Antipodal sets of compact Riemannian symmetric spaces and their applications

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Contents

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Introduction

(M, g) : a Riemannian manifold

(M, g) : **a Riemannian symmetric space**

$$\stackrel{\text{def}}{\Longleftrightarrow} \ {}^{\forall}x \in M, \ {}^{\exists}s_x : M \to M : \text{ an isometry}$$
$$\text{s.t. (i) } s_x^2 = \mathrm{id}_M$$
$$\text{(ii) } x \text{ is an isolated fixed point of } s_x$$

s_x is called **the geodesic symmetry** at x.

Remarks.

- $\gamma(t)$: a geodesic with $\gamma(0) = x \Longrightarrow s_x(\gamma(t)) = \gamma(-t)$
- s_x acts on $T_x(M)$ as $-\mathrm{id}$.
- If (M, g) is irreducible, g is unique up to constant.

Introduction

M : a Riemannian symmetric space
s_x : the geodesic symmetry at $x \in M$
$S \subset M$: a subset

S : **an antipodal set** $\overset{\text{def}}{\Longleftrightarrow}$ ${}^\forall x, y \in S,\ s_x(y) = y$
(Chen-Nagano 1988)

Remark. An antipodal set is finite.

Example 1. ${}^\forall p \in S^n(\subset \mathbb{R}^{n+1}),\ s_p = 1_{\langle p \rangle_{\mathbb{R}}} - 1_{p^\perp}$
$\Longrightarrow$ $\{p, -p\}$: an antipodal set

Example 2. For $x \in \mathbb{R}P^n$, s_x is induced by $1_x - 1_{x^\perp}$ on $\mathbb{R}^{n+1}$
$y \subset x^\perp$: 1-dim subspace $\Longrightarrow \{x, y\}$: an antipodal set
More generally,
$e_1, e_2, \ldots, e_{n+1}$: o.n.b. of $\mathbb{R}^{n+1}$
$\Longrightarrow \{\langle e_1 \rangle_{\mathbb{R}}, \ldots, \langle e_{n+1} \rangle_{\mathbb{R}}\}$: a (maximal) antipodal set

Introduction

M : a compact Riemannian symmetric space
the 2-**number** $\#_2 M$ of M

$$\#_2 M := \sup\{\#S \mid S \subset M : \text{an antipodal set}\}$$

(Chen-Nagano 1988)

Remark. $\#_2 M < \infty$

$S \subset M$: an antipodal set
S is **great** $\overset{\text{def}}{\Longleftrightarrow}$ $\#S = \#_2 M$ (Chen-Nagano 1988)

Remark. A great antipodal set S is maximal (i.e., $\nexists$ antipodal set S' satisfying $S \subsetneqq S'$) but the converse is not true in general.

Chen-Nagano gave $\#_2 M$ for compact irreducible Riemannian symmetric spaces M with some exceptions.

Introduction

Examples.

$\#_2 S^n = 2.$ $S = \{p, -p\}$ is a great antipodal set.

$\#_2 \mathbb{R}P^n = n + 1.$ $S = \{\langle e_1 \rangle_{\mathbb{R}}, \ldots, \langle e_{n+1} \rangle_{\mathbb{R}}\}$ is a great antipodal set.

$\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$
$G_k^{\mathbb{K}}(\mathbb{K}^n) = \{V \subset \mathbb{K}^n \mid V : \ \mathbb{K}\text{-subspace}, \ \dim_{\mathbb{K}} V = k\}$

$$\#_2 G_k^{\mathbb{K}}(\mathbb{K}^n) = \frac{n!}{k!(n-k)!}$$

$\{\langle e_{i_1}, \ldots, e_{i_r} \rangle_{\mathbb{K}} \in G_k^{\mathbb{K}}(\mathbb{K}^n) \mid 1 \leq i_1 < \cdots < i_r \leq n\}$

where $e_1, \ldots, e_n$ is the canonical basis of $\mathbb{K}^n$

Introduction

M : a Hermitian symmetric space of compact type
τ : an involutive anti-holomorphic isometry of M

$F(\tau, M) := \{x \in M \mid \tau(x) = x\}$: a **real form** of M if $F(\tau, M) \neq \emptyset$

Remarks.

- A real form is connected.
- A real form L is totally geodesic Lagrangian submanifold of M.
- Every real form is a symmetric R-space, and vice versa (Takeuchi).

A compact Riemannian symmetric space is called **a symmetric** ***R*****-space** if it is an orbit of a linear isotropy representation of Riemannian symmetric space of compact type.

Introduction

What we did are :

- to investigate the following fundamental properties of antipodal sets :
  (A) Any antipodal set is included in a great antipodal set.
  (B) Any two great antipodal sets are congruent.

