THE HARDY INEQUALITY FOR MEAN ZERO FUNCTIONS WITH LOGARITHMIC WEIGHT IN N-DIMENSIONS

JAEYOUNG BYEON, MEGUMI SANO, AND FUTOSHI TAKAHASHI

ABSTRACT. Let B be the unit ball in $\mathbb{R}^N$, $0 \in \Omega \subset B$ be a bounded domain with $R_\Omega = \sup_{x \in \Omega} |x| < 1$ and $\alpha, \beta \in \mathbb{R}$. In this paper, we study the Neumann version of the weighted Hardy inequality in N-dimensions, which involves the logarithmic weights. We show several results concerning the positivity, attainability, and non-attainability of the constant, according to the range $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$.

1. INTRODUCTION

Let B be the unit ball in $\mathbb{R}^N$, $0 \in \Omega \subset B$ be a bounded domain with $R_\Omega = \sup_{x \in \Omega} |x| < 1$ and $\alpha, \beta \in \mathbb{R}$. Put

$$Y_{\alpha,\beta}(\Omega) = \{ u \in C^\infty(\Omega) \mid \|u\|_{\alpha,\beta} < \infty \}, \quad Y_{0,\alpha,\beta}(\Omega) = \{ u \in C_0^\infty(\Omega) \mid \|u\|_{\alpha,\beta} < \infty \}$$

where

$$(1) \qquad \|u\|_{\alpha,\beta}^2 = \int_\Omega \left(|\nabla u|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} + \frac{u^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} \right) dx,$$

and define the spaces $\mathcal{H}_{\alpha,\beta}^{1,2}(\Omega)$ and $\mathcal{H}_{0,\alpha,\beta}^{1,2}(\Omega)$ by the completion of $Y_{\alpha,\beta}(\Omega)$ and $Y_{0,\alpha,\beta}(\Omega)$ with respect to the norm $\| \cdot \|_{\alpha,\beta}$ respectively. By this we mean that $u \in \mathcal{H}_{\alpha,\beta}^{1,2}(\Omega)$ (resp. $u \in \mathcal{H}_{0,\alpha,\beta}^{1,2}(\Omega)$) if and only if $u \in L^2(\Omega, \frac{1}{|x|^N (\log \frac{1}{|x|})^{2\alpha}})$, there exists a sequence $\{\phi_j\} \subset Y_{\alpha,\beta}(\Omega)$ (resp. $\{\phi_j\} \subset Y_{0,\alpha,\beta}(\Omega)$) and a $\mathbb{R}^N$-valued function $\vec{v} \in L^2(\Omega, |x|^{2-N} (\log \frac{1}{|x|})^{2\beta}; \mathbb{R}^N)$ such that

$$\int_\Omega \frac{|u - \phi_j|^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx \to 0 \quad (j \to \infty), \quad \text{and}$$

$$\int_\Omega |\vec{v} - \nabla \phi_j|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx \to 0 \quad (j \to \infty).$$

In Appendix, we prove that $\mathcal{H}_{\alpha,\beta}^{1,2}(\Omega) \subset L_{loc}^1(\Omega)$ and $\vec{v}$ coincides with the distributional derivative ∇u for any $\alpha, \beta \in \mathbb{R}$.

We consider the following minimization problems:

$$(2) \qquad H_{\alpha,\beta}(\Omega) := \inf \left\{ \int_\Omega |\nabla u|^2 |x|^{2-N} \left(\log \frac{1}{|x|} \right)^{2\beta} dx : u \in \mathcal{A}_D \right\}$$

$$(3) \qquad \mathcal{A}_D = \left\{ u \in \mathcal{H}_{0,\alpha,\beta}^{1,2}(\Omega) : \int_\Omega \frac{|u|^N}{|x|^N \left(\log \frac{1}{|x|} \right)^{2\alpha}} dx = 1 \right\}$$

and
(4)

$$N_{\alpha,\beta}(\Omega) := \inf \left\{ \int_\Omega |\nabla u|^2 |x|^{2-N} \left(\log \frac{1}{|x|} \right)^{2\beta} dx : u \in \mathcal{A}_N \right\}$$

(5)

$$\mathcal{A}_N = \left\{ u \in \mathcal{H}_{\alpha,\beta}^{1,2}(\Omega) : \int_\Omega \frac{|u|^2}{|x|^N \left(\log \frac{1}{|x|} \right)^{2\alpha}} dx = 1, \ \int_\Omega \frac{u}{|x|^N \left(\log \frac{1}{|x|} \right)^{2\alpha}} dx = 0 \right\}.$$

In this paper, we prove the following:

Theorem 1. $N_{\alpha,\beta}(\Omega) > 0$ *occurs if and only if* $\alpha + \beta \geq 1$ *and* $(\alpha, \beta) \neq (\frac{1}{2}, \frac{1}{2})$. *The same equivalence holds for* $H_{\alpha,\beta}(\Omega)$.

Theorem 2. *If* $\alpha + \beta > 1$, *then* $H_{\alpha,\beta}(\Omega)$ *is achieved.*

Theorem 3. *If* $\alpha + \beta > 1$ *and* $2\alpha + \beta > \frac{3}{2}$, *then* $N_{\alpha,\beta}(\Omega)$ *is achieved.*

Theorem 4. *If* $\alpha + \beta \geq 1$ *and* $2\alpha + \beta \leq \frac{3}{2}$, *then* $N_{\alpha,\beta}(\Omega)$ *is not achieved.*

Theorem 5. *If* $\alpha \geq \frac{1}{2}$, *then* $N_{\alpha,1-\alpha}(\Omega) \leq H_{\alpha,1-\alpha}(\Omega) = (\alpha - \frac{1}{2})^2$. *Furthermore, if*

$$N_{\alpha,1-\alpha}(\Omega) < H_{\alpha,1-\alpha}(\Omega),$$

then $N_{\alpha,1-\alpha}(\Omega)$ *is achieved.*

Let $\Omega = B_R$ denote a ball of radius $R \in (0, 1)$ with center 0,

Theorem 6. *For* $R \in (0, 1)$, *if*

$$\alpha > 1 + \sqrt{\frac{1}{4} + (N - 1)(\log \frac{1}{R})^2}$$

then $N_{\alpha,1-\alpha}(B_R)$ *is achieved.*

2. Results for $H_{\alpha,\beta}(\Omega)$.

In this section, we prove results concerning the constant $H_{\alpha,\beta}(\Omega)$ in (2). The following fact is now well-known.

Proposition 1. *Let* $\alpha \in \mathbb{R}$. *Then the inequality*
(6)

$$\left(\alpha - \frac{1}{2} \right)^2 \int_\Omega \frac{|u(x)|^2}{|x|^N \left(\log \frac{1}{|x|} \right)^{2\alpha}} dx \leq \int_\Omega \left| \frac{x}{|x|} \cdot \nabla u(x) \right|^2 |x|^{2-N} \left(\log \frac{1}{|x|} \right)^{2(1-\alpha)} dx$$

holds for any $u \in \mathcal{H}_{0,\alpha,1-\alpha}(\Omega)$. Moreover,

$$H_{\alpha,1-\alpha}(\Omega) = \left(\alpha - \frac{1}{2}\right)^2$$

it is not attained by any element in $\mathcal{H}_{0,\alpha,1-\alpha}(\Omega)$.

Proof. We give here a sketch of proof only. By density, it is enough to prove (6) for $u \in Y_{0,\alpha,1-\alpha}(\Omega)$. By a formula

$$\operatorname{div}\left(\frac{x}{|x|^N(\log\frac{1}{|x|})^{2\alpha-1}}\right) = \frac{2\alpha - 1}{|x|^N(\log\frac{1}{|x|})^{2\alpha}} \quad (x \in \mathbb{R}^N \setminus \{0\}),$$

we obtain

$$\left|(2\alpha - 1)\int_\Omega \frac{|u|^2}{|x|^N(\log\frac{1}{|x|})^{2\alpha}}dx\right| = \left|\int_\Omega |u|^2\operatorname{div}\left(\frac{x}{|x|^N(\log\frac{1}{|x|})^{2\alpha-1}}\right)dx\right|$$

$$= 2\left|\int_\Omega u\nabla u \cdot \frac{x}{|x|^N(\log\frac{1}{|x|})^{2\alpha-1}}dx\right|$$

$$\leq 2\left(\int_\Omega \frac{|u|^2}{|x|^N(\log\frac{1}{|x|})^{2\alpha}}dx\right)^{1/2}\left(\int_\Omega \left|\frac{x}{|x|}\cdot\nabla u\right|^2|x|^{2-N}\left(\log\frac{1}{|x|}\right)^{2(1-\alpha)}dx\right)^{1/2}.$$

By this, we have the inequality for $u \in Y_{0,\alpha,1-\alpha}(\Omega)$. The fact $H_{\alpha,1-\alpha}(\Omega) = \left(\alpha - \frac{1}{2}\right)^2$ is proven by a test function of the form

$$\psi_k(x) = \begin{cases} (\log\frac{1}{|x|})^{\alpha-1/2-\frac{1}{k}} - (\log\frac{1}{\delta})^{\alpha-1/2-\frac{1}{k}}, & 0 \leq |x| \leq \delta, \\ 0, & x \in \Omega \cap \{|x| \geq \delta\} \end{cases}$$

for $k \in \mathbb{N}$, where $\delta \in (0,1)$ is a number such that $B_\delta \subset\subset \Omega$. Note that ψ_k is positive and decreasing (resp. negative and increasing) when $\alpha > 1/2$ (resp. $\alpha < 1/2$) and $k \in \mathbb{N}$ large, We see that $\psi_k \in \mathcal{H}_{0,\alpha,1-\alpha}(\Omega)$; this is assured easily when $\alpha < 1/2$ and by Lemma 3 in Appendix and a standard truncation procedure when $\alpha > 1/2$.

If $(\alpha - \frac{1}{2})^2$ were attained, we see from the above inequality that u must satisfy

$$u_r r^{(2-N)/2}\left(\log\frac{1}{r}\right)^{1-\alpha} = \pm(\alpha - \frac{1}{2})u r^{-N/2}\left(\log\frac{1}{r}\right)^{-\alpha}.$$

This implies that for some $c \in \mathbb{R}$,

$$u = c(\log\frac{1}{r})^{\alpha-\frac{1}{2}}.$$

However, since $(\log\frac{1}{|x|})^{\alpha-\frac{1}{2}} \notin \mathcal{H}^{1,2}_{0,\alpha,1-\alpha}(\Omega)$, we conclude that $H_{\alpha,1-\alpha}(\Omega)$ is never achieved by an element in $\mathcal{H}^{1,2}_{0,\alpha,1-\alpha}(\Omega)$. $\square$

Next, we introduce a nonlinear scaling related to the Hardy inequality with logarithmic weight in $\mathbb{R}^N$: Let $\lambda > 0$ and for a function $u = u(x) \in \mathcal{H}^{1,2}_{0,\alpha,\beta}(\Omega)$, we define a new function

(7) $$u_\lambda(y) = u(x), \quad y = |x|^\lambda\frac{x}{|x|} \in \Omega_\lambda := \{|x|^\lambda\frac{x}{|x|} : x \in \Omega\}.$$

Note that if $\Omega = B_\delta$, then $(B_\delta)_\lambda = B_{\delta^\lambda}$ is a ball with radius δ^λ. In polar coordinate $x = r\omega$, $r = |x|$ and $\omega = \frac{x}{|x|} \in \mathbb{S}^{N-1}$,

$$u_\lambda(y) = u_\lambda(s,\omega) = u(r,\omega)$$

where $s = |y| = r^\lambda$. By a direct computation using polar coordinates, we have the following scaling property.

Proposition 2. *For $\lambda > 0$, define u_λ as in* (7). *Then we have*

$$\int_{\Omega_\lambda} \left| \frac{y}{|y|} \cdot \nabla u_\lambda(y) \right|^2 |y|^{2-N} \left(\log \frac{1}{|y|} \right)^{2\beta} dy = \lambda^{2\beta-1} \int_\Omega \left| \frac{x}{|x|} \cdot \nabla u(x) \right|^2 |x|^{2-N} \left(\log \frac{1}{|x|} \right)^{2\beta} dx,$$

$$\int_{\Omega_\lambda} \frac{|u_\lambda(y)|^2}{|y|^N \left(\log \frac{1}{|y|} \right)^{2\alpha}} dy = \lambda^{1-2\alpha} \int_\Omega \frac{|u(x)|^2}{|x|^N \left(\log \frac{1}{|x|} \right)^{2\alpha}} dx,$$

$$\int_{\Omega_\lambda} \frac{1}{|y|^N} \left| \nabla_{\mathbb{S}^{N-1}} u_\lambda(y) \right|^2 \left(\log \frac{1}{|y|} \right)^{2\beta} dy = \lambda^{2\beta+1} \int_\Omega \frac{1}{|x|^N} \left| \nabla_{\mathbb{S}^{N-1}} u(x) \right|^2 \left(\log \frac{1}{|x|} \right)^{2\beta} dx,$$

where $\nabla_{\mathbb{S}^{N-1}}$ denotes the gradient operator on $\mathbb{S}^{N-1}$.

Proof. By density, it is enough to prove the formulea for $u \in Y_{0,\alpha,\beta}(\Omega)$. By extending u_λ as $u_\lambda \equiv 0$ outside of $\Omega \subset\subset B$, we may consider that u_λ is a function on B. Then we have

$$\int_{\Omega_\lambda} \left| \frac{y}{|y|} \cdot \nabla u_\lambda(y) \right|^2 |y|^{2-N} \left(\log \frac{1}{|y|} \right)^{2\beta} dy = \int_B \left| \frac{y}{|y|} \cdot \nabla u_\lambda(y) \right|^2 |y|^{2-N} \left(\log \frac{1}{|y|} \right)^{2\beta} dy$$

$$= \int_{\mathbb{S}^{N-1}} \int_0^1 \left| \frac{\partial u_\lambda}{\partial s}(s,\omega) \right| (\log \frac{1}{s})^{2\beta} s \, ds dS_\omega$$

$$= \int_{\mathbb{S}^{N-1}} \int_0^1 \left| \frac{\partial u}{\partial r}(r,\omega) \underbrace{\left(\frac{ds}{dr} \right)^{-1}}_{=\frac{1}{\lambda} r^{1-\lambda}} \right|^2 (\log \frac{1}{r^\lambda})^{2\beta} r^\lambda \cdot \lambda r^{\lambda-1} \, dr dS_\omega$$

$$= \lambda^{2\beta-1} \int_{\mathbb{S}^{N-1}} \int_0^1 \left| \frac{\partial u}{\partial r}(r,\omega) \right|^2 (\log \frac{1}{r})^{2\beta} r \, dr dS_\omega = \lambda^{2\beta-1} \int_B \left| \frac{x}{|x|} \cdot \nabla u(x) \right|^2 |x|^{2-N} \left(\log \frac{1}{|x|} \right)^{2\beta} dx$$

$$= \lambda^{2\beta-1} \int_\Omega \left| \frac{x}{|x|} \cdot \nabla u(x) \right|^2 |x|^{2-N} \left(\log \frac{1}{|x|} \right)^{2\beta} dx.$$

Similarly, we see

$$\int_{\Omega_\lambda} \frac{|u_\lambda(y)|^2}{|y|^N \left(\log \frac{1}{|y|} \right)^{2\alpha}} dy = \int_B \frac{|u_\lambda(y)|^2}{|y|^N \left(\log \frac{1}{|y|} \right)^{2\alpha}} dy$$

$$= \int_{\mathbb{S}^{N-1}} \int_0^1 \frac{|u_\lambda(s,\omega)|^2}{s^N (\log \frac{1}{s})^{2\alpha}} s^{N-1} \, ds dS_\omega = \int_{\mathbb{S}^{N-1}} \int_0^1 \frac{|u(r,\omega)|^2}{r^{N\lambda}(\log \frac{1}{r^\lambda})^{2\alpha}} r^{\lambda(N-1)} \cdot \lambda r^{\lambda-1} \, dr dS_\omega$$

$$= \lambda^{1-2\alpha} \int_{\mathbb{S}^{N-1}} \int_0^1 \frac{|u(r,\omega)|^2}{r^N (\log \frac{1}{r})^{2\alpha}} r^{N-1} \, dr dS_\omega = \lambda^{1-2\alpha} \int_B \frac{|u(x)|^2}{|x|^N \left(\log \frac{1}{|x|} \right)^{2\alpha}} dx$$

$$= \lambda^{1-2\alpha} \int_\Omega \frac{|u(x)|^2}{|x|^N \left(\log \frac{1}{|x|} \right)^{2\alpha}} dx,$$

and

$$\int_{\Omega_\lambda} \frac{1}{|y|^N} \left|\nabla_{\mathbb{S}^{N-1}} u_\lambda(y)\right|^2 \left(\log \frac{1}{|y|}\right)^{2\beta} dy = \int_B \frac{1}{|y|^N} \left|\nabla_{\mathbb{S}^{N-1}} u_\lambda(y)\right|^2 \left(\log \frac{1}{|y|}\right)^{2\beta} dy$$

$$\int_{\mathbb{S}^{N-1}} \int_0^1 \frac{1}{s^N} \left|\nabla_{\mathbb{S}^{N-1}} u_\lambda(s,\omega)\right|^2 \left(\log \frac{1}{s}\right)^{2\beta} s^{N-1} ds dS_\omega$$

$$= \int_{\mathbb{S}^{N-1}} \int_0^1 \frac{1}{r^{N\lambda}} \left|\nabla_{\mathbb{S}^{N-1}} u(r,\omega)\right|^2 \left(\log \frac{1}{r^\lambda}\right)^{2\beta} r^{(N-1)\lambda} \cdot \lambda r^{\lambda-1} dS_\omega$$

$$= \lambda^{2\beta+1} \int_B \frac{1}{|x|^N} \left|\nabla_{\mathbb{S}^{N-1}} u(x)\right|^2 \left(\log \frac{1}{|x|}\right)^{2\beta} dx$$

$$= \lambda^{2\beta+1} \int_\Omega \frac{1}{|x|^N} \left|\nabla_{\mathbb{S}^{N-1}} u(x)\right|^2 \left(\log \frac{1}{|x|}\right)^{2\beta} dx.$$

