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Floer homology and Hamiltonian volume minimizing properties of real forms of complex hyperquadric

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What is the Hamiltonian volume minimization problem?

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Definition (Y.-G. Oh, 1990).

(M, ω, J) : Kähler manifold

$L \subset M$: closed Lagrangian submanifold

• L : **Hamiltonian volume minimizing**

$$\overset{\text{def}}{\Longleftrightarrow} \qquad \mathrm{vol}(\phi L) \geq \mathrm{vol}(L)$$

for any Hamiltonian diffeomorphism $\phi \in \mathrm{Ham}(M, \omega)$.

Example.

$M = (\mathbb{C}, dx \wedge dy, J_0)$

$L = S^1$: round circle

$\phi \in \mathrm{Ham}(\mathbb{C}, dx \wedge dy)$

$\iff \phi$: volume preserving diffeomorphism

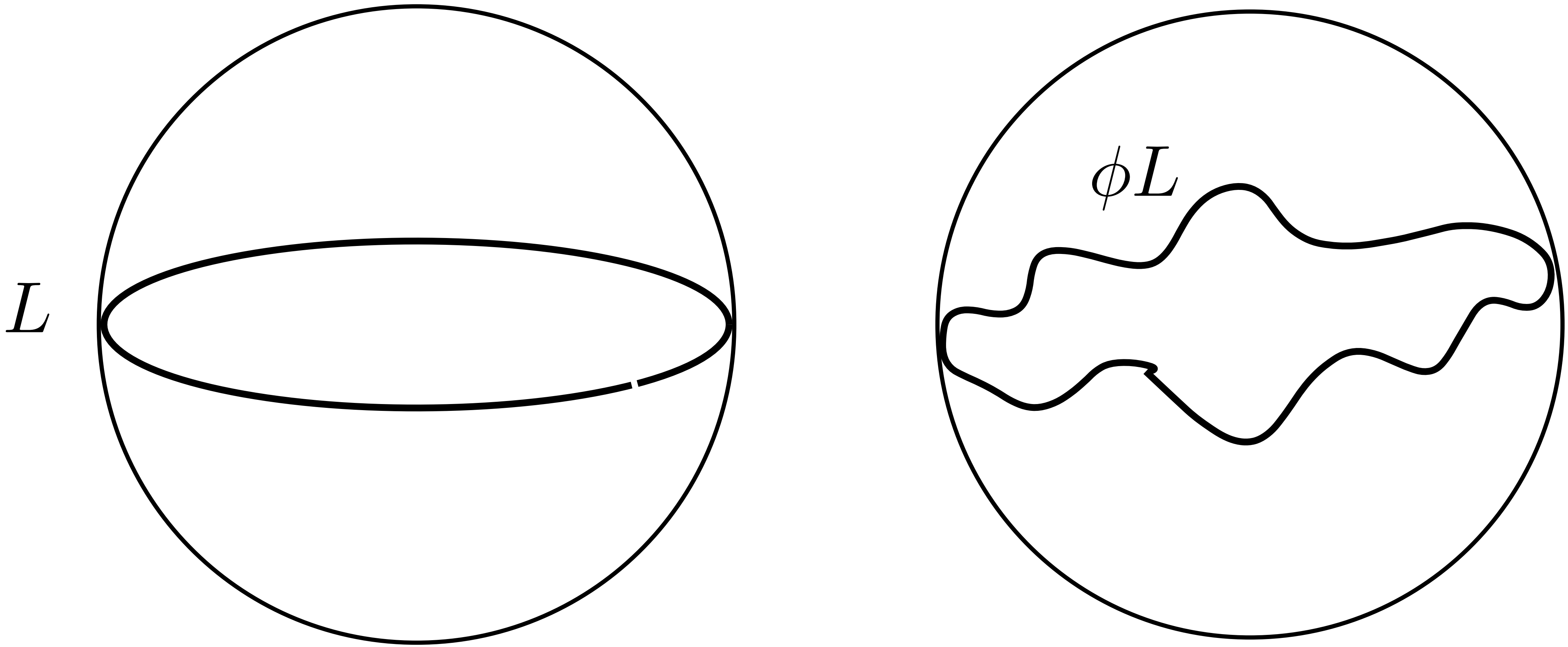
Then for any ϕ,

$$\mathrm{vol}(\phi S^1) \geq \mathrm{vol}(S^1)$$

"Isoperimetric inequality"

Example.

A great circle $L = \mathbb{R}P^1$ in $(\mathbb{C}P^1, \omega_{FS}, J_0)$ is Hamiltonian volume minimizing.



Theorem 1 (Kleiner-Oh, 1990)
$\mathbb{R}P^n$ in $\mathbb{C}P^n$ is Hamiltonian volume minimizing.

Conjecture (Oh, 1990).
M : Kähler-Einstein manifold with Ric > 0
σ : involutive anti-holomorphic isometry
$L = \mathrm{Fix}\ \sigma$ **"real form"**
(**totally geodesic Lagrangian submfd**)
Assume that L : Einstein with Ric > 0

$\Longrightarrow L$ is Hamiltonian volume minimizing.

Theorem 2 (Sakai-Tasaki-I., 2011)
$M = Q_n(\mathbb{C})$: n-dim complex hyperquadric
$L = S^n$: totally geodesic Lagrangian sphere

$\Longrightarrow L$ is Hamiltonian volume minimizing.

Note that

Theorem 3 (Gluck-Morgan-Ziller, 1989)
If n is **even**, then $S^n \subset Q_n(\mathbb{C})$ is homologically volume minimizing.

