Geometry of Conformal Field Theory

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Contents

0. Introduction

Conformal Field Theory

- **Representation thoery of ∞-dim. Lie alg.**

  (affine Lie alg., Virasoro alg.)

  + Moduli of Riemann surfaces,

  Moduli of G-bundles over Riemann surfaces

CFT and topology

- Tsuchiya-Kanie (1988): Jones representation

  $\Rightarrow$ Kohno, Drinfeld: quantum groups

  $\Rightarrow$ Witten: Chern-Simons gauge theory

- Categorification of Tsuchiya-Kanie ?

  Jones-Witten theory $\Longrightarrow$ Khovanov homology

CFT and 4-dim. gauge theory

- **Nakajima (1990's)**

  Instanton moduli and affine Lie alg.

- **Alday-Gaiotto-Tachikawa (2009)**

  Instanton moduli and Virasoro alg.

Geometric Langlands correspondence

- Non-abelian Class Field Theory

- Wakimoto modules, screening operators

- Kapustin-Witten (2006)   4-dim. gauge theory

References

1. E. Frenkel and D. Ben-Zvi,

   Vertex Algebras and Algebraic Curves, 2nd. ed., AMS, 2004.

2. K. Ueno,

   Conformal Field Theory with Gauge Symmetry, AMS, 2008.

1. Harish-Chandra pairs and D-modules

Representation theory of Lie algebras

- Representations of a Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ induced by those of the Lie group G can be understood geometrically.

- More generally, we study representations of $\mathfrak{g}$ whose restrictions on $\mathrm{Lie}(K) \subset \mathfrak{g}$ are induced by those of K.

Harish-Chandra pair $(\mathfrak{g}, K)$

Example G: Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$, $K \subset G$

Def. If K: Lie group, $\mathfrak{g}$: Lie algebra with a K-action with

K-equivariant embedding $\mathrm{Lie}(K) \subset \mathfrak{g}$,

$(\mathfrak{g}, K)$ is called **Harish-Chandra pair**.

<u>$(\mathfrak{g}, K)$-**modules**</u> Representation theory of HC pairs

Def. V: $(\mathfrak{g}, K)$-**module**

(1) V: K-module

(2) $\mathfrak{g} \to \mathrm{End}(V)$: K-equivariant

- To study $(\mathfrak{g}, K)$-modules geometrically,

  we needs the language of **sheaf.**

Structure sheaf

X: complex manifold

Example $U \subset X$ open set

$$\mathcal{O}_X(U) = \{\text{holomorphic functions on } U\}$$

$\mathcal{O}_X : U \longmapsto \mathcal{O}_X(U)$ **structure sheaf** of X

Definition of sheaf

Def. $\mathcal{F} : U \longmapsto \mathcal{F}(U)$ **sheaf** on X

$$\rho_{12} = \rho_{U_1 \subset U_2} : \mathcal{F}(U_2) \to \mathcal{F}(U_1), \quad \rho_{13} = \rho_{12}\rho_{23}$$

$$\mathcal{F}(\bigcup U_\alpha) \hookrightarrow \prod \mathcal{F}(U_\alpha) \rightrightarrows \prod \mathcal{F}(U_\alpha \cap U_\beta) \quad \text{exact}$$

$\mathcal{F}(U)$ set, abelian group, $\mathbb{C}$-vector space, $\mathbb{C}$-algebra, ...

$\mathcal{F}(U) = \Gamma(U, \mathcal{F})$

$\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U)$ **stalk** at $x \in X$

Examples of sheaves

(1) V: $\mathbb{C}$-vector space

$$V_X : \text{connected } U \longmapsto V \quad \textbf{constant sheaf}$$

(2) $\mathcal{F} : \mathcal{O}_X$-**module**

$\mathcal{F}(U)$: $\mathcal{O}_X(U)$-module + compatibility

(3) $\Theta_X(U) = \{\text{holomorphic vector fields on } U\}$

(4) $\Omega^q_X(U) = \{\text{holomorphic } q\text{-forms on } U\}$

Sheaf $\mathcal{D}_X$

$$\mathcal{D}_X(U) \subset \mathrm{End}_{\mathbb{C}}(\mathcal{O}_X(U))$$

the $\mathbb{C}$-algebra generated by $\mathcal{O}_X(U)$, $\Theta_X(U)$

$\mathcal{O}_X$ left $\mathcal{D}_X$-module

$K_X = \Omega_X^{\dim(X)}$ **canonical sheaf**, right $\mathcal{D}_X$-module

$$L_{f\xi}(\omega) = L_\xi(f\omega) \quad (f \in \mathcal{O}_X,\, \xi \in \Theta_X,\, \omega \in K_X)$$