□

Recall the orthogonal decomposition of the gradient operator ∇ in $\mathbb{R}^N$:

$$\nabla = \omega \frac{\partial}{\partial r} + \frac{1}{r} \nabla_{\mathbb{S}^{N-1}}, \quad \omega \in \mathbb{S}^{N-1}, \quad \omega \cdot \nabla_{\mathbb{S}^{N-1}} \equiv 0.$$

Thus if $Q_{\alpha,\beta}(u,\Omega)$ denotes the critical Hardy quotient

$$Q_{\alpha,\beta}(u,\Omega) = \frac{\int_\Omega |\nabla u(x)|^2 |x|^{2-N} \left(\log \frac{1}{|x|}\right)^{2\beta} dx}{\int_\Omega \frac{|u(x)|^2}{|x|^N \left(\log \frac{1}{|x|}\right)^{2\alpha}} dx},$$

then we have

(8)
$$\frac{\int_{\Omega_\lambda} |\nabla u_\lambda(y)|^2 |y|^{2-N} \left(\log \frac{1}{|y|}\right)^{2\beta} dy}{\int_{\Omega_\lambda} \frac{|u_\lambda(y)|^2}{|y|^N \left(\log \frac{1}{|y|}\right)^{2\alpha}} dy}$$

$$= \lambda^{2(\alpha+\beta-1)} \frac{\int_\Omega \left|\frac{x}{|x|} \cdot \nabla u(x)\right|^2 |x|^{2-N} \left(\log \frac{1}{|x|}\right)^{2\beta} dx}{\int_\Omega \frac{|u(x)|^2}{|x|^N \left(\log \frac{1}{|x|}\right)^{2\alpha}} dx}$$

$$+ \lambda^{2(\alpha+\beta)} \frac{\int_\Omega \left|\nabla_{\mathbb{S}^{N-1}} u(x)\right|^2 |x|^{-N} \left(\log \frac{1}{|x|}\right)^{2\beta} dx}{\int_\Omega \frac{|u(x)|^2}{|x|^N \left(\log \frac{1}{|x|}\right)^{2\alpha}} dx}.$$

For $\lambda \in (0,1)$, we have $\lambda^{2(\alpha+\beta-1)} > \lambda^{2(\alpha+\beta)}$ for any $\alpha, \beta \in \mathbb{R}$, thus we have an estimate

(9)
$$Q_{\alpha,\beta}(u_\lambda, \Omega_\lambda) \le \lambda^{2(\alpha+\beta-1)} Q_{\alpha,\beta}(u,\Omega), \quad \lambda \in (0,1).$$

Proposition 3. *If $\alpha + \beta < 1$, then $H_{\alpha,\beta}(\Omega) = 0$.*

Proof. Since the origin is in the interior of Ω, there is $\delta_0 \in (0,1)$ such that $B_{\delta_0} \subset\subset \Omega$. We choose a radial smooth function $\psi \in C_0^\infty(B_{\delta_0} \setminus \{0\})$ such that $\|\psi\|_{\alpha,\beta} < \infty$ and consider its scaled function $\psi_\lambda \in C_0^\infty((B_{\delta_0} \setminus \{0\})_\lambda)$. Since $(B_{\delta_0})_\lambda = B_{\delta_0^\lambda} \subset B_{\delta_0}$ for any $\lambda > 1$, we can consider that $\psi_\lambda \in C_0^\infty(\Omega)$ and $\|\psi_\lambda\|_{\alpha,\beta} < \infty$ by extending zero outside of $B_{\delta_0^\lambda}$. Thus by testing $H_{\alpha,\beta}(\Omega)$ by ψ_λ and using Proposition 2 (8), we see

$$H_{\alpha,\beta}(\Omega) \le Q(\psi_\lambda, \Omega) = Q(\psi_\lambda, (B_{\delta_0})_\lambda) = \lambda^{2(\alpha+\beta-1)} Q(\psi, B_{\delta_0}) \to 0$$

as $\lambda \to +\infty$ if $\alpha + \beta < 1$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proposition 4. *If $\alpha + \beta \geq 1$ and $\alpha \neq \frac{1}{2}$, then $H_{\alpha,\beta}(\Omega) > 0$.*

Proof. Let $u \in \mathcal{H}_{0,\alpha,\beta}^{1,2}(\Omega)$. When $\alpha + \beta \geq 1$, we see

$$\int_\Omega |\nabla u|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx = \int_\Omega |\nabla u|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2(1-\alpha)} (\log \frac{1}{|x|})^{2(\alpha+\beta-1)} dx$$

$$\geq \left(\log \frac{1}{R_\Omega} \right)^{2(\alpha+\beta-1)} \int_\Omega |\nabla u|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2(1-\alpha)} dx$$

where $R_\Omega = \sup_{x \in \Omega} |x| < 1$. Thus we see

$$\mathcal{H}_{0,\alpha,\beta}^{1,2}(\Omega) \hookrightarrow \mathcal{H}_{0,\alpha,1-\alpha}^{1,2}(\Omega)$$

and for any $u \in \mathcal{H}_{0,\alpha,\beta}^{1,2}(\Omega)$,

$$Q_{\alpha,\beta}(u,\Omega) \geq \left(\log \frac{1}{R_\Omega} \right)^{2(\alpha+\beta-1)} Q_{\alpha,1-\alpha}(u,\Omega) \geq \left(\log \frac{1}{R_\Omega} \right)^{2(\alpha+\beta-1)} \underbrace{H_{\alpha,1-\alpha}(\Omega)}_{=(\alpha-1/2)^2}.$$

Thus we have $H(\alpha,\beta)(\Omega) > 0$ if $\alpha + \beta \geq 1$ and $\alpha \neq \frac{1}{2}$. $\qquad\qquad\square$

Proposition 5. *$H_{\frac{1}{2},\beta}(\Omega) > 0$ for $\beta > \frac{1}{2}$.*

Proof. Note that when $\alpha = \frac{1}{2}$, the weight $\frac{1}{|x|^N (\log \frac{1}{|x|})} \notin L^1(\Omega)$. Then if $u \not\equiv 0$ and $u \in Y_{0,1/2,\beta}(\Omega)$, then u must satisfy that $u(0) = 0$. Thus by Taylor expansion, u can be written as $u(r\omega) = r^m \phi(\omega) + O(r^{m+1})$ near $r = |x| = 0$, where ϕ is a smooth function on $\mathbb{S}^{N-1}$ and $m \in \mathbb{N}$. Since $m \geq 1$, we see

$$\int_0^\delta \frac{r^{2m}}{r^N (\log \frac{1}{r})^{2(1-\beta)}} r^{N-1} dr < \infty$$

for any $\beta \in \mathbb{R}$ and for small $\delta \in (0,1)$, which implies that

$$\int_{B_\delta} \frac{|u(x)|^2}{|x|^N (\log \frac{1}{|x|})^{2(1-\beta)}} dx < \infty.$$

Also we see $\int_\Omega |\nabla u|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx < \infty$ by the definition of $Y_{0,1/2,\beta}(\Omega)$. This implies that

$$Y_{0,1/2,\beta}(\Omega) \subset Y_{0,1-\beta,\beta}(\Omega) \quad (\forall \beta \in \mathbb{R}).$$

Now we assume $\beta \geq \frac{1}{2}$. In this case, we have

$$\int_\Omega \frac{|u|^2}{|x|^N (\log \frac{1}{|x|})} dx = \int_\Omega \frac{|u|^2}{|x|^N (\log \frac{1}{|x|})^{2(1-\beta)}} (\log \frac{1}{|x|})^{1-2\beta} dx$$

$$\leq \left(\log \frac{1}{R_\Omega} \right)^{1-2\beta} \int_\Omega \frac{|u|^2}{|x|^N (\log \frac{1}{|x|})^{2(1-\beta)}} dx$$

for $u \in Y_{0,1/2,\beta}(\Omega) \subset Y_{0,1-\beta,\beta}(\Omega)$. Thus testing $H_{1-\beta,\beta}(\Omega)$ by u, we see

$$\int_\Omega |\nabla u|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx \geq H_{1-\beta,\beta}(\Omega) \int_\Omega \frac{|u|^2}{|x|^N (\log \frac{1}{|x|})^{2(1-\beta)}} dx$$

$$\geq H_{1-\beta,\beta}(\Omega) \left(\log \frac{1}{R_\Omega} \right)^{2\beta-1} \int_\Omega \frac{|u|^2}{|x|^N (\log \frac{1}{|x|})} dx.$$

From this, we have

$$Q_{\frac{1}{2},\beta}(u,\Omega) \geq \underbrace{H_{1-\beta,\beta}(\Omega)}_{=(\beta-1/2)^2} \left(\log \frac{1}{R_\Omega} \right)^{2\beta-1}$$

for $\beta \geq \frac{1}{2}$ and for any $u \in Y_{0,1/2,\beta}(\Omega)$. This implies that $H_{\frac{1}{2},\beta}(\Omega) > 0$ if $\beta > 1/2$. $\square$

Propositions 3, 4, 5 complete the proof of the latter half part of Theorem 1:

$$(10) \qquad H_{\alpha,\beta}(\Omega) > 0 \iff \begin{cases} \alpha + \beta \geq 1, \\ (\alpha, \beta) \neq (\frac{1}{2}, \frac{1}{2}). \end{cases}$$

Also if $\alpha + \beta > 1$, the scaling property (9) prevents the concentration of minimizing sequence of $H_{\alpha,\beta}(\Omega)$ at the origin. This gives the compactness for a minimizing sequence and enable us to prove Theorem 2:

Proof of Theorem 2.

Proof. Let $\{u_n\}$ be a minimizing sequence of $H_{\alpha,\beta}(\Omega)$ in $\mathcal{H}^{1,2}_{0,\alpha,\beta}(\Omega)$ such that

$$\int_\Omega \frac{u_n^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = 1, \quad \text{and} \quad \int_\Omega |\nabla u_n|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx = H_{\alpha,\beta}(\Omega) + o(1).$$

Then up to a subsequence, there is $u \in \mathcal{H}^{1,2}_{0,\alpha,\beta}(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } \mathcal{H}^{1,2}_{0,\alpha,\beta}(\Omega).$$

Since $u_n \to u$ in $L^2(\Omega \setminus B_\delta, \frac{1}{|x|^N (\log \frac{1}{|x|})^{2\alpha}})$ strongly as $n \to \infty$ for any small $\delta > 0$, we see that

$$\int_{B_\delta} \frac{u_n^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = \int_\Omega \frac{u_n^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx - \int_{\Omega \setminus B_\delta} \frac{u^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx + o_n(1)$$

$$(11) \qquad \geq 1 - \int_\Omega \frac{u^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx + o_n(1).$$

Now, we claim that

$$(12) \qquad \int_\Omega \frac{u^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = 1.$$

If (12) is the case, then $u \in \mathcal{A}_D$ defined in (3) and by the weak-lower semi-continuity of the norm, we have

$$1 + H_{\alpha,\beta}(\Omega) = \liminf_{n \to \infty} \|u_n\|^2_{\alpha,\beta} \geq \|u\|^2_{\alpha,\beta} = \underbrace{\int_\Omega \frac{u^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx}_{=1} + \int_\Omega |\nabla u|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx.$$

This implies that $\int_\Omega |\nabla u|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx \leq H_{\alpha,\beta}(\Omega)$ and u attains $H_{\alpha,\beta}(\Omega)$ in $\mathcal{A}_D$.

Now we show (12). Assume the contrary that $\int_\Omega \frac{u^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx < 1$ would hold. Then by (11), for arbitrary small $\delta > 0$, there is $n_0 = n_0(\delta) > 0$ such that

$$(13) \qquad \int_{B_\delta} \frac{u_n^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx \geq \frac{1}{2} \left(1 - \int_\Omega \frac{u^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx \right) > 0$$

for $n > n_0$.

Let $\delta_0 \in (0,1)$ be a number such that $B_{\delta_0} \subset\subset \Omega$. For $\delta \in (0, \delta_0)$ and fixed $s > 1$, we take a truncation function $g_{\delta,s}$ such that

$$(14) \qquad g_{\delta,s}(x) = \begin{cases} 1 & (0 \leq |x| \leq \delta^s), \\ \frac{1}{\log s} \left\{ \log\log \frac{1}{|x|} - \log\log \frac{1}{\delta} \right\} & (\delta^s \leq |x| \leq \delta), \\ 0 & (\Omega \cap \{\delta \leq |x|\}) \end{cases}$$

and define

$$(15) \qquad v_{n,\delta}(x) = g_{\delta,s}(x) u_n(x) \quad (x \in B_\delta)$$

$$(16) \qquad (v_{n,\delta})_\lambda(y) = v_{n,\delta}(x) \quad (y = |x|^\lambda \frac{x}{|x|} \in (B_\delta)_\lambda)$$

as in (7). Note that $g_{\delta,s}$ is a continuous function with compact support supp $g_{\delta,s} = \overline{B_\delta}$ and differentiable a.e.

$$(17) \qquad |\nabla g_{\delta,s}(x)|^2 = \begin{cases} 0 & (0 \leq |x| \leq \delta^s), \\ (\frac{1}{\log s})^2 \frac{1}{|x|^2 (\log \frac{1}{|x|})^2} & (\delta^s \leq |x| \leq \delta), \\ 0 & (\Omega \cap \{\delta \leq |x|\}). \end{cases}$$

By (17), we estimate

(18)

$$\int_{B_\delta} |\nabla v_{n,\delta}|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx$$

$$\leq 2 \int_{B_\delta} |\nabla u_n|^2 g_{\delta,s}^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx + 2 \int_{B_\delta} |u_n|^2 |\nabla g_{\delta,s}|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx$$

$$\leq 2 \int_{B_\delta} |\nabla u_n|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx + \frac{2}{(\log s)^2} \int_{B_\delta \setminus B_{\delta^s}} \frac{|u_n|^2}{|x|^2 (\log \frac{1}{|x|})^2} |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx$$

$$\leq 2 \int_{B_\delta} |\nabla u_n|^2 (\log \frac{1}{|x|})^{2\beta} dx + \frac{2}{(\log s)^2} \left(\log \frac{1}{\delta^s} \right)^{2(\alpha+\beta-1)} \int_{B_\delta \setminus B_{\delta^s}} \frac{|u_n|^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx$$

since $\alpha + \beta - 1 > 0$. Now, we take $\lambda \in (0,1)$ such that $\delta^\lambda = \delta_0$, which means that

$$(B_\delta)_\lambda = \{ y = |x|^\lambda \frac{x}{|x|} : x \in B_\delta \} = B_{\delta^\lambda} = B_{\delta_0}$$

and

$$\lambda = \frac{\log \frac{1}{\delta_0}}{\log \frac{1}{\delta}} \to 0 \iff \delta \to 0.$$

We test $H_{\alpha,\beta}(B_{\delta_0})$ by $(v_{n,\delta})_\lambda \in \mathcal{H}_{0,\alpha,\beta}^{1,2}(B_{\delta_0})$: Then by (9), (13), (18), and the absolute continuity of the integral

$$\int_{B_\delta \backslash B_{\delta^s}} \frac{|u|^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx \to 0 \quad (\delta \to 0)$$

for $u \in \mathcal{H}_{\alpha,\beta}^{1,2}(\Omega)$, we see

$$Q((v_{n,\delta})_\lambda, B_{\delta_0}) = Q((v_{n,\delta})_\lambda, (B_\delta)_\lambda) \overset{(9)}{\leq} \lambda^{2(\alpha+\beta-1)} Q(v_{n,\delta}, B_\delta)$$

$$= \lambda^{2(\alpha+\beta-1)} \frac{\int_{B_\delta} |\nabla v_{n,\delta}|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx}{\int_{B_\delta} \frac{v_{n,\delta}^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx}$$

$$\overset{(18)}{\leq} \lambda^{2(\alpha+\beta-1)} \frac{2\int_{B_\delta} |\nabla u_n|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx + \frac{2}{(\log s)^2} \left(\log \frac{1}{\delta^s}\right)^{2(\alpha+\beta-1)} \int_{B_\delta \backslash B_{\delta^s}} \frac{|u_n|^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx}{\int_{B_\delta} \frac{u_n^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx}$$

$$\overset{(13)}{\leq} \frac{C\lambda^{2(\alpha+\beta-1)} \sup_{n \in \mathbb{N}}\{\|u_n\|_{\alpha,\beta}^2\} + C \int_{B_\delta \backslash B_{\delta^s}} \frac{|u|^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx + o_n(1)}{\frac{1}{2}\left(1 - \int_\Omega \frac{u^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx\right)}.$$

Thus first by taking a limit $n \to \infty$, and later by taking a limit $\delta \to 0$, we obtain $H_{\alpha,\beta}(B_{\delta_0}) = 0$ when $\alpha+\beta > 1$; This is a contradiction to the fact that $H_{\alpha,\beta}(B_{\delta_0}) > 0$ when $\alpha + \beta > 1$, see Proposition 4. Thus we obtain (12), which completes the proof of Theorem 2. $\qquad\square$

3. RESULTS FOR $N_{\alpha,\beta}(\Omega)$ WHEN $\alpha + \beta > 1$.

Let us consider $N_{\alpha,\beta}(\Omega)$ defined in (4). First, we analyze the continuity property of the averaging operator

$$(19) \qquad T_0 : \mathcal{H}_{\alpha,\beta}^{1,2}(\Omega) \to \mathbb{R}, \quad T_0(u) = \int_\Omega \frac{u(x)}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx.$$

Proposition 6. *If $2\alpha+\beta > 3/2$ or $\alpha > 1/2$, then T_0 in (19) is a continuous linear functional on $\mathcal{H}_{\alpha,\beta}^{1,2}(\Omega)$.*

Proof. When $\alpha > 1/2$, then the weight function $\frac{1}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} \in L^1(\Omega)$ and the result is easily seen as

$$|T_0(u)| \leq \int_\Omega \frac{|u(x)|}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx \leq \left(\int_\Omega \frac{|u(x)|^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx\right)^{1/2} \underbrace{\left(\int_\Omega \frac{1}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx\right)^{1/2}}_{\leq C}$$

$$\leq C\|u\|_{\alpha,\beta}.$$

When $\alpha \leq 1/2$, we need a lemma below.

