

# Supplementary Material for “Linear Discriminative Image Processing Operator Analysis”

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## 4.2. $\alpha$ step

Suppose that a sample  $\mathbf{x}$  in class  $\omega_i$  is transformed to  $\mathbf{x}^{(k)}$  by  $G^{(k)}$  at the  $k$ th step as  $\mathbf{x}^{(k)} = G^{(k)}\mathbf{x}$ . Let

$$F^{(k)} = \sum_{l=0}^{k-1} G^{(l)}, \quad (1)$$

and

$$\bar{G}^{(k)} = \frac{1}{k+1} \sum_{l=0}^k G^{(l)}. \quad (2)$$

Now  $D = PA$  are known, and the scatter matrices  $\tilde{S}_W^{(k)}$ ,  $\tilde{S}_B^{(k)}$  are given as follows:

$$\begin{aligned} \tilde{S}_W^{(k)} &= D^T \bar{G}^{(k)} (S_W - R_W) \bar{G}^{(k)T} D \\ &\quad + \frac{1}{k+1} \sum_{l=0}^k D^T G^{(l)} R_W G^{(l)T} D \end{aligned} \quad (3)$$

$$\tilde{S}_B^{(k)} = D^T \bar{G}^{(k)} S_B \bar{G}^{(k)T} D \quad (4)$$

Now the criterion  $E(A, P, \boldsymbol{\alpha}^{(k)})$  can be rewritten as the ratio of

$$\text{tr} \left( \tilde{S}_W^{(k)} \right) = \boldsymbol{\alpha}^{(k)T} H_W^{(k)} \boldsymbol{\alpha}^{(k)} + 2\mathbf{q}_W^{(k)T} \boldsymbol{\alpha}^{(k)} + \pi_W^{(k)}, \quad (5)$$

$$\text{tr} \left( \tilde{S}_B^{(k)} \right) = \boldsymbol{\alpha}^{(k)T} H_B^{(k)} \boldsymbol{\alpha}^{(k)} + 2\mathbf{q}_B^{(k)T} \boldsymbol{\alpha}^{(k)} + \pi_B^{(k)}, \quad (6)$$

where

$$H_W^{(k)} = \left\{ h_{Wjl}^{(k)} \right\}, \quad (7)$$

$$h_{Wjl}^{(k)} = \frac{\text{tr} \left\{ D^T G_j (S_W + kR_W) G_l^T D \right\}}{(k+1)^2}, \quad (8)$$

$$\mathbf{q}_W^{(k)} = \frac{(q_{W1}^{(k)}, q_{W2}^{(k)}, \dots, q_{WJ}^{(k)})^T}{(k+1)^2}, \quad (9)$$

$$q_{Wj}^{(k)} = \text{tr} \left\{ D^T G_j (S_W - R_W) F^{(k)T} D \right\}, \quad (10)$$

$$\begin{aligned} \pi_W^{(k)} &= \frac{\text{tr} \left\{ D^T F^{(k)} (S_W - R_W) F^{(k)T} D \right\}}{(k+1)^2} \\ &\quad + \frac{\sum_{l=0}^{k-1} \text{tr} \left\{ D^T G^{(l)} R_W G^{(l)T} D \right\}}{k+1}, \end{aligned} \quad (11)$$

$$H_B^{(k)} = \left\{ h_{Bjl}^{(k)} \right\}, \quad (12)$$

$$h_{Bjl}^{(k)} = \frac{\text{tr} \left\{ D^T G_j S_B G_l^T D \right\}}{(k+1)^2}, \quad (13)$$

$$\mathbf{q}_B^{(k)} = \frac{(q_{B1}^{(k)}, q_{B2}^{(k)}, \dots, q_{BJ}^{(k)})^T}{(k+1)^2}, \quad (14)$$

$$q_{Bj}^{(k)} = \text{tr} \left\{ D^T G_j S_B F^{(k)T} D \right\}, \quad (15)$$

$$\pi_B^{(k)} = \frac{1}{(k+1)^2} \text{tr} \left\{ D^T F^{(k)} S_B F^{(k)T} D \right\}. \quad (16)$$

Since this is not a usual form of the Rayleigh quotient, we prove the following proposition.

**Proposition 3** Let  $\boldsymbol{\beta}^{(k)}$  be a  $J+1$  dimensional vector  $\boldsymbol{\beta}^{(k)} = (\boldsymbol{\alpha}^{(k)T}, 1)^T$ . Then, the solution that maximizes the ratio of the equations above is given by the solution to the eigenvalue problem for  $Q_W^{(k)-1} Q_B^{(k)}$ , which maximize

$$\frac{\boldsymbol{\beta}^{(k)T} Q_B^{(k)} \boldsymbol{\beta}^{(k)}}{\boldsymbol{\beta}^{(k)T} Q_W^{(k)} \boldsymbol{\beta}^{(k)}}, \quad (17)$$

where

$$Q_B^{(k)} = \begin{bmatrix} H_B^{(k)} & \mathbf{q}_B^{(k)} \\ \mathbf{q}_B^{(k)T} & \pi_B^{(k)} \end{bmatrix}, \quad Q_W^{(k)} = \begin{bmatrix} H_W^{(k)} & \mathbf{q}_W^{(k)} \\ \mathbf{q}_W^{(k)T} & \pi_W^{(k)} \end{bmatrix}. \quad (18)$$

**Proof** Let  $\mathbf{e}$  be the eigenvector of the largest eigenvalue of  $Q_W^{(k)-1} Q_B^{(k)}$ , and  $e_{\text{last}}$  its last element. Then,  $\frac{1}{e_{\text{last}}} \mathbf{e} = \boldsymbol{\beta}^{(k)} = (\boldsymbol{\alpha}^{(k)T}, 1)^T$  and  $\boldsymbol{\alpha}^{(k)}$  is obtained.