Contents

1 $Q_n(\mathbb{C})$ and its real forms

- $Q_n(\mathbb{C}) = \{[z] \in \mathbb{C}P^{n+1} \mid z_0^2 + \cdots + z_{n+1}^2 = 0\}$
  - Hermitian symmetric space of cpt type
  - Kähler-Einstein manifold with Ric > 0
  - $\pi_1(Q_n(\mathbb{C})) = 0$
  - $\mathrm{Ham}(Q_n(\mathbb{C}), \omega_0) = \mathrm{Symp}_0(Q_n(\mathbb{C}), \omega_0)$
- $Q_n(\mathbb{C}) \cong SO(n+2)/SO(2) \times SO(n)$

- If $z_i = x_i + \sqrt{-1}y_i$, then
$$0 = \sum_{i=0}^{n+1}(x_i^2 - y_i^2) + 2\sqrt{-1}\sum_{i=0}^{n+1} x_i y_i$$

- $x = (x_0, \ldots, x_{n+1}), y = (y_0, \ldots, y_{n+1}) \in \mathbb{R}^{n+2}$, then $|x| = |y|,\ x \cdot y = 0$.

- $\{x, y\}$ defines an oriented two-plane in $\mathbb{R}^{n+2}$.

$$\begin{array}{ccc} Q_n(\mathbb{C}) \cong \tilde{Gr}_2(\mathbb{R}^{n+2}) \subset \wedge^2\mathbb{R}^{n+2} \\ [z] \quad \mapsto \quad x \wedge y \end{array}$$

real forms of $Q_n(\mathbb{C})$

- $S^{k,n-k} = \{[x] \in \mathbb{R}P^{n+1} \mid x_0^2 + \cdots + x_k^2 - (x_{k+1}^2 + \cdots + x_{n+1}^2) = 0\} \cong (S^k \times S^{n-k})/\mathbb{Z}_2$

- $u_1, u_2, e_1, \ldots, e_n$: oriented o.n.b. of $\mathbb{R}^{n+2}$,

$$S^{k,n-k} = S^k(\mathbb{R}u_1 + \mathbb{R}e_1 + \cdots + \mathbb{R}e_k) \wedge S^{n-k}(\mathbb{R}u_2 + \mathbb{R}e_{k+1} + \cdots + \mathbb{R}e_n).$$