Lemma 1. *When $\alpha \leq 1/2$ and $2\alpha + \beta > 3/2$, there exists a constant $C_{\alpha,\beta} > 0$ such that for any $\delta \in (0,1)$ such that $B_\delta \subset\subset \Omega$ and $u \in \mathcal{H}_{\alpha,\beta}^{1,2}(\Omega)$,*

$$(20) \qquad \int_{B_\delta} \frac{|u(x)|}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx \leq C_{\alpha,\beta}\|u\|_{\alpha,\beta} \left(\log \frac{1}{\delta}\right)^{\frac{3}{2}-2\alpha-\beta}$$

holds.

Proof. By density, it is enough to prove (20) for $u \in Y_{\alpha,\beta}(\Omega)$. Since $\alpha \leq 1/2$, it is necessary $u(0) = 0$ in order to assure that the integral $\int_{\Omega} \frac{|u(x)|^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx$ is finite. In this case, we compute for $r \in (0, \delta)$ and $\omega \in \mathbb{S}^{N-1}$,

$$|u(r\omega)| = \left| \int_0^r \frac{\partial}{\partial t} \{u(t\omega)\} dt \right|$$

$$(21) \qquad \leq \left(\int_0^r |\frac{\partial u}{\partial r}(t\omega)|^2 t^{2-N} \left(\log \frac{1}{t} \right)^{2\beta} t^{N-1} dt \right)^{1/2} \left(\int_0^r \frac{dt}{t(\log \frac{1}{t})^{2\beta}} \right)^{1/2}$$

and the last integral is finite when $\beta > 1/2$:

$$\int_0^r \frac{dt}{t \left(\log \frac{1}{t} \right)^{2\beta}} = \frac{1}{2\beta - 1} \left(\log \frac{1}{r} \right)^{1-2\beta}.$$

Thus when $\beta > 1/2$, we see

$$(22) \qquad\qquad |u(r\omega)| \leq C \|u\|_{\alpha,\beta} (\log \frac{1}{r})^{\frac{1}{2} - \beta}$$

and

$$\int_0^\delta \frac{|u(r\omega)|}{r(\log \frac{1}{r})^{2\alpha}} dr \leq C \|u\|_{\alpha,\beta} \int_0^\delta \frac{dr}{r(\log \frac{1}{r})^{2\alpha+\beta-1/2}}.$$

The last integral is finite if $2\alpha + \beta > 3/2$ and in this case

$$\int_0^\delta \frac{dr}{r(\log \frac{1}{r})^{2\alpha+\beta-1/2}} = \frac{1}{2\alpha + \beta - \frac{3}{2}} \left(\log \frac{1}{\delta} \right)^{\frac{3}{2} - 2\alpha - \beta}.$$

Note that if $\alpha \leq 1/2$ and $2\alpha + \beta > 3/2$, we have automatically $\beta > 1/2$. Thus

$$\int_{B_\delta} \frac{|u(x)|}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = \int_{\mathbb{S}^{N-1}} \int_0^\delta \frac{|u(r\omega)|}{r^N (\log \frac{1}{r})^{2\alpha}} r^{N-1} dr dS_\omega$$

$$\leq C_{\alpha,\beta} \|u\|_{\alpha,\beta} \left(\log \frac{1}{\delta} \right)^{\frac{3}{2} - 2\alpha - \beta}.$$

This ends the proof of Lemma 1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

We return to the proof of Proposition 6. We use Lemma 1 when $\alpha \leq 1/2$. Fix $\delta \in (0, 1)$ such that $B_\delta \subset\subset \Omega$. Then

$$|T_0(u)| = \int_\Omega \frac{|u(x)|}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = \int_{B_\delta} (\cdots) dx + \int_{\Omega \setminus B_\delta} (\cdots) dx$$

$$\leq \underbrace{\int_{B_\delta} \frac{|u(x)|}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx}_{(20)} + \left(\int_\Omega \frac{|u(x)|^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx \right)^{1/2} \underbrace{\left(\int_{\Omega \setminus B_\delta} \frac{1}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx \right)^{1/2}}_{\leq C}$$

$$\leq \left(C_{\alpha,\beta} \left(\log \frac{1}{\delta} \right)^{\frac{3}{2} - 2\alpha - \beta} + \left(\int_{\Omega \setminus B_\delta} \frac{1}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx \right)^{1/2} \right) \|u\|_{\alpha,\beta}$$

when $2\alpha + \beta > 3/2$ and $\alpha \leq 1/2$. Tus T_0 in (19) is a bounded linear functional. $\square$

When $2\alpha + \beta < 3/2$ and $\alpha < 1/2$, then we can construct a sequence of functions in $\mathcal{H}^{1,2}_{0,\alpha,\beta}(\Omega)$ to show the unboundedness of T_0 in (19). As before, we may assume $B_{\delta_0} \subset\subset \Omega$ for some $\delta_0 \in (0,1)$.

Proposition 7. *If $2\alpha + \beta < 3/2$ and $\alpha < 1/2$, then there is a sequence $\{\psi_\delta\}_{\delta \in (0,\delta_0)}$ in $\mathcal{H}^{1,2}_{0,\alpha,\beta}(\Omega)$ such that*

$$\psi_\delta \geq 0, \quad \|\psi_\delta\|_{\alpha,\beta} \to 0 \quad (\delta \to 0), \quad and$$
$$T_0(\psi_\delta) \to +\infty \quad (\delta \to 0).$$

Proof. Let $s_1 > s_2 > s_3 > 1$ be fixed and thus $0 < \delta^{s_1} < \delta^{s_2} < \delta^{s_3} < \delta$ for $\delta \in (0,\delta_0)$. Define a radially symmetric cut-off function

$$(23) \quad \phi_\delta(x) = \begin{cases} 0 & (\delta \leq |x|), \\ \dfrac{1}{\log s_3}\left\{\log\log\dfrac{1}{|x|} - \log\log\dfrac{1}{\delta}\right\} & (\delta^{s_3} \leq |x| \leq \delta), \\ 1 & (\delta^{s_2} \leq |x| \leq \delta^{s_3}), \\ \dfrac{1}{\log s_2 - \log s_1}\left\{\log\log\dfrac{1}{|x|} - \log\log\dfrac{1}{\delta^{s_1}}\right\} & (\delta^{s_1} \leq |x| \leq \delta^{s_2}), \\ 0 & (0 \leq |x| \leq \delta^{s_1}). \end{cases}$$

Note that
$$(24)$$
$$|\nabla\phi_\delta|^2 = \left(\frac{1}{\log s_3}\right)^2 \frac{1}{|x|^2(\log\frac{1}{|x|})^2}\chi_{B_\delta\setminus B_{\delta^{s_3}}} + \left(\frac{1}{\log(s_2/s_1)}\right)^2 \frac{1}{|x|^2(\log\frac{1}{|x|})^2}\chi_{B_{\delta^{s_2}}\setminus B_{\delta^{s_1}}}$$

where χ_D is an indicator function of a set $D \subset \mathbb{R}^N$, so when $\beta \neq 1/2$, we have

$$(25)$$
$$\int_{B_\delta} |\nabla\phi_\delta|^2 |x|^{2-N}(\log\frac{1}{|x|})^{2\beta}dx \leq C\left(\int_{B_\delta\setminus B_{\delta^{s_3}}} + \int_{B_{\delta^{s_2}}\setminus B_{\delta^{s_1}}}\right)\frac{(\log\frac{1}{|x|})^{2\beta}}{|x|^N(\log\frac{1}{|x|})^2}dx$$
$$\leq C\left(\log\frac{1}{\delta}\right)^{2\beta-1}.$$

We construct a desired function ψ_δ of the form $\psi_\delta(x) = \left(\log\frac{1}{|x|}\right)^A \phi_\delta(|x|)$, where $A \in \mathbb{R}$ is chosen later. We have

$$T_0(\psi_\delta) = \int_{B_\delta} \frac{\psi_\delta(x)}{|x|^N(\log\frac{1}{|x|})^{2\alpha}}dx \geq \int_{B_{\delta^{s_3}}\setminus B_{\delta^{s_2}}} \frac{(\log\frac{1}{|x|})^A}{|x|^N(\log\frac{1}{|x|})^{2\alpha}}dx$$
$$= |\mathbb{S}^{N-1}|\int_{\delta^{s_2}}^{\delta^{s_3}} \frac{(\log\frac{1}{r})^A}{r(\log\frac{1}{r})^{2\alpha}}dr$$
$$= \frac{|\mathbb{S}^{N-1}|}{A-2\alpha+1}\left(s_2^{A-2\alpha+1} - s_3^{A-2\alpha+1}\right)\left(\log\frac{1}{\delta}\right)^{A-2\alpha+1} \to +\infty \quad (\delta \to 0)$$

when $A - 2\alpha + 1 > 0$. Even when $A - 2\alpha + 1 = 0$, we have

$$T_0(\psi_\delta) \geq \int_{B_{\delta^{s_3}} \setminus B_{\delta^{s_2}}} \frac{(\log \frac{1}{|x|})^A}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx$$

$$= |\mathbb{S}^{N-1}| \int_{\delta^{s_2}}^{\delta^{s_3}} \frac{dr}{r \log \frac{1}{r}}$$

$$= |\mathbb{S}^{N-1}| (\log s_2 - \log s_3) \log \log \frac{1}{\delta} \to +\infty \quad (\delta \to 0).$$

On the other-hand, if $2\alpha - 2A > 1$, we can compute that

$$\int_{B_\delta} \frac{|\psi_\delta(x)|^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx \leq \int_{B_\delta} \frac{(\log \frac{1}{|x|})^{2A}}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = |\mathbb{S}^{N-1}| \int_0^\delta \frac{(\log \frac{1}{r})^{2A}}{r(\log \frac{1}{r})^{2\alpha}} dr$$

$$= \frac{|\mathbb{S}^{N-1}|}{2\alpha - 1 - 2A} \left(\log \frac{1}{\delta} \right)^{2A - 2\alpha + 1} \to 0 \quad (\delta \to 0).$$

Also when $2A + 2\beta - 1 < 0$, we see by (24),

$$\int_{B_\delta} |\nabla \psi_\delta|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx$$

$$\leq 2A^2 \int_{B_\delta} \frac{(\log \frac{1}{|x|})^{2(A-1)+2\beta}}{|x|^2} \phi_\delta^2 |x|^{2-N} dx + 2 \int_{B_\delta} (\log \frac{1}{|x|})^{2A+2\beta} |\nabla \phi_\delta|^2 |x|^{2-N} dx$$

$$\leq C \left(\int_{\delta^{s_1}}^{\delta^{s_2}} + \int_{\delta^{s_3}}^\delta \right) \frac{(\log \frac{1}{r})^{2A+2\beta-2}}{r} dr$$

$$\leq C \left(\log \frac{1}{\delta} \right)^{2A+2\beta-1} \to 0 \quad (\delta \to 0).$$

Therefore, if we can choose $A \in \mathbb{R}$ so that

$$\left\{ \begin{array}{l} A - 2\alpha + 1 \geq 0, \\ 2\alpha - 2A > 1, \\ 2A - 2\beta - 1 < 0, \end{array} \right. \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} A \geq 2\alpha + 1, \\ A < \alpha - 1/2, \\ A < 1/2 - \beta, \end{array} \right.$$

$\psi_\delta(x) = \left(\log \frac{1}{|x|} \right)^A \phi_\delta(|x|)$ has desired properties: This is the case if $2\alpha + \beta < 3/2$ and $\alpha < 1/2$. Thus we have proven Proposition 7. $\qquad \square$

Proposition 8. *If $\alpha + \beta < 1$, then $N_{\alpha,\beta}(\Omega) = 0$.*

Proof. We may assume that $B_{\delta_0} \subset\subset \Omega$ for some $\delta_0 \in (0,1)$. Let $\zeta = \zeta(r) \in C_0^\infty(B_{\delta_0})$, $r = |x|$, denote a compact supported smooth radial function such that $0 \leq \zeta \leq 1$, $\zeta \equiv 1$ for $0 \leq r \leq \delta_0/2$, and $p_1, p_2 > 0$. Define

$$\phi(r) = \begin{cases} \zeta(r)(C_1 r^{p_1} - C_2 r^{p_2}) & (0 \leq r = |x| \leq \delta_0), \\ 0 & (x \in \Omega \setminus B_{\delta_0}), \end{cases}$$

where

$$C_i^{-1} = \int_0^{\delta_0} \frac{\zeta(r) r^{p_i}}{r(\log \frac{1}{r})^{2\alpha}} dr \quad (i = 1, 2).$$

Note that

$$\int_0^\delta \frac{dr}{r^A (\log \frac{1}{r})^B} = \lim_{\varepsilon \to 0} \int_\varepsilon^\delta \frac{dr}{r^A (\log \frac{1}{r})^B} = \begin{cases} < \infty & (A < 1, B \in \mathbb{R}), \\ < \infty & (A = 1, B > 1), \\ = \infty & (A = 1, B \le 1), \\ = \infty & (A > 1, B \in \mathbb{R}). \end{cases}$$

Thus C_i is well-defined and we have $\int_\Omega \frac{\phi}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = 0$. We also see

$$|\phi(r)|^2 \le C(r^{2p_1} + r^{2p_2}), \quad |\phi'(r)|^2 \le C(r^{2p_1} + r^{2p_2} + r^{2(p_1-1)} + r^{2(p_2-1)})$$

for some $C > 0$, thus $\|\phi\|_{\alpha,\beta} < \infty$ for any $\alpha, \beta \in \mathbb{R}$ when $p_i > 0$, $(i = 1, 2)$. We test $N_{\alpha,\beta}(\Omega)$ by a scaled function $\phi_\lambda \in \mathcal{H}_{0,\alpha,\beta}^{1,2}((B_{\delta_0})_\lambda)$, which can be regarded as a function in $\mathcal{H}_{\alpha,\beta}^{1,2}(\Omega)$ by zero-extension when $\lambda > 1$. By (8) for radial functions, we see

$$N_{\alpha,\beta}(\Omega) \le Q(\phi_\lambda, \Omega) = Q(\phi_\lambda, (B_{\delta_0})_\lambda) = \lambda^{2(\alpha+\beta-1)} Q(\phi, B_{\delta_0}),$$

and the right-hand side goes to 0 as $\lambda \to \infty$ when $\alpha + \beta < 1$. Thus we obtain $N_{\alpha,\beta}(\Omega) = 0$ if $\alpha + \beta < 1$. $\qquad\square$

Proposition 9. $N_{\alpha,\beta}(\Omega) = 0$ if $(\alpha, \beta) = (\frac{1}{2}, \frac{1}{2})$.