Vector bundles with flat connections

(E, ∇): vector bundle with a flat connection over X

$\mathcal{E}$: sheaf of local sections of E

$$\Theta_X \ni \xi \longmapsto (\nabla_\xi : \mathcal{E} \to \mathcal{E}), \quad \nabla_{[\xi,\eta]} = [\nabla_\xi, \nabla_\eta]$$

$\Longrightarrow \mathcal{E}$: left $\mathcal{D}_X$-module

$\mathcal{M}$: left $\mathcal{D}_X$-module, $F_p\mathcal{M} \subset F_{p+1}\mathcal{M}$,

$$\nabla_\xi(F_p\mathcal{M}) \subset F_{p+1}(\mathcal{M})$$

Harish-Chandra pair $(\mathfrak{g}, K)$ (over $\mathbb{C}$)

K: Lie group, $\mathfrak{g}$: Lie algebra with a K-action,

$\mathrm{Lie}(K) \subset \mathfrak{g}$: K-equivariant embedding

Def. V: $(\mathfrak{g}, K)$-**module**

V: K-module, $\mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(V)$: K-equivariant

$(\mathfrak{g}, K)$-action

Z: complex manifold with a K-action

$\mathrm{Lie}(K) \subset \mathfrak{g} \underset{a}{\longrightarrow} \Theta_Z$ K-equivariant, compatible

For example, Z is a K-invariant open subset of a G-space.

Setting

$\pi : Z \to S$ principal K-bundle

$a : \mathcal{O}_Z \otimes \mathfrak{g} \to \Theta_Z$ surjective

Localization functor

$\Delta : (\mathfrak{g},\, K)\text{-mod} \longrightarrow \mathcal{D}_S\text{-mod}$

$$\Delta(V) = (\pi_*(\mathcal{D}_Z \otimes_{U\mathfrak{g}} V))^K$$

the left adjoint of $\Gamma : \mathcal{D}_S\text{-mod} \longrightarrow (\mathfrak{g},\, K)\text{-mod}$

$$\Gamma(\mathcal{M}) = \Gamma(Z,\, \pi^*\mathcal{M})$$

$$\mathrm{Hom}_{\mathcal{D}_S}(\Delta(V),\, \mathcal{M}) \cong \mathrm{Hom}_{\mathfrak{g}}(V,\, \Gamma(\mathcal{M}))$$

Terminology

(1) $a : \mathfrak{g} \longrightarrow \Theta_Z$ induces $U\mathfrak{g} \longrightarrow \mathcal{D}_Z$

(2) $\pi : Z \longrightarrow S$, $\mathcal{F}$: sheaf on Z, $\mathcal{M}$: sheaf on S,

$(\pi_* \mathcal{F})(U) = \mathcal{F}(\pi^{-1}(U))$ **direct image**

$(\pi^{-1}\mathcal{M})_z = \mathcal{M}_{\pi(z)}$ **inverse image**

$\pi^* \mathcal{M} = \pi^{-1}\mathcal{M} \otimes_{\pi^{-1}\mathcal{O}_S} \mathcal{O}_Z$

Coinvariants

V: $(\mathfrak{g}, K)$-module

$a : \mathcal{O}_Z \otimes \mathfrak{g} \to \Theta_Z$ surjective

$V_{\text{coinv}} = \mathcal{O}_Z \otimes V / \text{Ker}(a) \cdot \mathcal{O}_Z \otimes V$

$V_{\text{coinv}} \cong \mathcal{D}_Z \otimes_{U\mathfrak{g}} V$ $\mathcal{D}_Z$-module