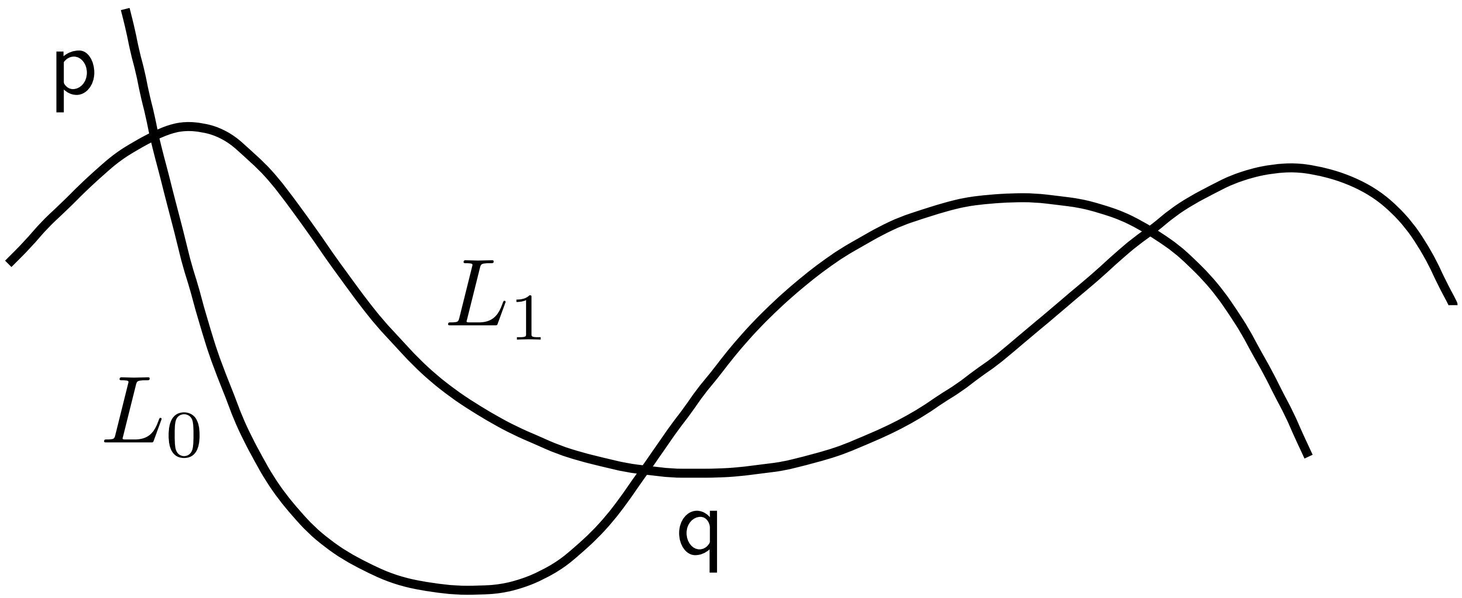
- $S^{0,n} = S^n$

2 Lagrangian Floer homology

(M, ω) : closed symplectic manifold

J_t : ω-compatible almost complex structures

L_0, L_1 : closed Lagrangian submfd, $L_0 \pitchfork L_1$

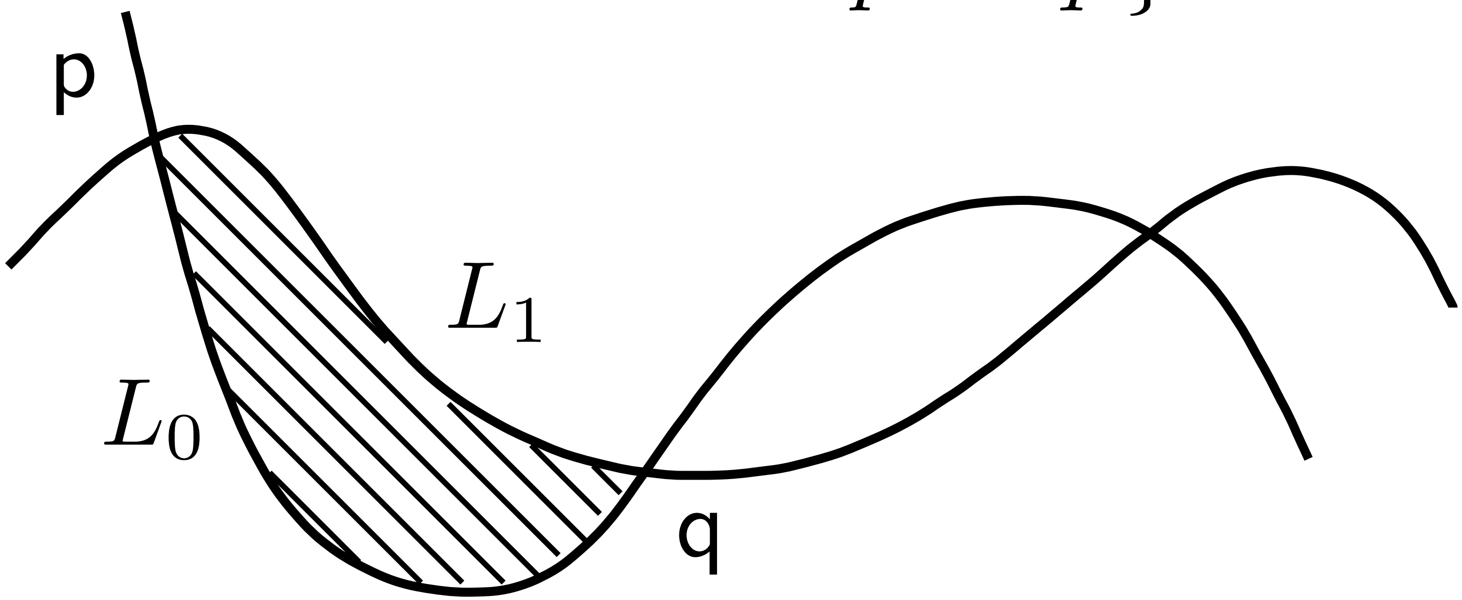


$CF(L_0, L_1)$: free $\mathbb{Z}_2$-module gen. by $L_0 \cap L_1$

$$\partial : CF(L_0, L_1) \longrightarrow CF(L_0, L_1)$$

$$\partial(p) = \sum_{q \in L_0 \cap L_1} n(p,q) \cdot q$$

$n(p,q) := \#_2\{$ isolated J-**holomorphic strip** from p to q $\}$



- $u : \mathbb{R} \times [0,1] \to M$ satisfying

$$\begin{cases} \frac{\partial u}{\partial s} + J_t(u)\frac{\partial u}{\partial t} = 0, \\ u(\cdot,0) \in L_0,\ u(\cdot,1) \in L_1, \\ u(-\infty,\cdot) = p,\ u(+\infty,\cdot) = q. \end{cases}$$

- $\partial^2 = 0 \implies HF(L_0, L_1) := \ker\partial/\mathrm{im}\partial$

Floer homology of the pair (L_0, L_1) with $\mathbb{Z}_2$-coefficients

$L \subset M$: closed Lagrangian submfd

- $I_\mu : \pi_2(M, L) \to \mathbb{Z}$: Maslov index
- $I_\omega : \pi_2(M, L) \to \mathbb{R}$,

  $I_\omega([u]) := \int_{D^2} u^*\omega$ for $u : D^2 \to (M, L)$.

Definition.

- L : **monotone**

$\overset{\text{def}}{\Longleftrightarrow} \exists \alpha > 0$: const. s.t. $I_\omega = \alpha I_\mu$.

- $\Sigma_L \geq 0$: **min. Maslov number** of L

$\Longleftrightarrow \{I_\mu(u) \mid [u] \in \pi_2(M, L)\} = \Sigma_L \cdot \mathbb{Z}$

Theorem 4 (Oh)

L_0, L_1 : monotone with $\Sigma_{L_0}, \Sigma_{L_1} \geq 3$

$\Longrightarrow$

- ∂ : well-defined.
- $\partial^2 = 0$.
- $HF(L_0, L_1 : \mathbb{Z}_2) \cong HF(L_0, \phi L_1 : \mathbb{Z}_2)$ for $\phi \in \mathrm{Ham}(M, \omega)$.

Hence if $L_0 \pitchfork \phi L_1$,

$$\#(L_0 \cap \phi L_1) \geq \mathrm{rank}\ HF(L_0, L_1 : \mathbb{Z}_2).$$

Remark.

(M, ω, J_0) : irreducible Hermitian symmetric space (**HSS**) of compact type

$\Longrightarrow$ a real form $L \subset M$: monotone

Theorem 5 (Oh)

(M, ω, J_0) : **HSS** of compact type

L_0, L_1 : real forms of M

$\Longrightarrow$ J_0 is regular.