Proof. For $\varepsilon > 0$, define

$$\phi_\varepsilon(x) = \left(\log \frac{1}{|x|}\right)^{-3\varepsilon} - C_0 \left(\log \frac{1}{|x|}\right)^{-2\varepsilon},$$

where C_0 is chosen so that $\int_\Omega \frac{\phi_\varepsilon}{|x|^N (\log \frac{1}{|x|})} dx = 0$, i.e.,

$$C_0 = \frac{\int_\Omega \frac{dx}{|x|^N (\log \frac{1}{|x|})^{1+3\varepsilon}}}{\int_\Omega \frac{dx}{|x|^N (\log \frac{1}{|x|})^{1+2\varepsilon}}} > 0.$$

Then we see that there exists $C > 0$ such that

$$\nabla \phi_\varepsilon = \frac{x}{|x|^2} \left\{ 3\varepsilon (\log \frac{1}{|x|})^{-3\varepsilon-1} - 2\varepsilon C_0 (\log \frac{1}{|x|})^{-2\varepsilon-1} \right\},$$

$$|\nabla \phi_\varepsilon|^2 \le C \frac{\varepsilon^2}{|x|^2} \left\{ (\log \frac{1}{|x|})^{-6\varepsilon-2} + (\log \frac{1}{|x|})^{-4\varepsilon-2} \right\},$$

$$|\phi_\varepsilon|^2 \le C \left\{ (\log \frac{1}{|x|})^{-6\varepsilon} + (\log \frac{1}{|x|})^{-4\varepsilon} \right\}$$

hold. From these, we easily see that $\|\phi_\varepsilon\|_{\frac{1}{2}, \frac{1}{2}} < \infty$, so Lemma 3 implies that $\phi_\varepsilon \in \mathcal{H}_{\alpha,\beta}^{1,2}(\Omega)$ for $(\alpha, \beta) = (1/2, 1/2)$ when $\varepsilon > 0$. We test $N_{\frac{1}{2}, \frac{1}{2}}(\Omega)$ by ϕ_ε. Since for $A > 0$,

$$\int_{B_{\delta_0}} \frac{dx}{|x|^N (\log \frac{1}{|x|})^{1+A\varepsilon}} = \frac{|\mathbb{S}^{N-1}|}{A\varepsilon} (\log \frac{1}{\delta_0})^{-A\varepsilon} = \frac{|\mathbb{S}^{N-1}|}{A\varepsilon} (1 + o(1))$$

as $\varepsilon \to 0$, we estimate

$$\int_\Omega \frac{\phi_\varepsilon^2}{|x|^N (\log \frac{1}{|x|})} dx \geq \int_{B_{\delta_0}} \frac{(\log \frac{1}{|x|})^{-6\varepsilon} - 2C_0 (\log \frac{1}{|x|})^{-5\varepsilon} + C_0^2 (\log \frac{1}{|x|})^{-4\varepsilon}}{|x|^N (\log \frac{1}{|x|})} dx$$

$$\geq \frac{|\mathbb{S}^{N-1}|}{\varepsilon} \left(\frac{1}{6}(1 + o(1)) - \frac{2C_0}{5}(1 + o(1)) + \frac{C_0^2}{4}(1 + o(1)) \right)$$

$$= \frac{|\mathbb{S}^{N-1}|}{\varepsilon} \left(\frac{1}{6} - \frac{2C_0}{5} + \frac{C_0^2}{4} + o(1) \right)$$

$$= \frac{|\mathbb{S}^{N-1}|}{\varepsilon} \left(\frac{1}{4}(C_0 - \frac{4}{5})^2 + \frac{1}{150} + o(1) \right) > 0.$$

On the other hand, let us take a ball B_R such that $\Omega \subset B_R \subset B$. Then we see

$$\int_\Omega |\nabla \phi_\varepsilon|^2 |x|^{2-N} (\log \frac{1}{|x|}) dx \leq C\varepsilon^2 \left(\int_{B_R} \frac{dx}{|x|^N (\log \frac{1}{|x|})^{6\varepsilon+1}} dx + \int_{B_R} \frac{dx}{|x|^N (\log \frac{1}{|x|})^{4\varepsilon+1}} dx \right)$$

$$= C\varepsilon^2 \left(\frac{|\mathbb{S}^{N-1}|}{6\varepsilon} + \frac{|\mathbb{S}^{N-1}|}{4\varepsilon} + o(1) \right) = O(\varepsilon).$$

Thus we have

$$N_{\frac{1}{2},\frac{1}{2}}(\Omega) \leq Q(\phi_\varepsilon, \Omega) = \frac{\int_\Omega |\nabla \phi_\varepsilon|^2 (\log \frac{1}{|x|}) dx}{\int_\Omega \frac{\phi_\varepsilon^2}{|x|^2 (\log \frac{1}{|x|})} dx} \leq \frac{O(\varepsilon)}{\varepsilon^{-1}} = O(\varepsilon^2) \to 0$$

as $\varepsilon \to 0$. This implies $N_{\frac{1}{2},\frac{1}{2}}(\Omega) = 0$. $\qquad\square$

Proposition 10. *If $H_{\alpha,\beta}(\Omega) > 0$, then $N_{\alpha,\beta}(\Omega) > 0$.*

Proof. We will prove a contraposition that $N_{\alpha,\beta}(\Omega) = 0$ implies $H_{\alpha,\beta}(\Omega) = 0$.
Let $\{u_n\}$ be a minimizing sequence for $N_{\alpha,\beta}(\Omega) = 0$ in $\mathcal{H}_{\alpha,\beta}^{1,2}(\Omega)$ such that

$$\int_\Omega \frac{u_n^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = 1, \quad \int_\Omega \frac{u_n}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = 0,$$

$$\text{and} \quad \int_\Omega |\nabla u_n|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx = o(1).$$

Then up to a subsequence, there is $u \in \mathcal{H}_{\alpha,\beta}^{1,2}(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } \mathcal{H}_{\alpha,\beta}^{1,2}(\Omega).$$

Since

$$o(1) = \int_\Omega |\nabla u_n|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx$$

$$= \int_\Omega |\nabla (u_n - u)|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx + \int_\Omega |\nabla u|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx$$

$$+ 2 \underbrace{\int_\Omega (\nabla (u_n - u) \cdot \nabla u) |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx}_{=o(1)},$$

we see $\int_\Omega |\nabla u|^2 (\log \frac{1}{|x|})^{2\beta} dx = 0$ and $\int_\Omega |\nabla (u - u_n)|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx = o(1)$ as $n \to \infty$. Form this, we obtain that $u \equiv C$ for a constant C.

When $\alpha > 1/2$, we know that the average operator $T_0 : \mathcal{H}^{1,2}_{\alpha,\beta}(\Omega) \to \mathbb{R}$,

$$T_0(u) = \int_\Omega \frac{u}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx$$

is continuous and linear, so $T_0 \in (\mathcal{H}^{1,2}_{\alpha,\beta}(\Omega))^*$. By the definition of weak convergence, we have

$$0 = T_0(u_n) \to T(u) = C \int_\Omega \frac{1}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx \quad (n \to \infty),$$

which implies $C \equiv 0$. When $\alpha \leq 1/2$, by the weak lower semi-continuity

$$\int_\Omega \frac{u^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx \leq \liminf_{n \to \infty} \int_\Omega \frac{u_n}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = 1$$

and $\int_\Omega \frac{1}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = \infty$ implies that $C \equiv 0$ again. We have proven that $u_n \rightharpoonup u \equiv 0$ weakly in $\mathcal{H}^{1,2}_{\alpha,\beta}(\Omega)$.

We may assume that $B_{\delta_0} \subset\subset \Omega$ for some $\delta_0 \in (0,1)$. As in the proof of Theorem 2, we take a cut-off function $g_{\delta_0,s}$ where $s > 1$ is a fixed number as in (14), and take a truncated sequence $\{v_{n,\delta_0}\} = \{g_{\delta_0,s} u_n\} \subset \mathcal{H}^{1,2}_{0,\alpha,\beta}(B_{\delta_0}) \subset \mathcal{H}^{1,2}_{0,\alpha,\beta}(\Omega)$ defined as in (15) with $\delta = \delta_0$. Then we have

(26)

$$Q(v_{n,\delta_0}, \Omega) = Q(v_{n,\delta_0}, B_{\delta_0}) = \frac{\int_{B_{\delta_0}} |\nabla v_{n,\delta_0}|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx}{\int_{B_{\delta_0}} \frac{v_{n,\delta_0}^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx}$$

$$\leq \frac{2 \int_{B_{\delta_0}} |\nabla u_n|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx + \frac{2}{(\log s)^2} \left(\log \frac{1}{\delta_0^s} \right)^{2(\alpha+\beta-1)} \int_{\Omega \setminus B_{\delta_0^s}} \frac{|u_n|^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx}{\int_{B_{\delta_0}} \frac{u_n^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx},$$

see the proof of Theorem 2. Since $u_n \to u \equiv 0$ in $L^2(\Omega \setminus B_{\delta_0}, \frac{1}{|x|^N (\log \frac{1}{|x|})^{2\alpha}})$ strongly as $n \to \infty$, we see that

$$\int_{\Omega \setminus B_{\delta_0}} \frac{u_n^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx \to \int_{\Omega \setminus B_{\delta_0}} \frac{u^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = 0 \quad (n \to \infty).$$

By this and $\int_\Omega \frac{u_n^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = 1$, we see $\int_{B_{\delta_0}} \frac{u_n^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = 1 + o(1)$. Thus taking a limit $n \to \infty$ in (26), we obtain that $H_{\alpha,\beta}(\Omega) = 0$. $\qquad \square$

Proof of Theorem 1.

Proof. The positivity of $H_{\alpha,\beta}(\Omega)$ has been proven in (10). Also, Propositions 8, 9, 10 give the necessary and sufficient condition for the positivity of $N_{\alpha,\beta}(\Omega)$. Thus, we have proven Theorem 1. $\qquad \square$

4. Proof of Theorem 3.

In this section, we prove Theorem 3. We may assume that $B_{\delta_0} \subset\subset \Omega$ for some $\delta_0 \in (0,1)$. Assume $\alpha + \beta > 1$ and $2\alpha + \beta > 3/2$. In this case, we know $N_{\alpha,\beta}(\Omega) > 0$ by Theorem 1. Let $\{u_n\}$ be a minimizing sequence of $N_{\alpha,\beta}(\Omega) > 0$ in $\mathcal{H}^{1,2}_{\alpha,\beta}(\Omega)$. Then $\{u_n\}$ satisfies

$$\int_\Omega \frac{u_n^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = 1, \quad \int_\Omega \frac{u_n}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = 0, \text{ and}$$

$$\int_\Omega |\nabla u_n|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx = N_{\alpha,\beta}(\Omega) + o_n(1)$$

as $n \to \infty$. Then there is $u \in \mathcal{H}^{1,2}_{\alpha,\beta}(\Omega)$ such that

$$u_n \rightharpoonup u \text{ in } \mathcal{H}^{1,2}_{\alpha,\beta}(\Omega)$$

up to a subsequence as $n \to \infty$. Take $\delta \in (0, \delta_0)$ and fix $s > 1$. Then by the strong convergence of u_n to u in $L^2(\Omega \setminus B_{\delta^s}, \frac{1}{|x|^N (\log \frac{1}{|x|})^{2\alpha}})$, we see that

$$\int_{B_{\delta^s}} \frac{u_n^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = 1 - \int_{\Omega \setminus B_{\delta^s}} \frac{u^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx + o_n(1)$$

(27)
$$> \frac{1}{2} \left(1 - \int_\Omega \frac{u^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx \right)$$

up to a subsequence, where $o_n(1) \to 0$ as $n \to \infty$ for a fixed $\delta \in (0, \delta_0)$.

We first claim that

(28)
$$\int_\Omega \frac{u}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = 0.$$

Indeed, when $2\alpha + \beta > 3/2$, the average operator $T_0 : \mathcal{H}^{1,2}_{\alpha,\beta}(\Omega) \to \mathbb{R}$, $T_0(u) = \int_\Omega \frac{u}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx$ is a continuous linear functional, so we obtain (28).

Next, we claim that

(29)
$$\int_\Omega \frac{u^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = 1.$$

If (29) is the case, then with (28) we easily see that u is a minimizer of $N_{\alpha,\beta}(\Omega)$ in $\mathcal{A}_N$. In the following, we will prove (29). By weak lower semi-continuity, we know that $\int_\Omega \frac{u^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx \leq 1$. Suppose the contrary that $\int_\Omega \frac{u^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx < 1$. then we find that for any $\delta > 0$, there exists $N = N(\delta) > 0$ such that (27) holds for $n > N$. Let $s = s_1 > s_2 > s_3 > 1$ and recall a truncation function $g_{\delta,s}$ in (14) and ϕ_δ in (23). Define

(30)
$$v_n = u_n g_{\delta,s} - C_n \phi_\delta$$

where $C_n \in \mathbb{R}$ is chosen so that $T_0(v_n) = 0$, that is,

(31)
$$C_n = \frac{\int_{B_\delta} \frac{u_n g_{\delta_s}}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx}{\int_{B_\delta} \frac{\phi_\delta}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx}.$$

We estimate the numerator of C_n from the above. When $\alpha > 1/2$, then

$$\int_{B_\delta} \frac{u_n g_{\delta,s}}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx \leq \left(\int_{B_\delta} \frac{u_n^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx \right)^{1/2} \left(\int_{B_\delta} \frac{g_{\delta,s}^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx \right)^{1/2}$$

$$\leq C \left(\int_{B_\delta} \frac{u_n^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx \right)^{1/2} \left(\int_0^\delta \frac{dr}{r (\log \frac{1}{r})^{2\alpha}} \right)^{1/2}$$

(32)

$$\leq C \left(\int_{B_\delta} \frac{u_n^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx \right)^{1/2} \left(\log \frac{1}{\delta} \right)^{\frac{1}{2}-\alpha} \leq C \|u_n\|_{\alpha,\beta} \left(\log \frac{1}{\delta} \right)^{\frac{1}{2}-\alpha}.$$

When $\alpha \leq 1/2$, then we can exploit Proposition 6 since now $2\alpha + \beta > 3/2$.

(33)
$$\int_{B_\delta} \frac{|u_n|}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx \leq C_{\alpha,\beta} \|u_n\|_{\alpha,\beta} \left(\log \frac{1}{\delta} \right)^{\frac{3}{2}-2\alpha-\beta}.$$

Next, we estimate the denominator of C_n from the below. When $\alpha \neq 1/2$, we have

$$\int_\Omega \frac{\phi_\delta}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx \geq \int_{B_{\delta^{s_3}} \setminus B_{\delta^{s_2}}} \frac{dx}{|x|^N (\log \frac{1}{|x|})^{2\alpha}}$$

$$= |\mathbb{S}^{N-1}| \int_{\delta^{s_2}}^{\delta^{s_3}} \frac{dr}{r (\log \frac{1}{r})^{2\alpha}}$$

(34)
$$= \frac{|\mathbb{S}^{N-1}|}{2\alpha - 1} (s_3^{1-2\alpha} - s_2^{1-2\alpha}) \left(\log \frac{1}{\delta} \right)^{1-2\alpha}.$$

Similarly, when $\alpha = 1/2$, we see

$$\int_\Omega \frac{\phi_\delta}{|x|^N (\log \frac{1}{|x|})} dx \geq \int_{B_{\delta^{s_3}} \setminus B_{\delta^{s_2}}} \frac{dx}{|x|^N (\log \frac{1}{|x|})} = 2\pi \int_{\delta^{s_2}}^{\delta^{s_3}} \frac{dr}{r (\log \frac{1}{r})}$$

$$= |\mathbb{S}^{N-1}| (\log \log \frac{1}{\delta^{s_3}} - \log \log \frac{1}{\delta^{s_2}})$$

(35)
$$= |\mathbb{S}^{N-1}| (\log s_3 - \log s_2).$$

By (32), (33), (34), (35), we have

(36)
$$C_n \leq \begin{cases} C(\delta) \left(\log \frac{1}{\delta} \right)^{\alpha - \frac{1}{2}} & (\alpha > 1/2), \\ C \left(\log \frac{1}{\delta} \right)^{\frac{1}{2}-\beta} & (\alpha \leq 1/2 \text{ and } 2\alpha + \beta > 3/2), \end{cases}$$

since $1/2 - \alpha - (1 - 2\alpha) = \alpha - 1/2$ and $3/2 - 2\alpha - \beta - (1 - 2\alpha) = 1/2 - \beta$, where

$$C(\delta) = C \left(\int_{B_\delta} \frac{u_n^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx \right)^{1/2} \to 0$$

as $\delta \to 0$ for fixed n.