CFT case (1)

$$\mathcal{O} = \mathbb{C}[[z]], \quad \mathcal{K} = \mathbb{C}(\!(z)\!) = \mathcal{O}[z^{-1}]$$

$$\mathrm{Der}(\mathcal{K}) = \mathbb{C}(\!(z)\!)\partial_z, \quad \mathrm{Der}_0(\mathcal{O}) = z\mathbb{C}[[z]]\partial_z$$

$$\mathrm{Aut}(\mathcal{O}) = \{z \mapsto a_1 z + a_2 z^2 + \cdots \mid a_1 \neq 0\}$$

$(\mathrm{Der}(\mathcal{K}),\, \mathrm{Aut}(\mathcal{O}))$: Harish-Chandra pair

C: compact Riemann surface, $\quad p \in C$

$$\mathrm{H}^1(C, \Theta_C) = \Theta_C(C \smallsetminus \{p\})\backslash \mathrm{Der}(\mathcal{K}_p)/\mathrm{Der}_0(\mathcal{O}_p)$$

CFT case (2)

G: reductive algebraic group, $\mathfrak{g} = \mathrm{Lie}(G)$

$(\mathfrak{g}(\mathcal{K}), G(\mathcal{O}))$: Harish-Chandra pair

$P \to C$: principal $G(\mathbb{C})$-bundle

$\mathrm{H}^1(C, \mathfrak{g}_P) = \Gamma(C \setminus \{p\}, \mathfrak{g}_P) \backslash \mathfrak{g}(\mathcal{K}_p) / \mathfrak{g}(\mathcal{O}_p)$

CFT case

$\mathcal{D}_S$-module structure on $\Delta(V)$

Knizhnik-Zamoldchikov connection

2. Families of stable curves

Stable curves

Let g, n be non-negative integers such that $2g - 2 + n > 0$, and $I = \{1, 2, \ldots, n\}$. An n-**pointed**, or I-**pointed stable curve of genus** g over a scheme B is a proper flat morphism $\pi : C \to B$ together with n sections $s_I = (s_i : B \to C)_{i \in I}$ such that

(i) The geometric fibers C_b of π at $b \in B$ are reduced and connected curves with at most ordinary double points.

(ii) C_b is smooth at $s_i(b)$.

(iii) $s_i(b) \neq s_j(b)$ for $i \neq j$.

(iv) Each non-singular rational component of C_b has at least 3 points which are sections or intersections with other components.

(v) $\dim \mathrm{H}^1(C_b, \mathcal{O}_{C_b}) = g$. $\square$

Stabilization

Let $I = \{1, 2, \ldots, n\}$, $I^+ = I \cup \{n+1\}$. An I-pointed stable curve $\mathcal{X} = (\pi : C \longrightarrow B,\ s_I = (s_i)_{i \in I})$ and another section $s_{n+1} : B \longrightarrow C$ are given. A pair $(\mathcal{X}^+, f)$ of an I^+-pointed stable curve $\mathcal{X}^+ = (\pi^+ : C^+ \longrightarrow B,\ s^+_{I^+} = (s^+_i)_{i \in I^+})$ and a morphism $f : C^+ \longrightarrow C$ over B is called a **stabilization** if

(i) $fs_i^+ = s_i \ (i \in I^+)$

(ii) There are two possible cases for a geometric fiber C_b^+:

a) If C_b is smooth at $s_{n+1}(b)$ and $s_{n+1}(b) \neq s_i(b) \ (i \in I)$, $f_b : C_b^+ \to C_b$ is an isomorphism.

b) If not, there is a rational component E of C_b^+ such that $s_{n+1}^+(b) \in E$ and $f_b(E) = \{s_{n+1}(b)\}$ and $f_b : C_b^+ \setminus E \to C_b \setminus \{s_{n+1}(b)\}$ is an isomorphism. $\square$

Remark

The stabilization $(\mathcal{X}^+, f)$ is unique up to isomorphisms for $\mathcal{X}$, s_{n+1}.