  Here subsets S_1 and S_2 in M are **congruent** if there exists $g \in I_0(M)$ such that $g(S_1) = S_2$
- to investigate the intersection of two real forms in a Hermitian symmetric space of compact type and we found that the intersection is an antipodal set.

Fundamental properties of antipodal sets

M : a Hermitian symmetric space of compact type

e.g. $G_k^{\mathbb{C}}(\mathbb{C}^n)$, $Q_n(\mathbb{C})$, $SO(2n)/U(n), Sp(n)/U(n)$, etc.

$M = \mathrm{Ad}(G)J \subset \mathfrak{g} = \mathrm{Lie}(G)$,

where G : a compact semisimple Lie group,

$$J(\neq 0) \in \mathfrak{g}, \ (\mathrm{ad}J)^3 = -\mathrm{ad}J$$

Fundamental properties of antipodal sets

Theorem 1 (Sánchez(1997), T.-Tasaki)

M : a Hermitian symmetric space of compact type

$M = Ad(G)J \subset \mathfrak{g}$

$\Longrightarrow$

(1) $X, Y \in M, \ \ s_X(Y) = Y \Longleftrightarrow [X, Y] = 0$

Moreover, the following conditions (A) and (B) hold.

(A) Any antipodal set is included in a great antipodal set.

(B) Any two great antipodal sets are congruent.

(2) $^{\forall}S$ *: a great antipodal set of M*

$^{\exists}\mathfrak{t}$ *: a maximal abelian subalgebra of $\mathfrak{g}$ s.t.* $S = M \cap \mathfrak{t}$

In particular, a great antipodal set is an orbit of the Weyl group of $\mathfrak{g}$.

Fundamental properties of antipodal sets

Theorem 2 (T.-Tasaki)

$M = Ad(G)J$: a Hermitian symmetric space of compact type
$L = F(\tau, M)$: a real form
(τ : an involutive anti-holomorphic isometry of M)
Assume $J \in L$
$I_\tau : G \to G, \quad I_\tau(g) := \tau g \tau^{-1} \ (g \in G)$
$\mathfrak{g} = \mathfrak{l} + \mathfrak{p}$: the decomposition w.r.t. dI_τ
$\Longrightarrow$

(1) $L = M \cap \mathfrak{p}$.
Moreover, (A) and (B) in Theorem 1 hold.

(2) *$^\forall S$: a great antipodal set of L*
$^\exists \mathfrak{a}$: a maximal abelian subspace of $\mathfrak{p}$ s.t. $S = M \cap \mathfrak{a}$
In particular, a great antipodal set is an orbit of the Weyl group of the symmetric pair determined by I_τ.

Fundamental properties of antipodal sets

Corollary 3

M : a symmetric R-space

$\Longrightarrow$

(A) Any antipodal set is included in a great antipodal set.

(B) Any two great antipodal sets are congruent.

Remark. $\mathrm{Ad}(SU(4)) \cong SU(4)/\mathbb{Z}_4$ does not satisfy (A). In fact, there exists a maximal antipodal set which is not great.

Polars

M : compact Riemannian symmetric space
$p \in M$

$$F(s_p, M) = \{x \in M \mid s_p(x) = x\}$$
$$= \bigcup_{j=1}^{r} M_j^+ \text{ : the disjoint union of the connected components}$$

where $M_1^+ = \{p\}$

M_j^+ is called **a polar** of M w.r.t. p.

(Chen-Nagano 1977, 1978, 1988)

Remark. A polar is a totally geodesic submanifold of M.

Example. $M = \mathbb{C}P^n$

$e_1, \ldots, e_{n+1}$: a unitary basis of $\mathbb{C}^{n+1}$, $\quad p := \langle e_1 \rangle_{\mathbb{C}}$

$$F(s_p, \mathbb{C}P^n) = \{p\} \cup \{V \subset \langle e_2, \ldots, e_{n+1} \rangle_{\mathbb{C}} \mid \dim V = 1\} (\cong \mathbb{C}P^{n-1})$$

Polars

M : a compact Riemannian symmetric space

$$F(s_p, M) = \bigcup_{j=1}^{r} M_j^+ \implies \#_2 M \leq \sum_{j=1}^{r} \#_2 M_j^+$$

Remark. S : an antipodal set, $p \in S \implies S \subset F(s_p, M)$

Theorem 4 (Chen-Nagano, 1988)

M : a compact Riemannian symmetric space
$\implies \#_2 M \geq \chi(M)$

M : a Hermitian symmetric space of compact type

$$\implies \#_2 M = \chi(M), \quad \#_2 M = \sum_{j=1}^{r} \#_2 M_j^+$$

Polars

Theorem 5 (Takeuchi,1989)

$$M \text{ : a symmetric } R\text{-space} \implies \#_2 M = \sum_{j=1}^{r} \#_2 M_j^+$$