Next we will estimate the critical Hardy quotient $Q(v_n, B_\delta)$ for v_n defined in (30) by using (36). By the elementary inequality $(a - b)^2 \geq \frac{1}{2}a^2 - b^2$, the denominator

of $Q(v_n, B_\delta)$ can be estimated from below:

$$
\int_{B_\delta} \frac{v_n^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = \int_{B_\delta} \frac{(u_n g_{\delta,s} - C_n \phi_\delta)^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx
$$

$$
\geq \frac{1}{2} \int_{B_\delta} \frac{u_n^2 g_{\delta,s^2}}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx - C_n^2 \int_{B_\delta} \frac{\phi_\delta^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx
$$

$$
\geq \frac{1}{2} \int_{B_{\delta^s}} \frac{u_n^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx - C_n^2 \int_{B_\delta} \frac{\phi_\delta^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx
$$

$$
(37) \qquad \geq \frac{1}{2}\left(1 - \int_\Omega \frac{u^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx\right) - C_n^2 \int_{B_\delta} \frac{\phi_\delta^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx.
$$

Now the first term of the right-hand side of (37) is positive by (27) and the contradiction assumption. Concerning the second term of (37), we have

$$
\int_{B_\delta} \frac{\phi_\delta^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx \leq \int_{B_\delta \setminus B_{\delta^{s_1}}} \frac{dx}{|x|^N (\log \frac{1}{|x|})^{2\alpha}}
$$

$$
= |\mathbb{S}^{N-1}| \int_{\delta^{s_1}}^\delta \frac{dr}{r(\log \frac{1}{r})^{2\alpha}}
$$

$$
= \frac{|\mathbb{S}^{N-1}|}{2\alpha - 1}(1 - s_1^{1-2\alpha})(\log \frac{1}{\delta})^{1-2\alpha}, \quad \text{if } \alpha \neq 1/2,
$$

$$
\int_{B_\delta} \frac{\phi_\delta^2}{|x|^N (\log \frac{1}{|x|})} dx \leq \int_{B_\delta \setminus B_{\delta^{s_1}}} \frac{dx}{|x|^N (\log \frac{1}{|x|})} = |\mathbb{S}^{N-1}| \int_{\delta^{s_1}}^\delta \frac{dr}{r(\log \frac{1}{r})}
$$

$$
= |\mathbb{S}^{N-1}| \log s_1, \quad \text{if } \alpha = 1/2.
$$

Together with the estimate of C_n in (36), we obtain

$$
(38)
$$
$$
\int_{B_\delta} \frac{v_n^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx \geq \begin{cases} C + C(\delta) & (\alpha > 1/2) \\ C + C(\log \frac{1}{\delta})^{2(1-\alpha-\beta)} & (\alpha \leq 1/2 \text{ and } 2\alpha + \beta > 3/2) \end{cases}
$$

where $C > 0$ is independent of $\delta \in (0, \delta_0)$.

On the other hand,

$$
\int_{B_\delta} |\nabla v_n|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx
$$

$$
\leq 2 \int_{B_\delta} |\nabla u_n|^2 g_{\delta,s}^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx + 2 \int_{B_\delta} |\nabla g_{\delta,s}|^2 u_n^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx
$$

$$
+ 2C_n^2 \int_{B_\delta} |\nabla \phi_\delta|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx
$$

$$
= (I) + (II) + (III).
$$

Now,

$$(I) = 2 \int_{B_\delta} |\nabla u_n|^2 g_{\delta,s}^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx \le 2 \int_{B_\delta} |\nabla u_n|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx \le 2\|u_n\|_{\alpha,\beta}^2,$$

$$(II) = 2 \int_{B_\delta} |\nabla g_{\delta,s}|^2 u_n^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx \le \frac{2}{(\log s)^2} \int_{B_\delta \setminus B_{\delta^s}} \frac{u_n^2}{|x|^2 (\log \frac{1}{|x|})^2} |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx$$

$$\le C \left(\log \frac{1}{\delta^s}\right)^{2(\alpha+\beta-1)} \int_{B_\delta \setminus B_{\delta^s}} \frac{u_n^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx$$

by (17) and $\alpha + \beta - 1 > 0$. Also we see when $\alpha > 1/2$ and $\beta \ne 1/2$,

$$(III) = 2C_n^2 \int_{B_\delta} |\nabla \phi_\delta|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx \overset{(36)}{\le} 2C(\delta)^2 \left(\log \frac{1}{\delta}\right)^{2\alpha-1} \underbrace{\int_{B_\delta} |\nabla \phi_\delta|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx}_{\le C(\log \frac{1}{\delta})^{2\beta-1}}$$

$$\le 2C(\delta)^2 \left(\log \frac{1}{\delta}\right)^{2(\alpha+\beta-1)}$$

by (25) and (36). When $\alpha \le 1/2$ and $2\alpha + \beta > 3/2$, we have $\beta \ne 1/2$ and

$$(III) = 2C_n^2 \int_{B_\delta} |\nabla \phi_\delta|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx \overset{(36),(25)}{\le} C \left(\log \frac{1}{\delta}\right)^{1-2\beta} \left(\log \frac{1}{\delta}\right)^{2\beta-1} = C.$$

By these estimates, we obtain
(39)
$$\int_{B_\delta} |\nabla v_n|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx \le \begin{cases} C\|u_n\|_{\alpha,\beta}^2 + C(\delta) \left(\log \frac{1}{\delta}\right)^{2(\alpha+\beta-1)} & (\alpha > 1/2) \\ C\|u_n\|_{\alpha,\beta}^2 + C & (\alpha \le 1/2 \text{ and } 2\alpha + \beta > 3/2) \end{cases}$$

Now, we introduce the scaled functions $(v_n)_\lambda$ as in (7):

$$(v_n)_\lambda(y) = v_n(x) \quad x \in B_\delta, \quad y = |x|^\lambda \frac{x}{|x|} \in (B_\delta)_\lambda = B_{\delta^\lambda}.$$

We take $\lambda \in (0,1)$ so that $\delta^\lambda = \delta_0$ as in the proof of Theorem 2. Then $\lambda = \frac{\log \frac{1}{\delta_0}}{\log \frac{1}{\delta}} \to 0$
as $\delta \to 0$. We test $N_{\alpha,\beta}(B_{\delta_0})$ by $(v_n)_\lambda$ to obtain $N_{\alpha,\beta}(B_{\delta_0}) = 0$; which yields a contradiction to the fact that $N_{\alpha,\beta} > 0$ if $\alpha + \beta > 1$ and $2\alpha + \beta > 3/2$. This is done by the scaling property (9) for $\lambda \in (0,1)$ and the estimates (38) and (39): Recall that we assume $\alpha + \beta > 1$. Thus when $\alpha > 1/2$, we see

$$N_{\alpha,\beta}(B_{\delta_0}) \le Q((v_n)_\lambda, B_{\delta_0}) = Q((v_n)_\lambda, (B_\delta)_\lambda) \le \lambda^{2(\alpha+\beta-1)} Q(v_n, B_\delta)$$

$$= \lambda^{2(\alpha+\beta-1)} \frac{C\|u_n\|_{\alpha,\beta}^2 + C(\delta) \left(\log \frac{1}{\delta}\right)^{2(\alpha+\beta-1)}}{C + C(\delta)}$$

$$\le C \frac{\lambda^{2(\alpha+\beta-1)} \sup_{n \in \mathbb{N}} \|u_n\|_{\alpha,\beta}^2 + C(\delta)}{C + C(\delta)} = o_\delta(1)$$

as $\delta \to 0$. When $\alpha \leq 1/2$ and $2\alpha + \beta > 3/2$, we see

$$N_{\alpha,\beta}(B_{\delta_0}) \leq Q((v_n)_\lambda, B_{\delta_0}) = Q((v_n)_\lambda, (B_\delta)_\lambda) \leq \lambda^{2(\alpha+\beta-1)}Q(v_n, B_\delta)$$

$$= \lambda^{2(\alpha+\beta-1)}\frac{C\|u_n\|_{\alpha,\beta}^2 + C}{C + C(\log\frac{1}{\delta})^{2(1-\alpha-\beta)}}$$

$$\leq \frac{C\lambda^{2(\alpha+\beta-1)}(\sup_{n\in\mathbb{N}}\|u_n\|_{\alpha,\beta}^2 + 1)}{C + o_\delta(1)} = o_\delta(1)$$

as $\delta \to 0$. This is a desired contradiction and we obtain (29). This completes the proof. $\qquad\square$

5. Proof of Theorem 4.

We will prove here that if $\alpha + \beta \geq 1$ and $2\alpha + \beta \leq \frac{3}{2}$, then $N_{\alpha,\beta}(\Omega)$ is not achieved. For this purpose, we introduce a new minimization problem as in [5].
(40)

$$\overline{N}_{\alpha,\beta}(\Omega) := \inf\left\{\int_\Omega |\nabla u|^2 |x|^{2-N}\left(\log\frac{1}{|x|}\right)^{2\beta} dx : u \in \mathcal{H}_{\alpha,\beta}^{1,2}(\Omega) : \int_\Omega \frac{|u|^2}{|x|^N\left(\log\frac{1}{|x|}\right)^{2\alpha}} dx = 1\right\}.$$

Since $\mathcal{H}_{0,\alpha,\beta}^{1,2}(\Omega) \subset \mathcal{H}_{\alpha,\beta}^{1,2}(\Omega)$, we have $\overline{N}_{\alpha,\beta}(\Omega) \leq H_{\alpha,\beta}(\Omega)$. Also it is trivial that $\overline{N}_{\alpha,\beta}(\Omega) \leq N_{\alpha,\beta}(\Omega)$. First, we need several properties for $\overline{N}_{\alpha,\beta}(\Omega)$.

Proposition 11. If $\alpha \leq 1/2$ and $H_{\alpha,\beta}(\Omega) > 0$, then $\overline{N}_{\alpha,\beta}(\Omega) > 0$.

Proof. Assume the contrary that there exists $\alpha \leq 1/2$ and $\beta \in \mathbb{R}$ such that $H_{\alpha,\beta}(\Omega) > 0$ and $\overline{N}_{\alpha,\beta}(\Omega) = 0$. Let $\{u_n\} \subset \mathcal{H}_{\alpha,\beta}^{1,2}(\Omega)$ denote a minimizing sequence for $\overline{N}_{\alpha,\beta}(\Omega) = 0$. Then we see

$$\int_\Omega \frac{u_n^2}{|x|^N(\log\frac{1}{|x|})^{2\alpha}} dx = 1, \quad \text{and} \quad \int_\Omega |\nabla u_n|^2 |x|^{2-N}(\log\frac{1}{|x|})^{2\beta} dx = o_n(1).$$

Then up to a subsequence, there is $u \in \mathcal{H}_{\alpha,\beta}^{1,2}(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } \mathcal{H}_{\alpha,\beta}^{1,2}(\Omega).$$

Then we have

$$\int_\Omega |\nabla u_n|^2 |x|^{2-N}(\log\frac{1}{|x|})^{2\beta} dx = \int_\Omega |\nabla(u_n - u)|^2 |x|^{2-N}(\log\frac{1}{|x|})^{2\beta} dx + \int_\Omega |\nabla u|^2 |x|^{2-N}(\log\frac{1}{|x|})^{2\beta} dx + o(1)$$

which implies $\int_\Omega |\nabla u|^2 |x|^{2-N}(\log\frac{1}{|x|})^{2\beta} dx = 0$ and u is a constant. Since no nonzero constant function belongs to $\mathcal{H}_{\alpha,\beta}^{1,2}(\Omega)$ when $\alpha \leq 1/2$, we have $u \equiv 0$ in this case.

Define $v_n = g_{\delta,s}u_n \in \mathcal{H}_{0,\alpha,\beta}^{1,2}(B_\delta)$ where $g_{\delta,s}$ is defined in (14) for some $s > 1$ and $\delta \in (0,1)$, and we consider v_n as an element of $\mathcal{H}_{0,\alpha,\beta}^{1,2}(\Omega)$ by zero extension. As in the estimate (18), we have

$$\int_\Omega |\nabla v_n|^2 |x|^{2-N}(\log\frac{1}{|x|})^{2\beta} dx \leq C\underbrace{\int_\Omega |\nabla u_n|^2 |x|^{2-N}(\log\frac{1}{|x|})^{2\beta} dx}_{=\overline{N}_{\alpha,\beta}(\Omega)+o_n(1)=o_n(1)}$$

$$+ C\left(\log\frac{1}{\delta}\right)^{2(\alpha+\beta-1)}\int_{\Omega\setminus B_{\delta^s}} \frac{|u_n|^2}{|x|^N(\log\frac{1}{|x|})^{2\alpha}} dx.$$

Also since $u_n \to u \equiv 0$ strongly in $L^2(\Omega \setminus B_{\delta^s}, \frac{1}{|x|^N (\log \frac{1}{|x|})^{2\alpha}})$ for any $\delta \in (0,1)$ fixed, we see

$$\int_\Omega \frac{v_n^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx \geq \int_{B_{\delta^s}} \frac{u_n^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = \left(\int_\Omega - \int_{\Omega \setminus B_{\delta^s}} \right)$$
$$= 1 + o_n(1) > 1/2.$$

Therefore, we test $H_{\alpha,\beta}(\Omega)$ by $v_n \in \mathcal{H}^{1,2}_{0,\alpha,\beta}(\Omega)$ to obtain

$$H_{\alpha,\beta}(\Omega) \leq Q(v_n, \Omega) = \frac{\int_\Omega |\nabla v_n|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx}{\int_\Omega \frac{v_n^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx}$$

$$\frac{C \int_\Omega |\nabla u_n|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx + C \left(\log \frac{1}{\delta} \right)^{2(\alpha+\beta-1)} \int_{\Omega \setminus B_{\delta^s}} \frac{|u_n|^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx}{1/2}$$

$$\to 0$$

as $n \to \infty$. Thus $H_{\alpha,\beta}(\Omega) = 0$, which contradicts the assumption $H_{\alpha,\beta}(\Omega) > 0$. $\square$

Proposition 12. *If $N_{\alpha,\beta}(\Omega) = \overline{N}_{\alpha,\beta}(\Omega)$, then $N_{\alpha,\beta}(\Omega)$ is not achieved.*

Proof. Assume the contrary that there exists $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$ such that $N_{\alpha,\beta}(\Omega) = \overline{N}_{\alpha,\beta}(\Omega)$ and $N_{\alpha,\beta}(\Omega)$ is achieved. Then the minimizer $u \in \mathcal{H}^{1,2}_{\alpha,\beta}(\Omega)$ of $N_{\alpha,\beta}(\Omega) = \overline{N}_{\alpha,\beta}(\Omega)$ must be a weak solution of the Euler-Lagrange equation

$$-\text{div} \left(|x|^{2-N} \left(\log \frac{1}{|x|} \right)^{2\beta} \nabla u \right) = \overline{N}_{\alpha,\beta}(\Omega) \frac{u}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} \quad \text{in } \Omega.$$

Note that $|u|$ satisfies $|u| \in \mathcal{H}^{1,2}_{\alpha,\beta}(\Omega)$ and $\int_\Omega \frac{|u|^2}{|x|^N \left(\log \frac{1}{|x|} \right)^{2\alpha}} dx = 1$, thus $|u|$ is a minimizer of $\overline{N}_{\alpha,\beta}(\Omega)$ and solves weakly the above Euler-Lagrange equation. Since the weight functions are locally smooth on $\Omega \setminus \{0\}$, standard regularity results imply that $|u| \in C^1(\Omega \setminus \{0\})$. Then the strong maximum principle assures that $|u| > 0$ on $\Omega \setminus \{0\}$, which leads to that u does not change its sign on $\Omega \setminus \{0\}$. However, this is a contradiction to the fact that $\int_\Omega \frac{u}{|x|^N \left(\log \frac{1}{|x|} \right)^{2\alpha}} dx = 0$. $\square$

Now we are going to prove Theorem 4.

Proof of Theorem 4.

Proof. By Proposition 12, it is enough to prove that $N_{\alpha,\beta}(\Omega) = \overline{N}_{\alpha,\beta}(\Omega)$ when $\alpha + \beta \geq 1$ and $2\alpha + \beta \leq \frac{3}{2}$. As noticed before, since $\overline{N}_{\alpha,\beta}(\Omega) \leq N_{\alpha,\beta}(\Omega)$ is trivial, we just need to prove that $N_{\alpha,\beta}(\Omega) \leq \overline{N}_{\alpha,\beta}(\Omega)$.