Tower

For an I-pointed stable curve $\mathcal{X} = (\pi : C \to B,\ s_I = (s_i)_{i \in I})$,

the first projection on the fiber product

$$\mathcal{X}^2 = (p_1 : C \times_B C \to C,\ \pi^* s_I = (\pi^* s_i)_{i \in I})$$

is an I-pointed stable curve over C.

We define an I^+-pointed stable curve

$$\mathcal{X}^{(2)} = (\pi^{(2)} : C^{(2)} \to C,\ s^{(2)}_{I^+})$$

over C as the stabilization of $\mathcal{X}^2$ for the diagonal section

$$\Delta : C \to C \times_B C.$$

Similarly we define $(n + k - 1)$-pointed stable curve

$$\mathcal{X}^{(k)} = (\pi^{(k)} : C^{(k)} \to C^{(k-1)},\ s^{(k)}_{I^{(k)}})$$

over $C^{(k-1)}$ as $\mathcal{X}^{(k)} = (\mathcal{X}^{(k-1)})^{(2)}$.

Regular family

An I-pointed stable curve $(\pi : C \longrightarrow B,\ s_I)$ is a **regular family** if

(1) C, B: non-singular

(2) The image of π of singular points in the fibers is a set of normal-crossing divisors in B.

Proposition

For a regular family $(\pi : C \longrightarrow B,\ s_I)$,

the variety $C^{(2)}$ is also non-singular.

Deformation theory

Let $(\pi : C \to B,\ s_I = (s_i)_{i \in I})$ be a regular family. We put

$S_I = \bigcup_{i \in I} s_i(B)$,

$\pi^*\Omega_B = \pi^{-1}\Omega_B \otimes_{\mathcal{O}_B} \mathcal{O}_C$,

$F = \mathcal{H}om_{\mathcal{O}_C}(-,\ \mathcal{O}_C(-S_I))$.

We apply a left exact functor $\pi_* \circ F$ to the short exact sequence

$$0 \to \pi^*\Omega_B \to \Omega_C \to \Omega_{C/B} \to 0.$$

Since π is proper,

$$\pi_* \circ F(\pi^* \Omega_B) = \pi_* \mathcal{H}om_{\mathcal{O}_C}(\pi^* \Omega_B,\, \mathcal{O}_C(-S_I))$$

$$= \mathcal{H}om_{\mathcal{O}_B}(\Omega_B,\, \mathcal{O}_B) = \Theta_B.$$

We obtain an exact sequence

$$0 \to \pi_* \Theta_{C/B}(-S_I) \to \pi_* \Theta_C(-S_I) \to \Theta_B$$

$$\xrightarrow[\rho_B]{} R^1(\pi_* \circ F)\Omega_{C/B} \to R^1(\pi_* \circ F)\Omega_C \to 0.$$

The map ρ_B is called a **Kodaira-Spencer map**.

3. Vertex algebras and chiral algebras

Taylor expansions

$\mathcal{O}_{z_1,\dots,z_n} = \mathbb{C}[[z_1, \dots, z_n]]$

$\mathcal{P}_{z_1,z_2} = (z_1 - z_2)^{-1}\mathcal{O}_{z_2}[(z_1 - z_2)^{-1}]$

We have a decomposition of $\mathcal{O}_{z_2}$-modules,

$$\mathcal{O}_{z_1,z_2}[(z_1 - z_2)^{-1}] = \mathcal{P}_{z_1,z_2} \oplus \mathcal{O}_{z_1,z_2}.$$

The first projection $\mu_0 : \mathcal{O}_{z_1, z_2}[(z_1 - z_2)^{-1}] \to \mathcal{P}_{z_1, z_2}$

is given by

$$\mu_0\left(\frac{f(z_1)g(z_2)}{(z_1 - z_2)^n}\right) = \sum_{k=0}^{n-1} \frac{1}{k!} \cdot \frac{\partial_{z_2}^k f(z_2) \cdot g(z_2)}{(z_1 - z_2)^{n-k}}$$

for $n \in \mathbb{N},\ f(z),\ g(z) \in \mathcal{O}_z$.

$\mathcal{P}_{z_1, z_2} \cong \mathcal{O}_{z_1, z_2}[(z_1 - z_2)^{-1}]/\mathcal{O}_{z_1, z_2}$

is an $\mathcal{O}_{z_1, z_2}$-module.