We can use J_0 for calculation of HF.

Theorem 6 (Sakai-Tasaki- I)
(M, ω, J_0) : monotone **HSS** of compact type
L_0, L_1 : real forms of M with $\Sigma_{L_0}, \Sigma_{L_1} \geq 3$

$$L_0 \pitchfork L_1$$

$$\Longrightarrow \quad HF(L_0, L_1 : \mathbb{Z}_2) \cong \bigoplus_{p \in L_0 \cap L_1} \mathbb{Z}_2[p].$$

Remark.

- $L_0 \cap L_1$: **antipodal set**
- $L_0 \cong L_1 \Longrightarrow HF \cong H_*(L_0 : \mathbb{Z}_2)$ **(Oh)**

3 Antipodal sets and HF

Definition (Chen-Nagano, 1988).

M : Riemannian symmetric space

s_x : geodesic symmetry at $x \in M$ $\quad (s_x^2 = id)$

$S \subset M$: subset

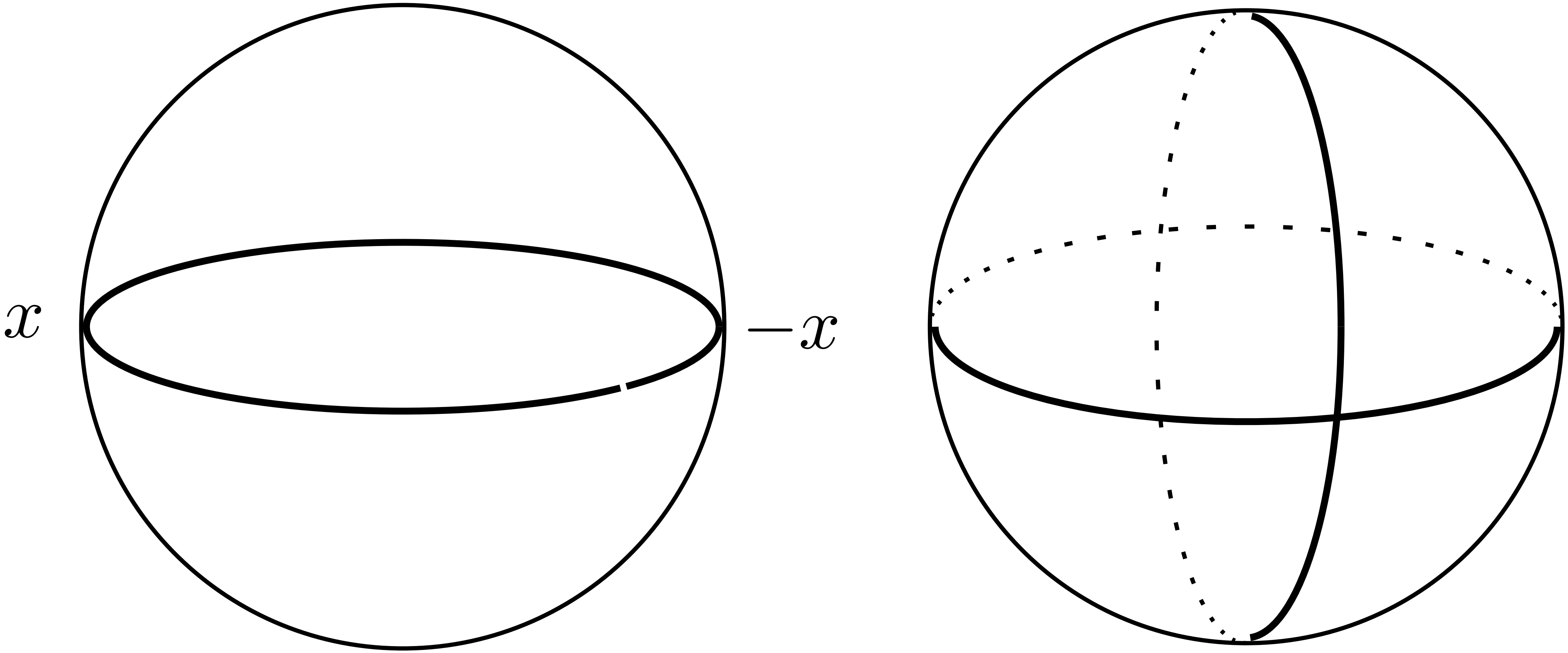
• S : **antipodal set**

$\overset{\text{def}}{\Longleftrightarrow}$ For any $x, y \in S$, we have $s_x y = y$.

Example.

Fix $x \in S^2 = \mathbb{C}P^1$. $\; s_x(x) = x,\; s_x(-x) = -x$.

$\{x, -x\}$ is an antipodal set of S^2.


x
−x

Theorem 7 (Tanaka-Tasaki, to appear)
(M, ω, J_0) : **HSS** of compact type
L_0, L_1 : real forms of M, $L_0 \pitchfork L_1$
$\Longrightarrow L_0 \cap L_1$: antipodal set of M.