For this purpose, let $\{u_n\} \subset \mathcal{H}^{1,2}_{\alpha,\beta}(\Omega)$ be a minimizing sequence of $\overline{N}_{\alpha,\beta}(\Omega) = 0$ such that

$$\int_\Omega \frac{u_n^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = 1, \quad \int_\Omega |\nabla u_n|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx = \overline{N}_{\alpha,\beta}(\Omega) + o(1).$$

In the following, we will construct a sequence of functions $\{v_n\} \subset \mathcal{H}^{1,2}_{\alpha,\beta}(\Omega)$ such that

$$\|u_n - v_n\|_{\alpha,\beta} \to 0 \quad (n \to \infty) \quad \text{and} \quad \int_\Omega \frac{v_n}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = 0.$$

If this will be done, then we see

$$N_{\alpha,\beta}(\Omega) \leq Q(v_n, \Omega) = \frac{\int_\Omega |\nabla v_n|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx}{\int_\Omega \frac{v_n^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx} = \frac{\int_\Omega |\nabla u_n|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx + o_n(1)}{\int_\Omega \frac{u_n^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx + o_n(1)}$$

$$= \overline{N}_{\alpha,\beta}(\Omega) + o_n(1),$$

which leads to the desired inequality $N_{\alpha,\beta}(\Omega) \leq \overline{N}_{\alpha,\beta}(\Omega)$. Take $\delta \in (0,1)$ sufficiently small and $s > 1$ and put $h_\delta(x) = 1 - g_{\delta,s}(x)$ where $g_{\delta,s}$ is a cut-off function in (14). Define

$$u_{n,\delta}(x) = u_n(x) h_\delta(x).$$

Note that $u_{n,\delta} = u_n$ on $(B_\delta)^c$, thus

$$\int_\Omega \frac{|u_n - u_{n,\delta}|^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = \int_{B_\delta} \frac{|u_n - u_{n,\delta}|^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = \int_{B_\delta} \frac{(1-h_\delta)^2 |u_n|^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx$$

$$\leq \int_{B_\delta} \frac{|u_n|^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = o_\delta(1)$$

as $\delta \to 0$ for fixed n, and by (17),

$$\int_\Omega |\nabla(u_n - u_{n,\delta})|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx$$

$$\leq 2 \int_{B_\delta} |\nabla u_n|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx + 2 \int_{B_\delta} |u_n|^2 |\nabla g_{\delta,s}|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx$$

$$\leq 2 \int_{B_\delta} |\nabla u_n|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx + C \int_{B_\delta \backslash B_{\delta^s}} \frac{|u_n|^2}{|x|^N (\log \frac{1}{|x|})^2} (\log \frac{1}{|x|})^{2\beta} dx$$

$$= o_\delta(1) + C \int_{B_\delta \backslash B_{\delta^s}} \frac{|u_n|^2}{|x|^N (\log \frac{1}{|x|})^2} (\log \frac{1}{|x|})^{2\beta} dx.$$

Now, the current assumption $\begin{cases} \alpha + \beta \geq 1 \\ 2\alpha + \beta \leq 3/2 \end{cases}$ implies $\alpha \leq 1/2$, so u_n satisfying $\int_\Omega \frac{|u_n|}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = 1$ must be $u_n(0) = 0$. Also we may assume $\beta > 1/2$, since if $\beta = 1/2$ then $\alpha = 1/2$ under the current assumption and the non-attainability of

$N_{1/2,1/2}(\Omega) = 0$ is easily checked. Thus by (21) in the proof of Lemma 1, we see

$$\int_{B_\delta \setminus B_{\delta^s}} \frac{|u_n|^2}{|x|^N (\log \frac{1}{|x|})^2} (\log \frac{1}{|x|})^{2\beta} dx$$

$$\leq \int_{\mathbb{S}^{N-1}} \int_{\delta^s}^\delta \frac{(\log \frac{1}{r})^{2\beta}}{r^N (\log \frac{1}{r})^2} \left(\int_0^r \left| \frac{\partial u_n}{\partial r} \right|^2 t^{2-N} (\log \frac{1}{t})^{2\beta} t^{N-1} dt \right) \left(\underbrace{\int_0^r \frac{dt}{t (\log \frac{1}{t})^{2\beta}}}_{= \frac{1}{2\beta-1} (\log \frac{1}{r})^{1-2\beta}} \right) r^{N-1} dr dS_\omega$$

$$\leq \int_{\mathbb{S}^{N-1}} \int_{\delta^s}^\delta \left(\frac{1}{2\beta - 1} \right) \frac{1}{r \log \frac{1}{r}} \left(\int_0^\delta \left| \frac{\partial u_n}{\partial r} \right|^2 t^{2-N} (\log \frac{1}{t})^{2\beta} t^{N-1} dt \right) dr dS_\omega$$

$$\leq \left(\frac{1}{2\beta - 1} \right) \left(\int_{B_\delta} |\nabla u_n|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx \right) \underbrace{\int_{\delta^s}^\delta \frac{dr}{r \log \frac{1}{r}}}_{= \log s}$$

$$\leq C \int_{B_\delta} |\nabla u_n|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2\beta} dx = o_\delta(1)$$

as $\delta \to 0$ for fixed n. Thus we have seen that $\|u_n - u_{n,\delta}\|_{\alpha,\beta} = o_\delta(1)$ as $\delta \to 0$ for fixed n. From this, we see there exists $\{\delta_n\} \subset \mathbb{R}_+$ such that $\delta_n \to 0$ and

$$\|u_n - u_{n,\delta_n}\|_{\alpha,\beta} \to 0$$

as $n \to \infty$.

Now, we divide the proof in two cases.

Case1: $2\alpha + \beta < 3/2$.

In this case, together with another assumption $\alpha + \beta \geq 1$, we automatically have $\alpha < 1/2$, so we can exploit Proposition 7. We construct desired functions v_n of the form

$$v_n(x) = u_{n,\delta_n}(x) - C_n \psi_{\varepsilon_n}(x)$$

where ψ_{ε_n} is a function in Proposition 7 with $\delta = \varepsilon_n$ and

$$C_n = \frac{\int_\Omega \frac{u_{n,\delta_n}}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx}{\int_\Omega \frac{\psi_{\varepsilon_n}}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx},$$

so that $\int_\Omega \frac{v_n}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = 0$. Here $\varepsilon_n > 0$ is chosen so that

$$|C_n| = \left| \frac{\int_\Omega \frac{u_{n,\delta_n}}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx}{\int_\Omega \frac{\psi_{\varepsilon_n}}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx} \right| \leq 1 \quad \text{and} \quad \varepsilon_n \to 0,$$

as $n \to \infty$. We know by Proposition 7 that

$$T_0(\psi_{\varepsilon_n}) = \int_\Omega \frac{\psi_{\varepsilon_n}}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx \to +\infty \quad (n \to \infty),$$

so we can choose such $\varepsilon_n > 0$. Also by Proposition 7, we see $\|\psi_{\varepsilon_n}\|_{\alpha,\beta} = o_n(1)$, thus

$$\|v_n - u_n\|_{\alpha,\beta} \le \underbrace{\|u_{n,\delta_n} - u_n\|_{\alpha,\beta}}_{=o_n(1)} + \underbrace{|C_n|}_{\le 1}\underbrace{\|\psi_{\varepsilon_n}\|_{\alpha,\beta}}_{=o_n(1)} \to 0$$

as $n \to \infty$.

Case2: $2\alpha + \beta = 3/2$.

In this case, we construct desired functions v_n of the form

$$v_n(x) = u_{n,\delta_n}(x) - C_n(\log \frac{1}{|x|})^{A_n}$$

where

$$A_n = 2\alpha - 1 - \delta_n$$

for $\delta_n > 0$ small and

$$C_n = \frac{\int_\Omega \frac{u_{n,\delta_n}}{|x|^N(\log \frac{1}{|x|})^{2\alpha}}dx}{\int_\Omega \frac{(\log \frac{1}{|x|})^{A_n}}{|x|^N(\log \frac{1}{|x|})^{2\alpha}}dx},$$

Note that the denominator of C_n is estimated from the below as

$$\int_\Omega \frac{(\log \frac{1}{|x|})^{A_n}}{|x|^N(\log \frac{1}{|x|})^{2\alpha}}dx \ge 2\pi \int_0^{\delta_n} \frac{dr}{r(\log \frac{1}{r})^{1+\delta_n}} = 2\pi\left(\frac{1}{\delta_n}\right)\left(\log \frac{1}{\delta_n}\right)^{-\delta_n},$$

since $2\alpha - A_n = 1 + \delta_n > 1$. Also the numerator of C_n is estimated from the above by using (22) in the proof of Lemma 1:

$$\int_\Omega \frac{u_{n,\delta_n}}{|x|^N(\log \frac{1}{|x|})^{2\alpha}}dx \le \int_{\Omega\setminus B_{\delta^s}} \frac{|u_n|}{|x|^N(\log \frac{1}{|x|})^{2\alpha}}dx$$

$$\overset{(22)}{\le} \int_{\Omega\setminus B_{\delta^s}} \frac{C\|u_n\|_{\alpha,\beta}(\log \frac{1}{|x|})^{1/2-\beta}}{|x|^N(\log \frac{1}{|x|})^{2\alpha}}dx$$

$$\le C\|u_n\|_{\alpha,\beta}\int_{\delta_n^s}^{R_\Omega} \frac{dr}{r(\log \frac{1}{r})^{2\alpha+\beta-\frac{1}{2}}}$$

$$\le C\sup_{n\in\mathbb{N}}\|u_n\|_{\alpha,\beta}\left[-\log\log\frac{1}{r}\right]_{\delta_n^s}^{R_\Omega}$$

$$\le C|\log\log\frac{1}{\delta_n}|.$$

Thus we have the estimate

$$(41) \qquad |C_n| \le C\frac{|\log\log\frac{1}{\delta_n}|}{\left(\frac{1}{\delta_n}\right)\left(\log\frac{1}{\delta_n}\right)^{-\delta_n}} = C\delta_n\left(\log\log\frac{1}{\delta_n}\right).$$

Now, the assumption implies $\alpha \leq 1/2$ and in this case we have

$$\int_\Omega \frac{\left(\log \frac{1}{|x|}\right)^{2A_n}}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx \leq |\mathbb{S}^{N-1}| \int_0^{R_\Omega} \frac{(\log \frac{1}{r})^{2A_n - 2\alpha}}{r} dr$$

$$= \left(\frac{1}{2\delta_n + 1 - 2\alpha}\right) \left(\log \frac{1}{R_\Omega}\right)^{2\alpha - 1 - 2\delta_n} \leq \frac{C}{\delta_n}.$$

Also we see

$$\int_\Omega \left| \nabla \left(\left(\log \frac{1}{|x|}\right)^{A_n} \right) \right|^2 |x|^{2-N} \left(\log \frac{1}{|x|}\right)^{2\beta} dx = A_n^2 \int_\Omega \frac{(\log \frac{1}{|x|})^{2(A_n - 1) + 2\beta}}{|x|^N} dx$$

$$\leq |\mathbb{S}^{N-1}| A_n^2 \int_0^{R_\Omega} \frac{(\log \frac{1}{r})^{2A_n + 2\beta - 2}}{r} dr$$

$$= |\mathbb{S}^{N-1}| A_n^2 \underbrace{\left(\frac{1}{1 - 2A_n - 2\beta}\right)}_{= \frac{1}{2\delta_n}} \left(\log \frac{1}{R_\Omega}\right)^{2A_n + 2\beta - 1}$$

since $2A_n + 2\beta - 2 < -1 \iff 2\alpha + \beta < 3/2 + \delta_n$ is satisfied. By a standard truncation procedure with Lemma 3 implies $(\log \frac{1}{|x|})^{A_n} \in \mathcal{H}_{\alpha,\beta}^{1,2}(\Omega)$ and

(42)
$$\|(\log \frac{1}{|x|})^{A_n}\|_{\alpha,\beta}^2 < C \left(\frac{1}{\delta_n}\right).$$

Thus by (41) and (42), we see

$$\|v_n\|_{\alpha,\beta} \leq \|u_{n,\delta_n} - u_n\|_{\alpha,\beta} + |C_n| \|(\log \frac{1}{|x|})^{A_n}\|_{\alpha,\beta}$$

$$\leq o_n(1) + C\delta_n \left(\log\log \frac{1}{\delta_n}\right) \sqrt{\left(\frac{1}{\delta_n}\right)}$$

$$\to 0$$

as $n \to \infty$. Thus v_n is the desired function in this case.

In both cases, by testing $N_{\alpha,\beta}(\Omega)$ by v_n above, we obtain $N_{\alpha,\beta}(\Omega) \leq \overline{N}_{\alpha,\beta}(\Omega)$. This completes the proof of Theorem 4. $\qquad \square$

6. Proof of Theorem 5 and Theorem 6.

In this section, we will prove Theorem 5 and Theorem 6.

Proof of Theorem 5.

Proof. First, we claim that

(43)
$$N_{\alpha,1-\alpha}(\Omega) \leq H_{\alpha,1-\alpha}(\Omega) = (\alpha - 1/2)^2$$

if $\alpha \geq 1/2$.

We have already seen that $N_{1/2,1/2}(\Omega) = 0 = H_{1/2,1/2}(\Omega)$, so we assume $\alpha > 1/2$. Also we may assume $B_{\delta_0} \subset\subset \Omega$ for some $\delta_0 \in (0,1)$. For $0 < \varepsilon < \alpha - 1/2$ small, we

define

$$u_\varepsilon(x) = \begin{cases} (\log \frac{1}{|x|})^{A_\varepsilon} - (\log \frac{1}{\delta_0})^{A_\varepsilon}, & 0 \le |x| \le \delta_0, \\ 0, & \{\delta_0 \le |x|\} \cap \Omega, \end{cases}$$

$$v_\varepsilon(x) = u_\varepsilon(x) - C_\varepsilon,$$

where $A_\varepsilon = \alpha - 1/2 - \varepsilon > 0$ and

$$C_\varepsilon = \frac{\int_\Omega \frac{u_\varepsilon}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx}{\int_\Omega \frac{dx}{|x|^N (\log \frac{1}{|x|})^{2\alpha}}}.$$

Note that the denominator of C_ε is finite since $\alpha > 1/2$. By the choice of C_ε, we see

(44)
$$\int_\Omega \frac{v_\varepsilon}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = 0.$$

We compute

$$\int_{B_{\delta_0}} \frac{(\log \frac{1}{|x|})^{A_\varepsilon}}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = \left(\frac{|\mathbb{S}^{N-1}|}{\alpha - \frac{1}{2} + \varepsilon} \right) \left(\log \frac{1}{\delta_0} \right)^{-\alpha + \frac{1}{2} - \varepsilon},$$

$$\int_{B_{\delta_0}} \frac{(\log \frac{1}{|x|})^{2A_\varepsilon}}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = \left(\frac{|\mathbb{S}^{N-1}|}{2\varepsilon} \right) \left(\log \frac{1}{\delta_0} \right)^{-2\varepsilon},$$

$$\int_{B_{\delta_0}} \frac{dx}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = \left(\frac{|\mathbb{S}^{N-1}|}{2\alpha - 1} \right) \left(\log \frac{1}{\delta_0} \right)^{1-2\alpha}$$

for $A_\varepsilon = \alpha - 1/2 - \varepsilon$. By these we have

$$\int_\Omega \frac{u_\varepsilon}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = \int_{B_{\delta_0}} \frac{(\log \frac{1}{|x|})^{A_\varepsilon}}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx - (\log \frac{1}{\delta_0})^{A_\varepsilon} \int_{B_{\delta_0}} \frac{dx}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx$$

$$= |\mathbb{S}^{N-1}| (\log \frac{1}{\delta_0})^{-\alpha + \frac{1}{2} - \varepsilon} \left(\frac{1}{\alpha - \frac{1}{2} + \varepsilon} - \frac{1}{2\alpha - 1} \right)$$

$$= C (\log \frac{1}{\delta_0})^{-\alpha + \frac{1}{2} - \varepsilon},$$

$$\int_\Omega \frac{u_\varepsilon^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx$$

$$= \int_{B_{\delta_0}} \frac{(\log \frac{1}{|x|})^{2A_\varepsilon}}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx - 2(\log \frac{1}{\delta_0})^{A_\varepsilon} \int_{B_{\delta_0}} \frac{(\log \frac{1}{|x|})^{A_\varepsilon}}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx + (\log \frac{1}{\delta_0})^{2A_\varepsilon} \int_{B_{\delta_0}} \frac{dx}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx$$

$$= |\mathbb{S}^{N-1}| (\log \frac{1}{\delta_0})^{-2\varepsilon} \left(\frac{1}{2\varepsilon} - \frac{2}{\alpha - \frac{1}{2} + \varepsilon} + \frac{1}{2\alpha - 1} \right)$$

$$= |\mathbb{S}^{N-1}| (\log \frac{1}{\delta_0})^{-2\varepsilon} \left(\frac{1}{2\varepsilon} + O(1) \right).$$

Also we have

$$|\nabla u_\varepsilon|^2 = A_\varepsilon^2 \frac{(\log \frac{1}{|x|})^{2(A_\varepsilon - 1)}}{|x|^2} \chi_{B_{\delta_0}},$$

thus

$$\int_\Omega |\nabla u_\varepsilon|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2(1-\alpha)} dx = A_\varepsilon^2 \int_{B_{\delta_0}} \frac{(\log \frac{1}{|x|})^{2(A_\varepsilon-1)+2(1-\alpha)}}{|x|^2} |x|^{2-N} dx$$

$$= |\mathbb{S}^{N-1}| A_\varepsilon^2 \frac{(\log \frac{1}{\delta_0})^{2A_\varepsilon-2\alpha+1}}{2\alpha-1-2A_\varepsilon}$$

$$= \frac{|\mathbb{S}^{N-1}|}{2\varepsilon} A_\varepsilon^2 (\log \frac{1}{\delta_0})^{-2\varepsilon}.$$

From these, we see

$$\int_\Omega |\nabla v_\varepsilon|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2(1-\alpha)} dx \overset{(44)}{=} \frac{|\mathbb{S}^{N-1}|}{2\varepsilon} A_\varepsilon^2 (\log \frac{1}{\delta_0})^{-2\varepsilon},$$

$$\int_\Omega \frac{v_\varepsilon^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = \int_\Omega \frac{u_\varepsilon^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx - C_\varepsilon \int_\Omega \frac{u_\varepsilon}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx$$

$$= |\mathbb{S}^{N-1}| (\log \frac{1}{\delta_0})^{-2\varepsilon} \left(\frac{1}{2\varepsilon} + O(1) \right) - \frac{\left(C(\log \frac{1}{\delta_0})^{-\alpha+\frac{1}{2}-\varepsilon} \right)^2}{C(\log \frac{1}{\delta_0})^{1-2\alpha}}$$

$$= C(\log \frac{1}{\delta_0})^{-2\varepsilon} \left(\frac{1}{\varepsilon} + O(1) \right) - C(\log \frac{1}{\delta_0})^{-2\varepsilon}$$

By (44), we may test $N_{\alpha,1-\alpha}(\Omega)$ by v_ε. Then we have

$$N_{\alpha,1-\alpha}(\Omega) \le Q(v_\varepsilon, \Omega) = \frac{\int_\Omega |\nabla v_\varepsilon|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2(1-\alpha)} dx}{\int_\Omega \frac{v_\varepsilon^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx}$$

$$= \frac{\frac{|\mathbb{S}^{N-1}|}{2\varepsilon} A_\varepsilon^2 (\log \frac{1}{\delta_0})^{-2\varepsilon}}{C(\log \frac{1}{\delta_0})^{-2\varepsilon} \left(\frac{1}{\varepsilon} + O(1) \right) + C(\log \frac{1}{\delta_0})^{-2\varepsilon}}$$

$$= A_\varepsilon^2 \frac{1}{(1+O(\varepsilon)) + C\varepsilon}$$

$$\to \lim_{\varepsilon \to 0} A_\varepsilon^2 = (\alpha - 1/2)^2$$

as $\varepsilon \to 0$. Thus we obtain that $N_{\alpha,1-\alpha}(\Omega) \le (\alpha-1/2)^2 = H_{\alpha,1-\alpha}(\Omega)$ when $\alpha \ge 1/2$. This completes the proof of the claim (43).