M : a Hermitian symmetric space of compact type

$\Longrightarrow$

M_j^+ : a Hermitian symmetric space of compact type if $\dim M_j^+ > 0$

Lemma 6

M : a Hermitian symmetric space of compact type
L : a real form of M, $\quad o \in L$
M^+ : a polar of M w.r.t. o, $\ M^+ \cap L \neq \emptyset$
$\Longrightarrow$ *$M^+ \cap L$ is a real form of M^+*

Polars

Lemma 7

M : a Hermitian symmetric space of compact type, $o \in M$

$$F(s_o, M) = \bigcup_{j=1}^{r} M_j^+$$

$\Longrightarrow$

(1) L : a real form of M, $o \in L$

$$F(s_o, L) = \bigcup_{j=1}^{r} L \cap M_j^+, \qquad \#_2 L = \sum_{j=1}^{r} \#_2(L \cap M_j^+)$$

(2) L_1, L_2 : real forms of M, $o \in L_1 \cap L_2$

$$L_1 \cap L_2 = \bigcup_{j=1}^{r} \{(L_1 \cap M_j^+) \cap (L_2 \cap M_j^+)\}$$

$$\#(L_1 \cap L_2) = \sum_{j=1}^{r} \#\{(L_1 \cap M_j^+) \cap (L_2 \cap M_j^+)\}$$

Intersections of two real forms

Simple example.

$S^2 = \mathbb{C}P^1$ is a Hermitian symmetric space of compact type.
A real form of S^2 is a great circle S^1, and vice versa.
Any two great circles intersect in two points which are antipodal to each other, if they intersect transversally.

More generally,
$M = \mathbb{C}P^n$, $L = \mathbb{R}P^n$: a real form of $\mathbb{C}P^n$
$g \in I_0(M)$, L and $g(L)$ intersect transversally
$\Longrightarrow$
${}^{\exists}u_1, \ldots, u_{n+1}$: a unitary basis of $\mathbb{C}^{n+1}$
s.t. $L \cap g(L) = \{\langle u_1 \rangle_{\mathbb{C}}, \ldots, \langle u_{n+1} \rangle_{\mathbb{C}}\}$ (Howard, 1993)

In particular, $L \cap g(L)$ is a great antipodal set of L.

Intersections of two real forms

Theorem 8 (T.-Tasaki)

M : a Hermitian symmetric space of compact type
L_1, L_2 : real forms of M, $L_1 \pitchfork L_2$

$\Longrightarrow L_1 \cap L_2$ *is an antipodal set of* L_1 *and* L_2.

Theorem 9 (T.-Tasaki)

M : a Hermitian symmetric space of compact type
L_1, L_2 : congruent real forms of M, $L_1 \pitchfork L_2$

$\Longrightarrow L_1 \cap L_2$ *is a great antipodal set of* L_1 *and* L_2,
i.e., $\#(L_1 \cap L_2) = \#_2 L_1 = \#_2 L_2$.

Intersections of two real forms

(Outline of Proof)

$L_1 \cap L_2 \neq \emptyset$ (Tasaki)
$o, p \in L_1 \cap L_2$
$\Longrightarrow$ $\exists$ closed geodesic on which o and p are antipodal, since M has a cubic unit lattice.
(Here we need to investigate the intersection of maximal tori $A_1 \subset L_1$ and $A_2 \subset L_2$ satisfying $o, p \in A_1 \cap A_2$.)

$\Longrightarrow$ Thm 8

By Lemma 7, $L_1 \cap L_2 = \bigcup_{j=1}^{r} \{(L_1 \cap M_j^+) \cap (L_2 \cap M_j^+)\}$
(Case 1) $L_1 \cap M_j^+ = L_2 \cap M_j^+ = \emptyset$
(Case 2) $L_1 \cap M_j^+ = L_2 \cap M_j^+ = \{\text{a point}\}$
(Case 3) $L_1 \cap M_j^+, L_2 \cap M_j^+$: congruent real forms of M_j^+ with

$L_1 \cap M_j^+ \pitchfork L_2 \cap M_j^+$

Intersections of two real forms

(Case 1) and (Case 2)
$\Longrightarrow\ \#(L_1 \cap M_j^+) = \#(L_2 \cap M_j^+) = \#_2(L_1 \cap M_j^+) = \#_2(L_2 \cap M_j^+)$
where $\#_2\emptyset := 0$

(Case 3)
$\Longrightarrow$
By taking a polar M_{jk}^+ of M_j^+ and repeating this argument a finite number, (Case 3) reduces to (Case 1) or (Case 2), since $\dim M_j^+ < \dim M$.