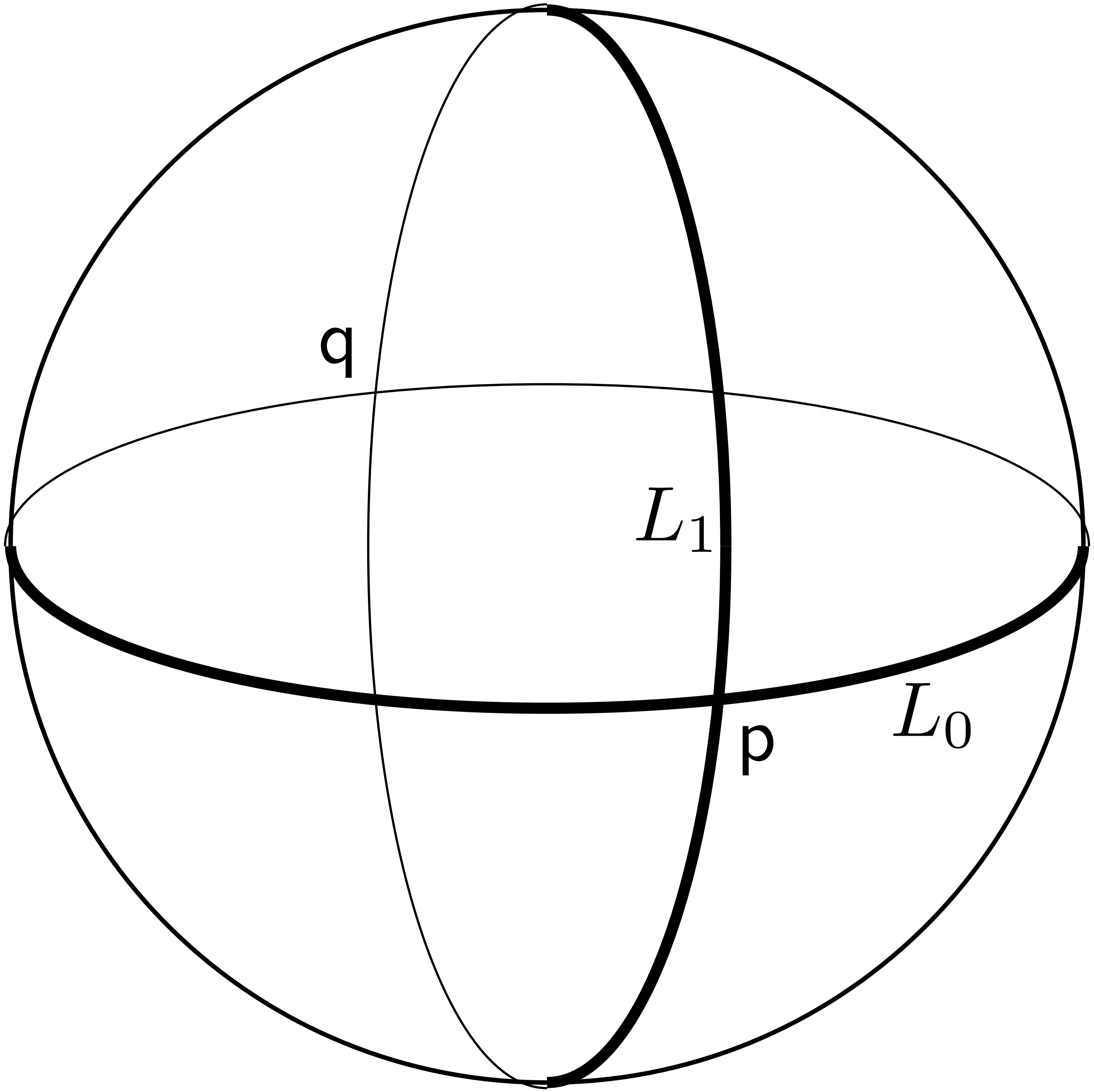
Outline of the Proof of Theorem 6

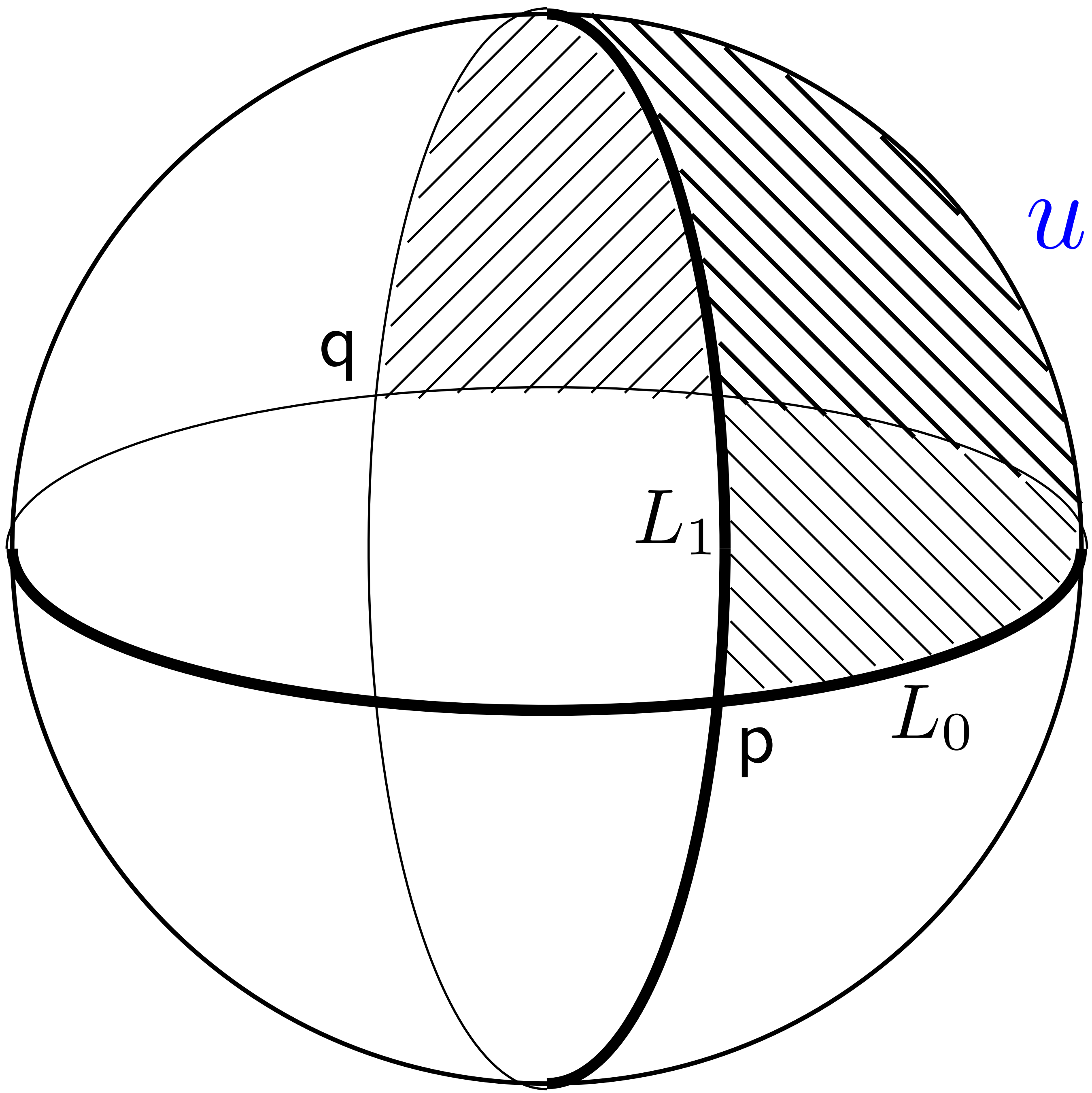
We want to show