Next, we will prove that if $N_{\alpha,1-\alpha}(\Omega) < H_{\alpha,1-\alpha}(\Omega)$, then $N_{\alpha,1-\alpha}(\Omega)$ is achieved. Here, we provide a proof of this claim without the use of concentration compactness lemma type argument; see also Theorem 2 in [23].

First, we prove the following inequality:

Proposition 13. *Let $\alpha \in \mathbb{R}$ be such that $H_{\alpha,1-\alpha}(\Omega) > 0$ (i.e., $\alpha \ne 1/2$). Then for any $\varepsilon > 0$ and $\delta \in (0,1)$ such that $B_\delta \subset\subset \Omega$, there exists $C(\varepsilon,\delta) > 0$ such that*

(45)

$$\int_\Omega \frac{|u|^2}{|x|^N \left(\log \frac{1}{|x|} \right)^{2\alpha}} dx \le \left(\frac{1}{H_{\alpha,1-\alpha}(\Omega)} + \varepsilon \right) \int_\Omega |\nabla u|^2 |x|^{2-N} \left(\log \frac{1}{|x|} \right)^{2(1-\alpha)} dx$$

(46)

$$+ C(\varepsilon,\delta) \int_{\Omega \setminus B_{\delta/2}} u^2 dx$$

holds for any $u \in \mathcal{H}^{1,2}_{\alpha,1-\alpha}(\Omega)$.

Proof. First, since

$$\infty > \int_\Omega \frac{|u|^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx \geq \inf_{x \in \Omega} \left(\frac{1}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} \right) \int_\Omega |u|^2 dx$$

for $u \in \mathcal{H}^{1,2}_{\alpha,1-\alpha}(\Omega)$, $\mathcal{H}^{1,2}_{\alpha,1-\alpha}(\Omega) \hookrightarrow L^2(\Omega)$ follows. Take a cut-off function $\phi \in C_0^\infty(\Omega)$ such that $\phi \equiv 1$ for $0 \leq |x| \leq \delta/2$, $\phi \equiv 0$ on $\Omega \setminus B_\delta$, $0 \leq \phi \leq 1$. Then $\text{supp}\,(\phi u) \subset\subset \Omega$ and

$$\int_\Omega \frac{|\phi u|^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx \leq \int_\Omega \frac{|u|^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx < \infty,$$

$$\int_\Omega |\nabla(\phi u)|^2 |x|^{2-N} \left(\log \frac{1}{|x|} \right)^{2(1-\alpha)} dx$$

$$\leq 2 \int_\Omega |\nabla u|^2 |\phi|^2 |x|^{2-N} \left(\log \frac{1}{|x|} \right)^{2(1-\alpha)} dx + 2 \int_{B_\delta \setminus B_{\delta/2}} |\nabla \phi|^2 |u|^2 |x|^{2-N} \left(\log \frac{1}{|x|} \right)^{2(1-\alpha)} dx$$

$$\leq 2 \int_\Omega |\nabla u|^2 |x|^{2-N} \left(\log \frac{1}{|x|} \right)^{2(1-\alpha)} dx$$

$$+ 2\|\nabla \phi\|^2_{L^\infty(\Omega)} \max_{x \in B_\delta \setminus B_{\delta/2}} \left\{ |x|^2 \left(\log \frac{1}{|x|} \right)^2 \right\} \int_{B_\delta \setminus B_{\delta/2}} \frac{|u|^2}{|x|^N \left(\log \frac{1}{|x|} \right)^{2\alpha}} dx < \infty.$$

Thus $\phi u \in \mathcal{H}^{1,2}_{0,\alpha,1-\alpha}(\Omega)$ and applying the critical Hardy inequality (6) to ϕu, we obtain

$$\int_\Omega \frac{|u|^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = \int_\Omega \frac{|\phi u|^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx + \int_\Omega \frac{(1-\phi^2)|u|^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx$$

$$\leq \frac{1}{H} \int_\Omega |\nabla(\phi u)|^2 |x|^{2-N} \left(\log \frac{1}{|x|} \right)^{2(1-\alpha)} dx + \int_\Omega \frac{(1-\phi^2)|u|^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx$$

where $H := H_{\alpha,1-\alpha}(\Omega)$. By Young inequality, we see

$$|\nabla(\phi u)|^2 \leq (1 + H\varepsilon) |\nabla u|^2 \phi^2 + \left(1 + \frac{1}{H\varepsilon} \right) |\nabla \phi|^2 u^2$$

and

$$\frac{1}{H} \int_\Omega |\nabla(\phi u)|^2 |x|^{2-N} \left(\log \frac{1}{|x|} \right)^{2(1-\alpha)} dx$$

$$\leq \left(\frac{1}{H} + \varepsilon \right) \int_\Omega |\nabla u|^2 |x|^{2-N} \left(\log \frac{1}{|x|} \right)^{2(1-\alpha)} dx + \left(\frac{1}{H} + \frac{1}{H^2 \varepsilon} \right) \int_\Omega |\nabla \phi|^2 u^2 |x|^{2-N} \left(\log \frac{1}{|x|} \right)^{2(1-\alpha)} dx.$$

From the above, we obtain

$$\int_\Omega \frac{|u|^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx \leq (\frac{1}{H} + \varepsilon) \int_\Omega |\nabla u|^2 |x|^{2-N} \left(\log \frac{1}{|x|} \right)^{2(1-\alpha)} dx$$

$$+ \int_\Omega |u|^2 \underbrace{\left(\frac{1-\phi^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} + \left(\frac{1}{H} + \frac{1}{H^2 \varepsilon} \right) |\nabla \phi|^2 |x|^{2-N} \left(\log \frac{1}{|x|} \right)^{2(1-\alpha)} \right)}_{=:\Phi(x)} dx.$$

Note that

$$\Phi(x) := \frac{1 - \phi(x)^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} + \left(\frac{1}{H} + \frac{1}{H^2 \varepsilon}\right) |\nabla \phi(x)|^2 |x|^{2-N} \left(\log \frac{1}{|x|}\right)^{2(1-\alpha)} \equiv 0 \quad \text{on } B_{\delta/2}.$$

Thus we have

$$\int_\Omega \frac{|u|^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx \leq (H^{-1} + \varepsilon) \int_\Omega |\nabla u|^2 |x|^{2-N} \left(\log \frac{1}{|x|}\right)^{2(1-\alpha)} dx$$

$$+ \max_{x \in \Omega \setminus B_{\delta/2}} |\Phi(x)| \int_{\Omega \setminus B_{\delta/2}} |u|^2 dx$$

for $u \in \mathcal{H}^{1,2}_{\alpha,1-\alpha}(\Omega)$. This proves Proposition 13. $\qquad\square$

Let us return to the proof of Theorem 5. Take a minimizing sequence $\{u_n\} \subset \mathcal{A}_N$ for $N_{\alpha,1-\alpha}(\Omega)$ where $\mathcal{A}_N$ is defined as in (5) for $\beta = 1 - \alpha$. Thus

$$\int_\Omega \frac{u_n^2}{||x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = 1, \quad \int_\Omega \frac{u_n}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = 0,$$

$$\text{and} \quad \int_\Omega |\nabla u_n|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2(1-\alpha)} dx = N_{\alpha,1-\alpha}(\Omega) + o(1).$$

Then there exists a subsequence and $u \in \mathcal{H}^{1,2}_{\alpha,1-\alpha}(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } \mathcal{H}^{1,2}_{\alpha,1-\alpha}(\Omega),$$

$$u_n \to u \quad \text{strongly in } L^2(\Omega \setminus B_{\delta_0}, \frac{1}{|x|^N (\log \frac{1}{|x|})^{2\alpha}}),$$

$$u_n \to u \quad \text{strongly in } L^2(\Omega \setminus B_{\delta_0})$$

for any small $\delta_0 \in (0, 1)$. Now the assumption $\alpha \geq 1/2$ and $N_{\alpha,1-\alpha}(\Omega) < (\alpha - \frac{1}{2})^2$ implies that $\alpha > 1/2$. Therefore the average functional $T_0 : \mathcal{H}^{1,2}_{\alpha,1-\alpha}(\Omega) \to \mathbb{R}$, $T_0(u) = \int_\Omega \frac{u}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx$ is continuous by Proposition 6. This assures the weak limit u must satisfy that

$$T_0(u) = \int_\Omega \frac{u}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = \lim_{n \to \infty} T(u_n) = 0.$$

Now, we claim that

$$(47) \qquad \int_\Omega \frac{|u|^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = 1$$

holds. If this is true, then it is easy to see that $u \in \mathcal{A}_N$ is a minimizer for $N_{\alpha,1-\alpha}(\Omega)$ and the proof will be completed.

In order to prove (47), first we see the weak limit $u \not\equiv 0$. Indeed, if $u \equiv 0$, then Proposition 13 for u_n and $u_n \to u \equiv 0$ strongly in $L^2(\Omega \setminus B_{\delta_0})$ implies

$$1 = \int_\Omega \frac{|u_n|^2}{|x|^N \left(\log \frac{1}{|x|}\right)^{2\alpha}} dx \le \left(\frac{1}{H_{\alpha,1-\alpha}(\Omega)} + \varepsilon\right) \underbrace{\int_\Omega |\nabla u_n|^2 |x|^{2-N} \left(\log \frac{1}{|x|}\right)^{2(1-\alpha)} dx}_{=N_{\alpha,1-\alpha}(\Omega)+o_n(1)}$$

$$+ C(\varepsilon, \delta_0) \underbrace{\int_{\Omega \setminus B_{\delta_0}} u_n^2 dx}_{=o_n(1)}$$

$$\le \left(\frac{1}{H_{\alpha,1-\alpha}(\Omega)} + \varepsilon\right) (N_{\alpha,1-\alpha}(\Omega) + o_n(1))$$

as $n \to \infty$. This implies that $H_{\alpha,1-\alpha}(\Omega) \le N_{\alpha,1-\alpha}(\Omega)$, which contradicts to the assumption $H_{\alpha,1-\alpha}(\Omega) > N_{\alpha,1-\alpha}(\Omega)$. Thus we have proven that $u \not\equiv 0$.

Now, we will prove (47). By the weak lower semi-continuity, we see that $\int_\Omega \frac{|u|^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx \le 1$. Assume the contrary that $\int_\Omega \frac{|u|^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx < 1$. Then

$$0 < 1 - \int_\Omega \frac{u^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = \int_\Omega \frac{u_n^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx - \int_\Omega \frac{u^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx$$

$$= \int_\Omega \frac{|u_n - u|^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx + \underbrace{\int_\Omega \frac{2u(u_n - u)}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx}_{=o_n(1)}$$

$$= \int_\Omega \frac{|u_n - u|^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx + o_n(1)$$

because $u_n \to u$ weakly in $L^2(\Omega, \frac{1}{|x|^N (\log \frac{1}{|x|})^{2\alpha}})$. Again, we apply the Neumann Hardy inequality (45) to the last term to obtain

$$\int_\Omega \frac{|u_n - u|^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx \le \left(\frac{1}{H_{\alpha,1-\alpha}(\Omega)} + \varepsilon\right) \int_\Omega |\nabla(u_n - u)|^2 |x|^{2-N} \left(\log \frac{1}{|x|}\right)^{2(1-\alpha)} dx$$

$$+ C(\varepsilon, \delta_0) \underbrace{\int_{\Omega \setminus B_{\delta_0}} |u_n - u|^2 dx}_{=o_n(1)}$$

$$\le \left(\frac{1}{H_{\alpha,1-\alpha}(\Omega)} + \varepsilon\right) \left(\int_\Omega |\nabla u_n|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2(1-\alpha)} dx - \int_\Omega |\nabla u|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2(1-\alpha)} dx + o_n(1)\right)$$

$$\le \left(\frac{1}{H_{\alpha,1-\alpha}(\Omega)} + \varepsilon\right) \left(N_{\alpha,1-\alpha}(\Omega) + o_n(1) - \int_\Omega |\nabla u|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2(1-\alpha)} dx\right)$$

$$\le \left(\frac{1}{H_{\alpha,1-\alpha}(\Omega)} + \varepsilon\right) \left(N_{\alpha,1-\alpha}(\Omega) + o_n(1) - N_{\alpha,1-\alpha}(\Omega) \int_\Omega \frac{|u|^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx\right),$$

since

$$N_{\alpha,1-\alpha}(\Omega) \int_\Omega \frac{|u|^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx \le \int_\Omega |\nabla u|^2 |x|^{2-N} \left(\log \frac{1}{|x|}\right)^{2(1-\alpha)} dx$$

holds for $u \in \mathcal{H}^{1,2}_{\alpha,1-\alpha}(\Omega)$ such that $T_0(u) = 0$. Thus we obtain

$$0 < 1 - \int_\Omega \frac{u^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx \leq \left(\frac{1}{H_{\alpha,1-\alpha}(\Omega)} + \varepsilon \right) N_{\alpha,1-\alpha}(\Omega) \left(1 - \int_\Omega \frac{|u|^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx \right)$$

when $n \to \infty$. Deviding the both sides by $1 - \int_\Omega \frac{u^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx > 0$ and taking a

limit $\varepsilon \to 0$, we obtain $1 \leq \frac{N_{\alpha,1-\alpha}(\Omega)}{H_{\alpha,1-\alpha}(\Omega)}$, which is a contradiction to the assumption.

$\square$

Proof of Theorem 6.