$\Longrightarrow$
$\#(L_1 \cap L_2) = \#_2 L_1 = \#_2 L_2$

$\Longrightarrow$ Thm 9

Intersections of two real forms

Theorem 10 (T.-Tasaki)

M : a Hermitian symmetric space of compact type
L_1, L_2, L_1', L_2' : real forms of M, $L_1 \pitchfork L_2$, $L_1' \pitchfork L_2'$
L_i and L_i' are congruent $(i = 1, 2)$
$\Longrightarrow \#(L_1 \cap L_2) = \#(L_1' \cap L_2')$

Remark. L_1 and L_2 (L_1' and L_2') are not necessarily congruent.

Corollary 11

M : a Hermitian symmetric space of compact type
L_1, L_2, L_1', L_2' : same as Thm 10
$\#(L_1 \cap L_2) = \min\{\#_2 L_1, \#_2 L_2\}$
(i.e., $L_1 \cap L_2$ is a great antipodal set of L_1 or L_2.)
$\Longrightarrow$ *$L_1 \cap L_2$ and $L_1' \cap L_2'$ are congruent.*

Intersections of two real forms

M : a Hermitian symmetric space
L : a Lagrangian submanifold
L : **globally tight**
$\overset{\text{def}}{\Longleftrightarrow}$ $\#(L \cap g(L)) = \dim H_*(L, \mathbb{Z}_2)$ for $^\forall g \in I_0(M)$ with $L \pitchfork g(L)$
(Y.-G Oh, 1991)

$$\begin{aligned} \#(L \cap g(L)) &= \#_2 L && \text{(Thm 9)} \\ &= \dim H_*(L, \mathbb{Z}_2) && \text{(Takeuchi)} \end{aligned}$$

Corollary 12 (T.-Tasaki)

Any real form of a Hermitian symmetric space of compact type is a globally tight Lagrangian submanifold.

Intersections of two real forms

Remark. The classification of real forms is obtained by D. P. S. Leung (1979) and M. Takeuchi (1984).

Example. $M = G_k^{\mathbb{C}}(\mathbb{C}^n)$

$$L \cong \begin{cases} G_k^{\mathbb{R}}(\mathbb{R}^n) \\ G_l^{\mathbb{H}}(\mathbb{H}^m) \text{ if } k = 2l, \ n = 2m \\ U(k) \text{ if } n = 2k \end{cases}$$

Intersections of two real forms

Theorem 13 (T.-Tasaki)

M : an irreducible Hermitian symmetric space of compact type
L_1, L_2 : real forms of M, $L_1 \pitchfork L_2$, $\#_2 L_1 \leq \#_2 L_2$

(1) $(M, L_1, L_2) = (G_{2m}^{\mathbb{C}}(\mathbb{C}^{4m}), G_m^{\mathbb{H}}(\mathbb{H}^{2m}), U(2m)) \quad (m \geq 2)$

$\Longrightarrow \#(L_1 \cap L_2) = 2^m < \binom{2m}{m} = \#_2 L_1 < 2^{2m} = \#_2 L_2$

In particular, $L_1 \cap L_2$ is not a great antipodal set of L_1 (and not of L_2).

(2) Otherwise, $\#(L_1 \cap L_2) = \#_2 L_1$
i.e., $L_1 \cap L_2$ is a great antipodal set of L_1.

Intersections of two real forms

Example (non-irreducible case).

$M = \mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$
$\tau_1, \tau_2 : \mathbb{C}P^1 \to \mathbb{C}P^1$: involutive anti-holomorphic isometries
s.t. real forms determined by τ_1, τ_2 intersect transversally
$L_1 = \{(x, y, \tau_1(x), \tau_1(y)) \mid x, y \in \mathbb{C}P^1\}$
$L_2 = \{(x, \tau_2(x), y, \tau_2(y)) \mid x, y \in \mathbb{C}P^1\}$

$\Longrightarrow L_1, L_2$: real forms of M, $L_1 \pitchfork L_2$

$$\#(L_1 \cap L_2) = 2 < 4 = \#_2 L_1 = \#_2 L_2$$

Intersections of two real forms

Application.

Theorem 14 (Iriyeh-Sakai-Tasaki)

M : an irreducible Hermitian symmetric space of compact type
L_1, L_2 : real forms of M, $L_1 \pitchfork L_2$
$\Longrightarrow$
(1) $M = G_{2m}^{\mathbb{C}}(\mathbb{C}^{4m})$ $(m \geq 2)$
L_1 : congruent to $G_m^{\mathbb{H}}(\mathbb{H}^{2m})$
L_2 : congruent to $U(2m)$
$\Longrightarrow HF(L_1, L_2 : \mathbb{Z}_2) \cong (\mathbb{Z}_2)^{2^m}$
(2) Otherwise, $HF(L_1, L_2 : \mathbb{Z}_2) \cong (\mathbb{Z}_2)^{min\{\#_2 L_1, \#_2 L_2\}}$.