$$\partial(p) = \sum_{q \in L_0 \cap L_1} n(p,q) \cdot q = 0$$

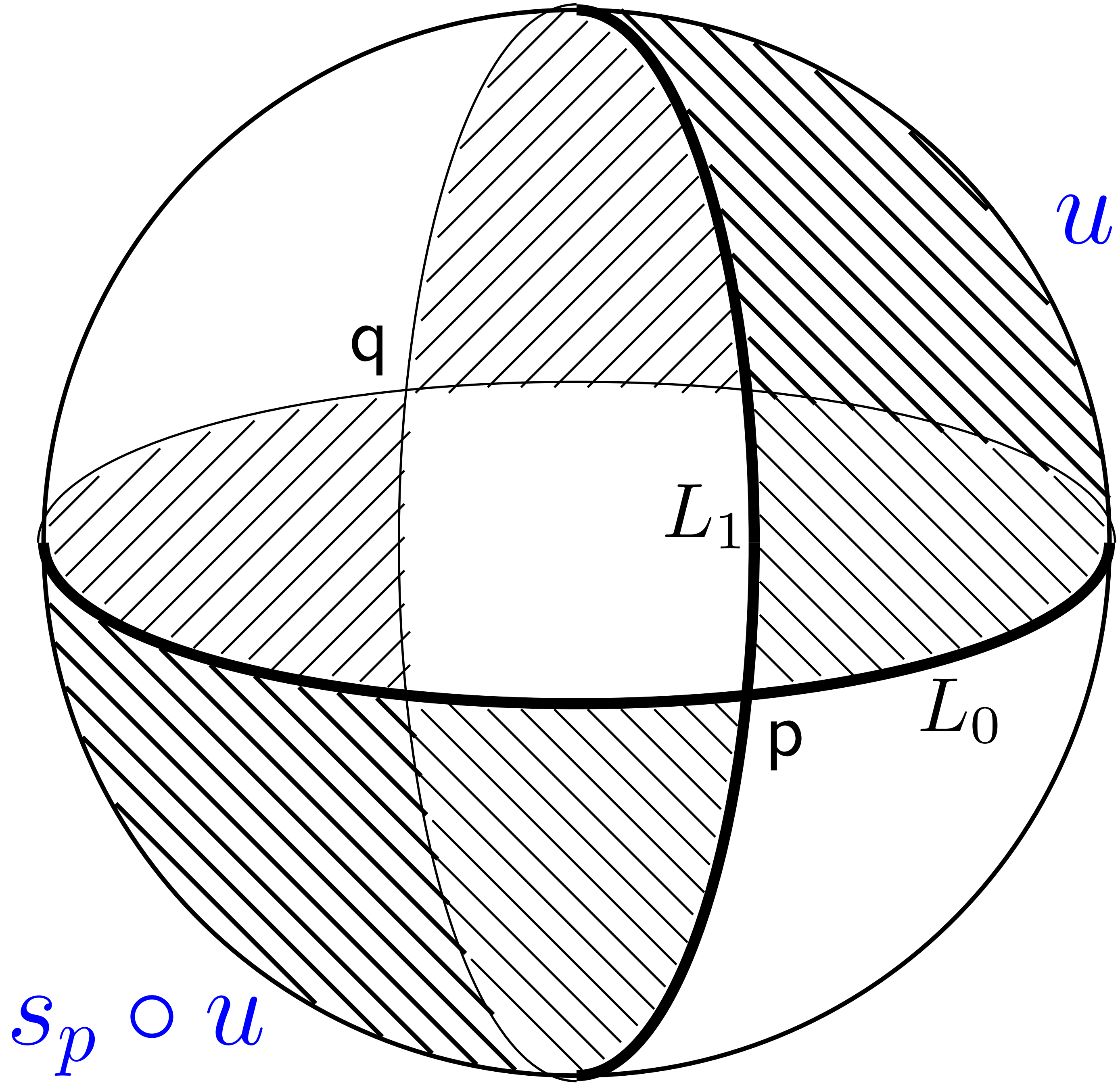
for any $p \in L_0 \cap L_1$.