Proof. Let $R \in (0,1)$ be given. In order to prove that $N_{\alpha,1-\alpha}(B_R)$ is achieved, from Theorem 5, it is sufficient to find a condition on $\alpha > 1/2$ to assure that $N_{\alpha,1-\alpha}(B_R) < (\alpha - 1/2)^2$. As in [10], [5], we use a test function of the form

$$\phi(x) = \frac{x_1}{|x|} = \omega_1$$

for $x = (x_1, \ldots, x_N) = r\omega$, where $\omega = (\omega_1, \ldots, \omega_N) \in \mathbb{S}^{N-1}$. Then easily we have

$$T_0(\phi) = \int_{B_R} \frac{\phi}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = \left(\int_{\mathbb{S}^{N-1}} \omega_1 dS_\omega \right) \int_0^R \frac{dr}{r(\log \frac{1}{r})^{2\alpha}} = 0.$$

$$\int_{B_R} \frac{\phi^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx = \left(\int_{\mathbb{S}^{N-1}} \omega_1^2 dS_\omega \right) \int_0^R \frac{dr}{r(\log \frac{1}{r})^{2\alpha}} = \frac{|\mathbb{S}^{N-1}|}{N} \left(\frac{1}{2\alpha - 1} \right) (\log \frac{1}{R})^{1-2\alpha},$$

since $\int_{\mathbb{S}^{N-1}} \omega_1^2 dS_\omega = \frac{1}{N} |\mathbb{S}^{N-1}|$. Also since

$$\nabla \phi = \left(\omega \frac{\partial}{\partial r} + \frac{1}{r} \nabla_{\mathbb{S}^{N-1}} \right) (\omega_1) = \frac{1}{r} \nabla_{\mathbb{S}^{N-1}} \omega_1,$$

we compute

$$\int_{B_R} |\nabla \phi|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2(1-\alpha)} dx = \left(\int_{\mathbb{S}^{N-1}} |\nabla_{\mathbb{S}^{N-1}} \omega_1|^2 dS_\omega \right) \int_0^R \frac{(\log \frac{1}{r})^{2(1-\alpha)}}{r} dr$$

$$= \left(\frac{N-1}{N} \right) |\mathbb{S}^{N-1}| \left(\frac{1}{2\alpha - 3} \right) (\log \frac{1}{R})^{3-2\alpha}$$

when $\alpha > 3/2$. Here, we have used the fact that $-\Delta_{\mathbb{S}^{N-1}} \omega_1 = (N-1)\omega_1$. From these estimates, we see

$$N_{\alpha,1-\alpha}(B_R) \leq \frac{\int_{B_R} |\nabla \phi|^2 |x|^{2-N} (\log \frac{1}{|x|})^{2(1-\alpha)} dx}{\int_{B_R} \frac{\phi^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx} = \frac{\left(\frac{N-1}{N} \right) |\mathbb{S}^{N-1}| \left(\frac{1}{2\alpha - 3} \right) (\log \frac{1}{R})^{3-2\alpha}}{\left(\frac{1}{N} \right) |\mathbb{S}^{N-1}| \left(\frac{1}{2\alpha - 1} \right) (\log \frac{1}{R})^{1-2\alpha}}$$

$$= (N-1) \left(\frac{2\alpha - 1}{2\alpha - 3} \right) \left(\log \frac{1}{R} \right)^2$$

when $\alpha > 3/2$. Therefore, if

(48)
$$(N-1) \left(\frac{2\alpha - 1}{2\alpha - 3} \right) \left(\log \frac{1}{R} \right)^2 < (\alpha - 1/2)^2$$

holds, then $N_{\alpha,1-\alpha}(B_R)$ is achieved by Theorem 5. The quadratic inequality (48) for $\alpha > 3/2$ is easily solved and is equivalent to $\alpha > 1 + \sqrt{\frac{1}{4} + (N-1)\left(\log \frac{1}{R}\right)^2}$. This proves Theorem 6 $\qquad\square$

7. Appendix.

Recall
$$\mathcal{H}_{\alpha,\beta}^{1,2}(\Omega) = \overline{Y_{\alpha,\beta}(\Omega)}^{\|\cdot\|_{\alpha,\beta}} = \overline{\{u \in C^\infty(\Omega) \mid \|u\|_{\alpha,\beta} < \infty\}}^{\|\cdot\|_{\alpha,\beta}},$$
$$\mathcal{H}_{0,\alpha,\beta}^{1,2}(\Omega) = \overline{Y_{0,\alpha,\beta}(\Omega)}^{\|\cdot\|_{\alpha,\beta}} = \overline{\{u \in C_0^\infty(\Omega) \mid \|u\|_{\alpha,\beta} < \infty\}}^{\|\cdot\|_{\alpha,\beta}},$$
where
$$\|u\|_{\alpha,\beta}^2 = \int_\Omega \left(|\nabla u|^2 |x|^{2-N}(\log \frac{1}{|x|})^{2\beta} + \frac{u^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}}\right) dx.$$

Therefore, $u \in \mathcal{H}_{\alpha,\beta}^{1,2}(\Omega)$ if and only if $u \in L^2(\Omega, \frac{1}{|x|^N (\log \frac{1}{|x|})^{2\alpha}})$, and there exists a sequence $\{\phi_j\} \subset Y_{\alpha,\beta}(\Omega)$, $\mathbb{R}^N$-valued function $\vec{v} \in L^2(\Omega, |x|^{2-N}(\log \frac{1}{|x|})^{2\beta}; \mathbb{R}^N)$ such that
$$\int_\Omega \frac{|u - \phi_j|^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx \to 0 \quad (j \to \infty), \quad \text{and}$$
$$\int_\Omega |\vec{v} - \nabla \phi_j|^2 |x|^{2-N}(\log \frac{1}{|x|})^{2\beta} dx \to 0 \quad (j \to \infty).$$

$\vec{v}$ above is called a *strong* derivative of $u \in \mathcal{H}_{\alpha,\beta}^{1,2}(\Omega)$.

First, we show that any $u \in \mathcal{H}_{\alpha,\beta}^{1,2}(\Omega)$ is in $L^1(\Omega)$ for any $(\alpha,\beta) \in \mathbb{R} \times \mathbb{R}$ and its strong derivative $\vec{v}$ coincides with its weak (distributional) derivative ∇u for any α, β.

Lemma 2. *There holds $\mathcal{H}_{\alpha,\beta}^{1,2}(\Omega) \subset L^1(\Omega)$ for any $\alpha, \beta \in \mathbb{R}$ and $\vec{v} = \nabla u$ for any $u \in \mathcal{H}_{\alpha,\beta}^{1,2}(\Omega)$, where $\vec{v}$ is a strong derivative and ∇u is a distributional (weak) derivative of $u \in \mathcal{H}_{\alpha,\beta}^{1,2}(\Omega)$ respectively.*

Proof. Since
$$\int_\Omega |u| dx \leq \left(\int_\Omega \frac{|u|^2}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} dx\right)^{1/2} \left(\int_\Omega |x|^N (\log \frac{1}{|x|})^{2\alpha} dx\right)^{1/2}$$
and
$$\int_\Omega |x|^N (\log \frac{1}{|x|})^{2\alpha} dx \leq |\mathbb{S}^{N-1}| \int_0^{R_\Omega} r^{2N-1}(\log \frac{1}{r})^{2\alpha} dr < \infty,$$
we see $u \in L^1(\Omega)$ and $\|u\|_{L^1(\Omega)} \leq C\|u\|_{\alpha,\beta}$. Similarly, we have
$$\int_\Omega |\vec{v}| dx \leq \left(\int_\Omega |\vec{v}|^2 |x|^{2-N}(\log \frac{1}{|x|})^{2\beta} dx\right)^{1/2} \left(\int_\Omega \frac{dx}{|x|^{2-N}(\log \frac{1}{|x|})^{2\beta}}\right)^{1/2} \leq C\|u\|_{\alpha,\beta}$$
since $\int_\Omega \frac{dx}{|x|^{2-N}(\log \frac{1}{|x|})^{2\beta}} \leq |\mathbb{S}^{N-1}| \int_0^{R_\Omega} (\log \frac{1}{r})^{2\beta} r \, dr < +\infty$. Thus we have the continuous embedding
$$\mathcal{H}_{\alpha,\beta}^{1,2}(\Omega) \hookrightarrow L^1(\Omega)$$

for any $(\alpha, \beta) \in \mathbb{R}^2$. Let $\{\phi_j\} \in Y_{\alpha,\beta}(\Omega)$ such that

$$\phi_j \to u \quad \text{in } L^2(\Omega, \frac{1}{|x|^N (\log \frac{1}{|x|})^{2\alpha}}) \hookrightarrow L^1(\Omega),$$

$$\nabla \phi_j \to \vec{v} \quad \text{in } L^2(\Omega, |x|^{2-N} (\log \frac{1}{|x|})^{2\beta}; \mathbb{R}^N) \hookrightarrow L^1(\Omega; \mathbb{R}^N),$$

where $\vec{v} = (v_1, \ldots, v_N)$ is a strong derivative of $u \in \mathcal{H}^{1,2}_{\alpha,\beta}(\Omega)$. Then for any $\phi \in C_0^\infty(\Omega)$ and $i = 1, 2$, we see

$$\left| \int_\Omega u \frac{\partial \phi}{\partial x_i} + \phi v_i dx \right| = \left| \int_{\text{supp}\phi} (u - \phi_j) \frac{\partial \phi}{\partial x_i} + v_i \phi + \phi_j \frac{\partial \phi}{\partial x_i} dx \right|$$

$$= \left| \int_{\text{supp}\phi} (u - \phi_j) \frac{\partial \phi}{\partial x_i} + v_i \phi + \phi_j \frac{\partial \phi}{\partial x_i} dx \right|$$

$$\to 0$$

as $j \to \infty$. Thus

$$\int_\Omega u \frac{\partial \phi}{\partial x_i} dx = - \int_\Omega \phi v_i dx$$

and $v_i = \frac{\partial u}{\partial x_i}$ in the distributional sense. $\qquad\qquad\square$

Lemma 3. *Let $u \in C_0(\Omega)$ and assume that u is differentiable (in the classical sense) a.e. Ω with $|\nabla u| \in L^\infty(D)$ and $\frac{1}{|x|^N (\log \frac{1}{|x|})^{2\alpha}} \in L^1(D)$ where D is an open set such that $\text{supp}(u) \subset D \subset\subset \Omega$. Then $u \in \mathcal{H}^{1,2}_{\alpha,\beta}(\Omega)$.*

For example, Lipshitz continuous functions are differentiable a.e. on Ω (Rademacher's theorem: see [15]).

Proof. Put $w_\alpha(x) = \frac{1}{|x|^N (\log \frac{1}{|x|})^{2\alpha}}$ and $w_\beta(x) = |x|^{2-N} (\log \frac{1}{|x|})^{2\beta}$ for $\alpha, \beta \in \mathbb{R}$. Note that $w_\beta \in L^1(\Omega)$ for any $\beta \in \mathbb{R}$. By taking $\psi \in C_0^\infty(\Omega)$ such that $\psi \equiv 1$ on $\text{supp}(u)$ and considering ψu instead of u, we can assume that u is bounded continuous on $\mathbb{R}^N$ with compact support in Ω and has a classical derivative ∇u a.e. on $\mathbb{R}^N$. We take the standard mollifier η_j, $0 \leq \eta_j \leq 1$, $\int_{\mathbb{R}^N} \eta_j dx = 1$, and put $u_j = \eta_j * u \in C^\infty(\mathbb{R}^N)$. Then we know that u_j has a compact support in D for j large, $u_j \to u$ uniformly on D and $\nabla u_j = \eta_j * \nabla u \to \nabla u$ uniformly on D. Moreover, we have

$$\|u_j\|_{L^\infty(D)} \leq \|u\|_{L^\infty(D)}, \quad \|\nabla u_j\|_{L^\infty(D)} \leq \|\nabla u\|_{L^\infty(D)}.$$

For the fundamental properties of the mollifier, see the book [14]. Thus

$$|u_j - u|^2 w_\alpha(x) \to 0, \quad |\nabla u_j - \nabla u|^2 w_\beta(x) \to 0 \quad a.e. \ x \in D,$$

$$|u_j - u|^2 w_\alpha(x) \leq 4\|u\|^2_{L^\infty(D)} w_\alpha(x) \in L^1(D),$$

$$|\nabla u_j - \nabla u|^2 w_\beta(x) \leq 4\|\nabla u\|^2_{L^\infty(D)} w_\beta(x) \in L^1(D).$$

Then Lebesgue's dominated convergence theorem implies that

$$\int_D |u_j - u|^2 w_\alpha dx \to 0, \quad \int_D |\nabla u_j - \nabla u|^2 w_\beta dx \to 0$$

as $j \to \infty$. Thus $u \in \mathcal{H}^{1,2}_{\alpha,\beta}(\Omega)$ by definition. $\qquad\qquad\square$

We define another type of weighted space using distributional derivatives. Define

$$\mathcal{W}_{\alpha,\beta}^{1,2}(\Omega) = \left\{ u \in L_{loc}^1(\Omega) : u \in L^2(\Omega, \frac{1}{|x|^N (\log \frac{1}{|x|})^{2\alpha}}), \quad |\nabla u| \in L^2(\Omega, |x|^{2-N}(\log \frac{1}{|x|})^{2\beta}) \right\}$$

where ∇u is the distributional derivative of $u \in L_{loc}^1(\Omega)$. For $u \in \mathcal{W}_{\alpha,\beta}^{1,2}(\Omega)$, we define the norm (1) as before.

Lemma 4. *For any $\alpha, \beta \in \mathbb{R}$, the space $\mathcal{W}_{\alpha,\beta}^{1,2}(\Omega)$ is a Banach space under the norm* (1).

Proof. Put $w_\alpha(x) = \frac{1}{|x|^N (\log \frac{1}{|x|})^{2\alpha}}$ and $w_\beta(x) = |x|^{2-N}(\log \frac{1}{|x|})^{2\beta}$ for $\alpha, \beta \in \mathbb{R}$. Since $w_\beta \in L^1(\Omega)$ for any $\beta \in \mathbb{R}$, it is easy to see that $L^2(\Omega, w_\beta dx)$ is a Banach space. Also if $\{u_j\} \subset L^2(\Omega, w_\alpha dx)$ is a Cauchy sequence, then $\{u_j w_\alpha^{1/2}\} \subset L^2(\Omega)$ is Cauchy, so it has a limit $v \in L^2(\Omega)$. Then $\{u_j\}$ converges to $u = v w_\alpha^{-1/2} \in L^2(\Omega, w_\alpha dx)$. Thus $L^2(\Omega, w_\alpha dx)$ is a Banach space for any $\alpha \in \mathbb{R}$.

Take a Cauchy sequence $\{u_j\}$ in $\mathcal{W}_{\alpha,\beta}^{1,2}(\Omega)$. Then $\{u_j\}$ and $\{\nabla u_j\}$ are Cauchy sequence in $L^2(\Omega, w_\alpha)$ and $L^2(\Omega, w_\beta; \mathbb{R}^N)$, respectively, so they have limits:

$$u_j \to u \quad \text{in } L^2(\Omega, w_\alpha), \quad \nabla u_j \to \vec{v} \quad \text{in } L^2(\Omega, w_\beta; \mathbb{R}^N)$$

as $j \to \infty$. Since $u_j \in L_{loc}^1(\Omega)$, it defines a distribution $T_{u_j} : \mathcal{D}(\Omega) \to \mathbb{R}$ such that $T_{u_j}(\phi) = \int_\Omega u_j \phi dx$. Also we can define a distribution $T_{\partial_i u_j}(\phi) = \int_\Omega \frac{\partial u_j}{\partial x_i} \phi dx$ for $i = 1, 2$. Now, we see

$$|T_{u_j}(\phi) - T_u(\phi)| \leq \int_\Omega |u_j - u||\phi| dx \leq \|\phi\|_{L^\infty(\Omega)} \int_{supp\phi} |u_j - u| dx \to 0,$$

$$|T_{\partial_i u_j}(\phi) - T_{\partial_i u}(\phi)| \leq \int_\Omega \left| \frac{\partial u_j}{\partial x_i} - \frac{\partial u}{\partial x_i} \right| |\phi| dx \leq \|\phi\|_{L^\infty(\Omega)} \int_{supp\phi} |\nabla u_j - \nabla u| dx \to 0$$

as $j \to \infty$, since $L^2(\Omega, w_\alpha) \hookrightarrow L_{loc}^1(\Omega)$ and $L^2(\Omega, w_\beta) \hookrightarrow L_{loc}^1(\Omega)$, respectively. Thus

$$T_{v_i}(\phi) = \lim_{j \to \infty} T_{\partial_i u_j}(\phi) = \lim_{j \to \infty} (-1) T_{u_j}(\partial_i \phi) = (-1) T_u(\partial_i \phi).$$

Thus we see $\nabla u = \vec{v}$ and $\|u_j - u\|_{\alpha,\beta} \to 0$ as $j \to \infty$. $\qquad \square$

By Lemma 4 and the definition of $\mathcal{H}_{\alpha,\beta}^{1,2}(\Omega)$, we see the following corollary.

Corollary 1. *For any $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$, $\mathcal{H}_{\alpha,\beta}^{1,2}(\Omega)$ is a Banach space (as a closed subspace of $\mathcal{W}_{\alpha,\beta}^{1,2}(\Omega)$) and the inclusion*

$$\mathcal{H}_{\alpha,\beta}^{1,2}(\Omega) \hookrightarrow \mathcal{W}_{\alpha,\beta}^{1,2}(\Omega)$$

is continuous.

We do not know Meyers-Serrin type "$H = W$" theorem ([21]) holds or not for general $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$.

Data availability Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Conflict of Interest. The authors have no conflict of interest.

References

[1] J. Byeon and S. Jin: *The Legendre Hardy inequality on bounded domains*, Transactions of the American Mathematical Society Series B **9** (2022), 208–257.

[2] J. Byeon, E. Jeon, and F. Takahashi: *The Hardy inequality on bounded domains for mean zero functions*, to appear in Commun. Contemp. Math.

[3] J. Chabrowski, I. Peral and B. Ruf; *On an eigenvalue problem involving the Hardy potential*, Communications in Contemporary Mathematics **12**(2010), 953–975

[4] L.C. Evans, and R.F. Gariepy: *Measure theory and fine properties of functions,* Studies in Advanced Mathematics, CRC Press (1992)

[5] H. Federer: *Geometric Measure Theorey*, Springer

[6] N. Meyers, and J. Serrin: $H = W$, Proc. Nat. Acad. Sci. USA. **51** (1964), 1055–1056.

[7] M. Sano, and F. Takahashi: *On eigenvalue problems involving the critical Hardy potential and Sobolev type inequalities with logarithmic weights in two dimensions*, J. Geometric Anal., **34**, no.4, Paper No. 112 (32 pages), (2024)

DEPARTMENT OF MATHEMATICAL SCIENCES, KAIST, 291 DAEHAK-RO, YUSEONG-GU, DAEJEON 34141, REPUBLIC OF KOREA,
  Email address: byeon@kaist.ac.kr,

DEPARTMENT OF MATHEMATICS, NARA WOMEN UNIVERSITY,
  Email address: sano@cc.nara-wu.ac.jp,

DEPARTMENT OF MATHEMATICS, OSAKA METROPOLITAN UNIVERSITY, 3-3-138, SUMIYOSHI-KU, SUGIMOTO-CHO, OSAKA, JAPAN,
  Email address: futoshi@omu.ac.jp,