Vertex algebra

A **vertex algebra** $(V, Y, |0\rangle)$ is a collection of data:

- a complex vector space V
- a linear map $Y : V \to \mathrm{Hom}(V, V(\!(z)\!))$ written as

$$Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$$
$$(a \in V, \quad a_{(n)} = \mathrm{Res}_{z=0}(Y(a, z) z^n \mathrm{d}z) \in \mathrm{End}(V))$$

- a vector $|0\rangle \in V$ (called **vacuum vector**)

satisfying the following conditions:

- (Locality) $Y(a, z)$, $Y(b, w)$ are mutually local for any $a, b \in V$. In other words, there exists a linear map

$$Y^2 : V \otimes V \to \mathrm{Hom}(V, V[[z, w]][z^{-1}, w^{-1}])[(z-w)^{-1}]$$

such that

$$\begin{aligned} Y(a, z)Y(b, w) &= \varepsilon_\infty Y^2(a, b; z, w), \\ \varepsilon_\infty (z-w)^{-1} &= \sum_{n \geq 0} w^n z^{-n-1}, \\ Y(b, w)Y(a, z) &= \varepsilon_0 Y^2(a, b; z, w), \\ \varepsilon_0 (z-w)^{-1} &= -\sum_{n<0} w^n z^{-n-1} \end{aligned}$$

- (Vacuum) $Y(a, z)|0\rangle \in V[[z]],\ Y(a, z)|0\rangle|_{z=0} = a\ (a \in V)$. In other words, $a_{(-1)} = a$, $a_{(n)}|0\rangle = 0\ (n \geq 0)$.

- (Translation) There exists a linear map $T : V \to V$ such that

$$T|0\rangle = 0, \qquad [T, Y(a, z)] = \partial_z Y(a, z) \quad (a \in V).$$

$\mathcal{D}$-module

$\Theta_{z_1,\,\ldots,\,z_n} = \bigoplus_{i=1}^{n} \mathcal{O}_{z_1,\,\ldots,\,z_n}\partial_{z_i}$

$\mathcal{D}_{z_1,\,\ldots,\,z_n}$ the $\mathbb{C}$-algebra generated by $\mathcal{O}_{z_1,\,\ldots,\,z_n}$, $\Theta_{z_1,\,\ldots,\,z_n}$.

The ring $\mathcal{O}_{z_1,\,\ldots,\,z_n}$ is a left $\mathcal{D}_{z_1,\,\ldots,\,z_n}$-module.

The projection $\mu_0 : \mathcal{O}_{z_1,\,z_2}[(z_1 - z_2)^{-1}] \to \mathcal{P}_{z_1,\,z_2}$

is a homomorphisms of left $\mathcal{D}_{z_1,\,z_2}$-modules.

The Lie algebra $\Theta_{z_1,\,\dots,\,z_n}$ acts on

$$\omega_{z_1,\,\dots,\,z_n} = \mathcal{O}_{z_1,\,\dots,\,z_n} \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n$$

by Lie derivative. It makes $\omega_{z_1,\,\dots,\,z_n}$ a right $\mathcal{D}_{z_1,\,\dots,\,z_n}$-module.

The map

$$\mu_0^{\mathrm{R}} : \mathcal{O}_{z_1,\,z_2}[(z_1 - z_2)^{-1}]\mathrm{d}z_1 \otimes \mathrm{d}z_2 \to \mathcal{P}_{z_1,\,z_2}\mathrm{d}z_1 \wedge \mathrm{d}z_2$$

induced by μ_0 is a homomorphism of right $\mathcal{D}_{z_1,\,z_2}$-modules.