- Assume $\exists J_0$-holomorphic strip u.
- Consider s_p. Since $L_0 \cap L_1$: antipodal set, $s_p(p) = p,\ s_p(q) = q$.
- Consider $s_p \circ u$.


q
L_1
p
L_0



u
q
L_1
p
L_0



u
q
L_1
p
L_0
$s_p \circ u$

Hence, $\partial(p) = 0$.

$$HF(L_0, L_1 : \mathbb{Z}_2) \cong \bigoplus_{p \in L_0 \cap L_1} \mathbb{Z}_2[p].$$

4 Proof of Theorem 2

Theorem 8 (Tasaki, 2010)

L_0, L_1 : real forms of $Q_n(\mathbb{C})$

$L_0 \cong S^{k,n-k}, L_1 \cong S^{l,n-l} \quad (k \leq l \leq [n/2])$

$L_0 \pitchfork L_1$

$\Longrightarrow$

$L_0 \cap L_1 \cong \{\pm u_1 \wedge u_2, \pm e_1 \wedge e_{k+1}, \ldots, \pm e_k \wedge e_{2k}\}.$

Remark.

$$S^{k,n-k} = S^k(\mathbb{R}u_1 + \mathbb{R}e_1 + \cdots + \mathbb{R}e_k) \wedge S^{n-k}(\mathbb{R}u_2 + \mathbb{R}e_{k+1} + \cdots + \mathbb{R}e_n).$$

- $L_0 \cong S^{k,n-k}, L_1 \cong S^{l,n-l} \quad (k \leq l \leq [n/2])$

By Theorem 6 and Theorem 8, we have

$$HF(L_0, L_1 : \mathbb{Z}_2) \cong (\mathbb{Z}_2)^{2(k+1)}.$$

Hence if $L_0 \pitchfork \phi L_1$,

$$\#(L_0 \cap \phi L_1) \geq 2(k+1).$$

Here, put $k = 0$,

$$\#(S^n \cap \phi S^{l,n-l}) \geq 2.$$

Integrating over $SO(n+2)$,

$$\int_{SO(n+2)} \#(S^n \cap g\phi S^{l,n-l}) d\mu \geq 2\,\mathrm{vol}(SO(n+2)).$$

Crofton type formula (Lê Hông Vân, 1993).

$N \subset Q_n(\mathbb{C})$: n-dim. submfd

$$\int_{SO(n+2)} \#(S^n \cap gN) d\mu \leq 2 \frac{\mathrm{vol}(SO(n+2))}{\mathrm{vol}(S^n)} \mathrm{vol}(N).$$

Put $N = \phi S^{l,n-l}$,

$$2\,\frac{\mathrm{vol}(SO(n+2))}{\mathrm{vol}(S^n)}\mathrm{vol}(\phi S^{l,n-l}) \geq 2\,\mathrm{vol}(SO(n+2)).$$

Formula

Any real form $S^{l,n-l} \subset Q_n(\mathbb{C})$ satisfies

$$\mathrm{vol}(\phi S^{l,n-l}) \geq \mathrm{vol}(S^n)$$

for any $\phi \in \mathrm{Ham}(Q_n(\mathbb{C}), \omega)$.

