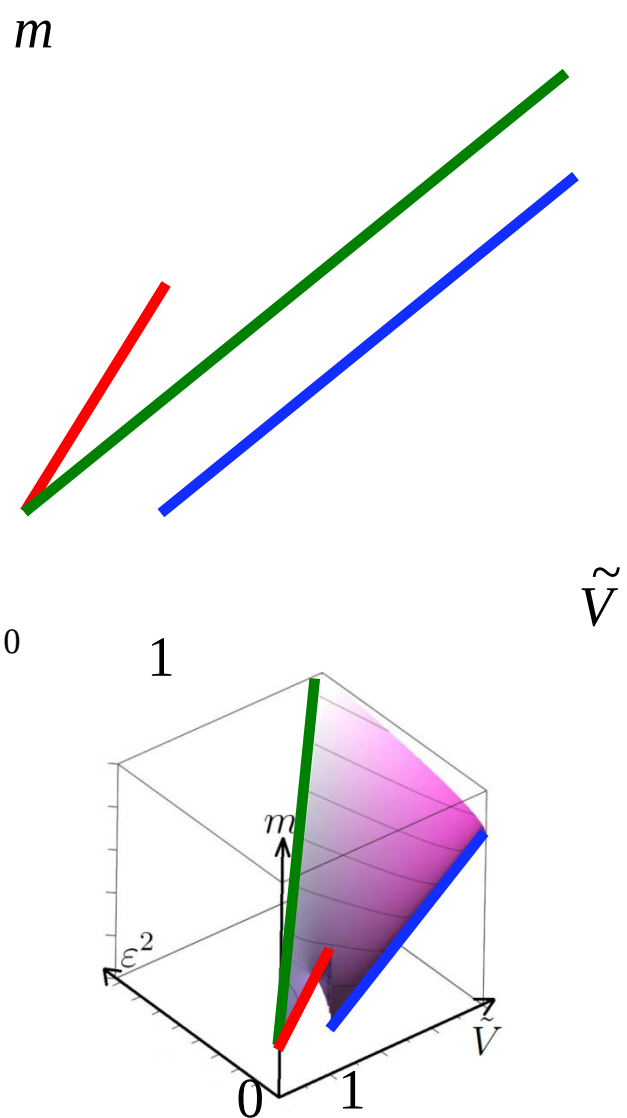
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