Jacobi identity

$$\mathcal{O} = \mathcal{O}_{z_1, z_2, z_2} = \mathbb{C}[[z_1,\ z_2,\ z_3]][z_1^{-1},\ z_2^{-1},\ z_3^{-1}]$$

$$t_{ij} = (z_i - z_j)^{-1}.$$

Since $t_{ij}t_{jk} = t_{ij}t_{ik} + t_{ik}t_{jk}$ for distinct $i,\ j,\ k$, we deduce

$$\mathcal{O}[t_{ij},\ t_{ik},\ t_{jk}] = \mathcal{O}[t_{ij},\ t_{ik}] + \mathcal{O}[t_{ik},\ t_{jk}].$$

Hence the natural maps

$$\frac{\mathcal{O}[t_{ij},\, t_{ik}]}{\mathcal{O}[t_{ik}]} \longrightarrow \frac{\mathcal{O}[t_{ij},\, t_{ik},\, t_{jk}]}{\mathcal{O}[t_{ik},\, t_{jk}]}$$

$$\frac{\mathcal{O}[t_{ij},\, t_{ik}]}{\mathcal{O}[t_{ij}] + \mathcal{O}[t_{ik}]} \longrightarrow \frac{\mathcal{O}[t_{ij},\, t_{ik},\, t_{jk}]}{\mathcal{O}[t_{ik},\, t_{jk}] + \mathcal{O}[t_{ij}]}$$

are isomorphisms.

Let

$$\mu_{1[23]} : \mathcal{O}[t_{12},\, t_{13},\, t_{23}] \longrightarrow \frac{\mathcal{O}[t_{12},\, t_{13},\, t_{23}]}{\mathcal{O}[t_{12},\, t_{13}] + \mathcal{O}[t_{23}]}$$

be the projection.

$$\mu_{[12]3},\, \mu_{2[13]} : \mathcal{O}[t_{12},\, t_{13},\, t_{23}] \longrightarrow \frac{\mathcal{O}[t_{12},\, t_{13},\, t_{23}]}{\mathcal{O}[t_{12},\, t_{13}] + \mathcal{O}[t_{23}]}$$

are the compositions

$$\begin{aligned}
\mathcal{O}[t_{12},\, t_{13},\, t_{23}] &\to \frac{\mathcal{O}[t_{12},\, t_{13},\, t_{23}]}{\mathcal{O}[t_{13},\, t_{23}] + \mathcal{O}[t_{12}]} \\
&\xleftarrow{\sim} \frac{\mathcal{O}[t_{12},\, t_{23}]}{\mathcal{O}[t_{12}] + \mathcal{O}[t_{23}]} \xrightarrow{\sim} \frac{\mathcal{O}[t_{12},\, t_{13},\, t_{23}]}{\mathcal{O}[t_{12},\, t_{13}] + \mathcal{O}[t_{23}]}, \\
\mathcal{O}[t_{12},\, t_{13},\, t_{23}] &\to \frac{\mathcal{O}[t_{12},\, t_{13},\, t_{23}]}{\mathcal{O}[t_{12},\, t_{23}] + \mathcal{O}[t_{13}]} \\
&\xleftarrow{\sim} \frac{\mathcal{O}[t_{13},\, t_{23}]}{\mathcal{O}[t_{13}] + \mathcal{O}[t_{23}]} \xrightarrow{\sim} \frac{\mathcal{O}[t_{12},\, t_{13},\, t_{23}]}{\mathcal{O}[t_{12},\, t_{13}] + \mathcal{O}[t_{23}]}
\end{aligned}$$

$\mu_{1[23]} = \mu_{[12]3}$, $\mu_{2[13]} = 0$ on $\mathcal{O}[t_{12}, t_{23}]$,

$\mu_{1[23]} = \mu_{2[13]}$, $\mu_{[12]3} = 0$ on $\mathcal{O}[t_{13}, t_{23}]$.

Therefore $\mu_{1[23]} = \mu_{[12]3} + \mu_{2[13]}$.

Chiral algebra

C: compact Riemann surface

$\mathcal{A}$: right $\mathcal{D}_C$-module

$\mathcal{A} \boxtimes \mathcal{A}(\infty\Delta) \to \Delta_! \mathcal{A}$: $\mathcal{D}_C$-module hom

unit: $\Omega_C \hookrightarrow \mathcal{A}$ compatible with μ_Ω

skew-symmetry: $\mu = -\sigma_{12} \circ \mu \circ \sigma_{12}$

Jacobi identity: $\mu_{1[23]} = \mu_{[12]3} + \mu_{2[13]}$