- $l = 0:$ $\mathrm{vol}(\phi S^n) \geq \mathrm{vol}(S^n)$. **Ham. vol. min.**

5 A generalization to contact isotopies

Theorem 9

$\pi : P \to Q_n(\mathbb{C})$: pre-quantization bundle

i.e., (P, θ) : contact manifold, $d\theta = \pi^*\omega_0$

L : a Legendrian lift of a real form S^n

$n \geq 3 + \sigma$

$$\Longrightarrow \qquad \mathrm{vol}(\pi(\psi L)) \geq \mathrm{vol}(L)$$

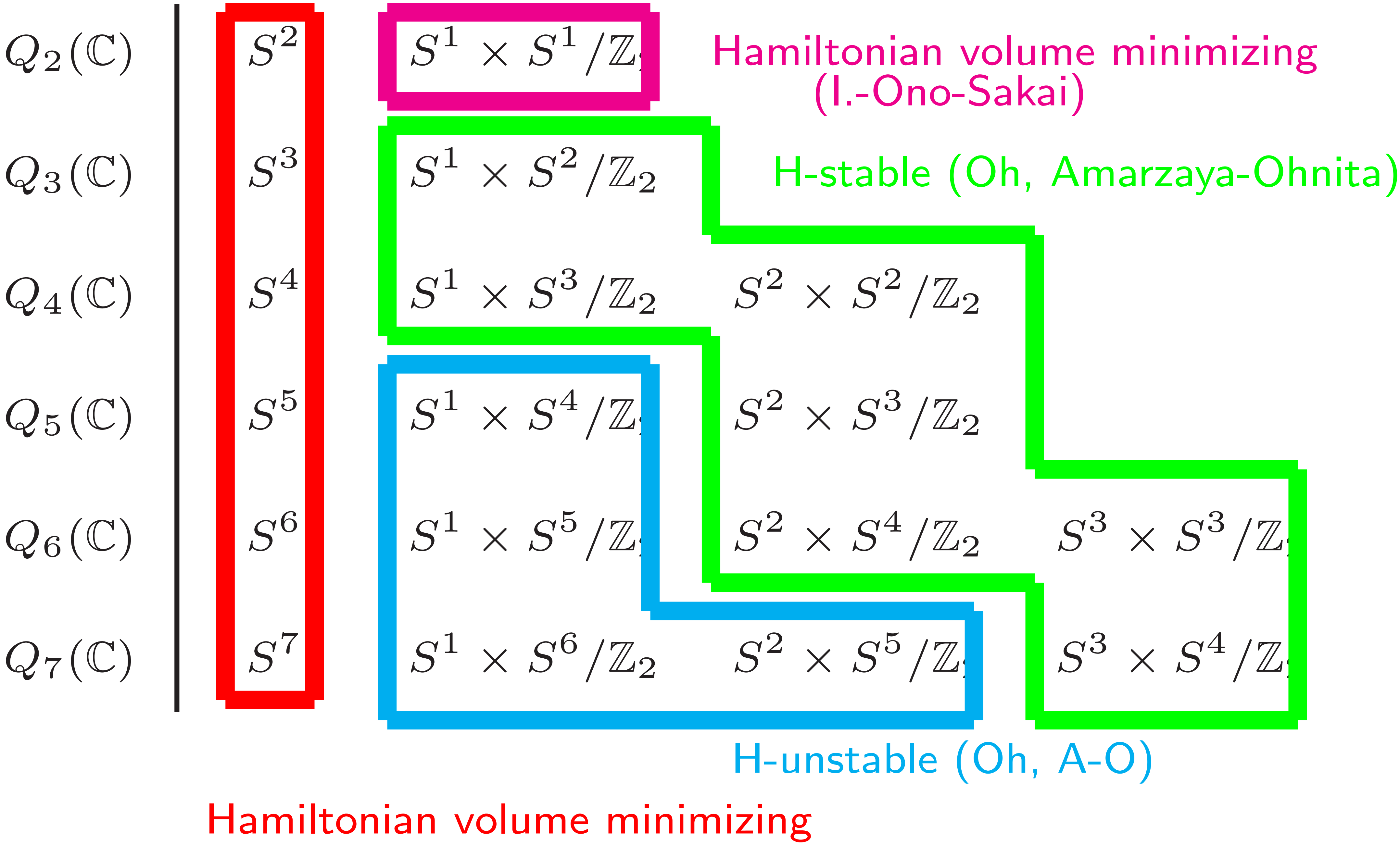
for any $\psi \in \mathrm{Cont}(P)$: contact diffeo. of P

$Q_2(\mathbb{C})$	S^2	$S^1 \times S^1/\mathbb{Z}_2$		
$Q_3(\mathbb{C})$	S^3	$S^1 \times S^2/\mathbb{Z}_2$		
$Q_4(\mathbb{C})$	S^4	$S^1 \times S^3/\mathbb{Z}_2$	$S^2 \times S^2/\mathbb{Z}_2$	
$Q_5(\mathbb{C})$	S^5	$S^1 \times S^4/\mathbb{Z}_2$	$S^2 \times S^3/\mathbb{Z}_2$	
$Q_6(\mathbb{C})$	S^6	$S^1 \times S^5/\mathbb{Z}_2$	$S^2 \times S^4/\mathbb{Z}_2$	$S^3 \times S^3/\mathbb{Z}_2$
$Q_7(\mathbb{C})$	S^7	$S^1 \times S^6/\mathbb{Z}_2$	$S^2 \times S^5/\mathbb{Z}_2$	$S^3 \times S^4/\mathbb{Z}_2$

$Q_2(\mathbb{C})$	S^2	$S^1 \times S^1/\mathbb{Z}_2$		
$Q_3(\mathbb{C})$	S^3	$S^1 \times S^2/\mathbb{Z}_2$		
$Q_4(\mathbb{C})$	S^4	$S^1 \times S^3/\mathbb{Z}_2$	$S^2 \times S^2/\mathbb{Z}_2$	
$Q_5(\mathbb{C})$	S^5	$S^1 \times S^4/\mathbb{Z}_2$	$S^2 \times S^3/\mathbb{Z}_2$	
$Q_6(\mathbb{C})$	S^6	$S^1 \times S^5/\mathbb{Z}_2$	$S^2 \times S^4/\mathbb{Z}_2$	$S^3 \times S^3/\mathbb{Z}_2$
$Q_7(\mathbb{C})$	S^7	$S^1 \times S^6/\mathbb{Z}_2$	$S^2 \times S^5/\mathbb{Z}_2$	$S^3 \times S^4/\mathbb{Z}_2$

Hamiltonian volume minimizing
(I.-Ono-Sakai)

Hamiltonian volume minimizing



$Q_2(\mathbb{C})$
$Q_3(\mathbb{C})$
$Q_4(\mathbb{C})$
$Q_5(\mathbb{C})$
$Q_6(\mathbb{C})$
$Q_7(\mathbb{C})$
S^2
S^3
S^4
S^5
S^6
S^7
$S^1 \times S^1/\mathbb{Z}$
$S^1 \times S^2/\mathbb{Z}_2$
$S^1 \times S^3/\mathbb{Z}_2$
$S^2 \times S^2/\mathbb{Z}_2$
$S^1 \times S^4/\mathbb{Z}$
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$S^3 \times S^3/\mathbb{Z}$
$S^1 \times S^6/\mathbb{Z}_2$
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$S^3 \times S^4/\mathbb{Z}$
Hamiltonian volume minimizing
(I.-Ono-Sakai)
H-stable (Oh, Amarzaya-Ohnita)
H-unstable (Oh, A-O)
Hamiltonian volume minimizing