Sequences of maximal antipodal sets of oriented real Grassmann manifolds

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1 Introduction

Antipodal sets in Riemannian symmetric spaces

: indtroduced by Chen-Nagano

M : a compact Riemannian symmetric space

s_x : the geodesic symmetry at $x \in M$

$A \subset M$: an antipodal set

$\Leftrightarrow s_x(y) = y \quad (x, y \in A)$

$\#_2 M = \max\{\#A \mid A \subset M : \text{antipodal}\}$

the 2-number of M a geometric invariant

$\boldsymbol{M}$: a Hermitian symmetric space

$\Rightarrow \#_2 \boldsymbol{M}$ = the Euler characteristic number

$\#_2 \boldsymbol{M} = \mathbf{dim}\, \boldsymbol{H}_*(\boldsymbol{M}; \mathbb{Z}_2)$

(Takeuchi) $\quad \boldsymbol{M}$: a symmetric $\boldsymbol{R}$-space

$\Rightarrow \#_2 \boldsymbol{M} = \mathbf{dim}\, \boldsymbol{H}_*(\boldsymbol{M}; \mathbb{Z}_2)$

$\boldsymbol{A} \subset \boldsymbol{M}$: a great antipodal set

$\Leftrightarrow \boldsymbol{A}$: antipodal, $\#\boldsymbol{A} = \#_2 \boldsymbol{M}$

$\boldsymbol{A}_1, \boldsymbol{A}_2 \subset \boldsymbol{M}$: congruent

$\Leftrightarrow \exists \boldsymbol{g} \in \boldsymbol{I}_0(\boldsymbol{M})\ \boldsymbol{A}_2 = \boldsymbol{g}\boldsymbol{A}_1$

(Tanaka-T.) In a symmetric $\boldsymbol{R}$-space

(A) $\forall$ antipodal set $\subset \exists$ a great antipodal set

(B) $\forall$ great antipodal sets are congruent

In a symmetric $\boldsymbol{R}$-space

Great antipodal sets $=$ Maximal antipodal sets

In general

Great antipodal sets $\subsetneqq$ Maximal antipodal sets

$\mathbf{2}$-numbers

$$\Downarrow$$

All congruent classes of maximal antipodal sets

2 Oriented real Grassmann mfds

$G_k(\mathbb{R}^n) = \{k\text{-dim subspaces in } \mathbb{R}^n\}$

$\tilde{G}_k(\mathbb{R}^n) = \{\text{oriented } k\text{-dim subspaces in } \mathbb{R}^n\}$

All of $G_k(\mathbb{R}^n)$: symmetric R-spaces

$\min\{k, n-k\} \leq 2$

$\Rightarrow \tilde{G}_k(\mathbb{R}^n)$: a symmetric R-space

$\min\{k, n-k\} > 2$

$\Rightarrow \tilde{G}_k(\mathbb{R}^n)$: not a symmetric R-space

All congruent classes of maximal antipodal sets are not known.

$$P_k(n) = \{\alpha \subset \{1, \ldots, n\} \mid \#\alpha = k\}$$

$\alpha - \beta = \{i \in \alpha \mid i \notin \beta\}$ for $\alpha, \beta \in P_k(n)$

α, β : antipodal $\Leftrightarrow \#(\alpha - \beta)$: even

$A \subset P_k(n)$: antipodal

$\Leftrightarrow \alpha, \beta$: antipodal for any $\alpha, \beta \in A$

$A_1, A_2 \subset P_k(n)$: congruent

$\Leftrightarrow \exists g \in \mathrm{Sym}(n)\ A_2 = gA_1$

$\boldsymbol{P_k(n) \ni \alpha = \{\alpha_1, \ldots, \alpha_k\}}$

$(\boldsymbol{\alpha_1 < \cdots < \alpha_k})$

$\boldsymbol{v = (v_1, \ldots, v_n)}$: an orthonormal basis of $\mathbb{R}^{\boldsymbol{n}}$

$\boldsymbol{A \subset P_k(n)}$: antipodal

$\boldsymbol{v(A) = \{\pm \mathrm{span}\{v_{\alpha_1}, \ldots, v_{\alpha_k}\} \mid \alpha \in A\}}$

Theorem (T.)

"$\boldsymbol{A \mapsto v(A)}$" induces a bijection from

{cong. classes of MAS in $\boldsymbol{P_k(n)}$} to

{cong. classes of MAS in $\boldsymbol{\tilde{G}_k}(\mathbb{R}^{\boldsymbol{n}})$}

3 Antipodal subsets of $P_k(n)$

In the case $k = 1$

$\{\{1\}\}$: MAS in $P_1(n)$

$\{\pm v\}$: MAS in $\tilde{G}_1(\mathbb{R}^{n+1}) = S^n$

In the case $k = 2$

$A(2, 2l) = \{\{1, 2\}, \{3, 4\}, \dots, \{2l - 1, 2l\}\}$

: MAS in $P_2(2l), P_2(2l + 1)$

$v(A(2, 2l))$: MAS in $\tilde{G}_2(\mathbb{R}^{2l}), \tilde{G}_2(\mathbb{R}^{2l+1})$

In the cases $\boldsymbol{k = 3, 4}$

All congruent classes of MAS in $\boldsymbol{P_k(n)}$ are classified. (T.)

$$\boldsymbol{A(3, 2l+1) = \{\alpha \cup \{2l+1\} \mid \alpha \in A(2, 2l)\}}$$
$$\subset \boldsymbol{P_3(2l+1)}$$

$$\boldsymbol{A(4, 2l) = \{\alpha \cup \beta \mid \alpha, \beta \in A(2, 2l), \alpha \neq \beta\}}$$
$$\subset \boldsymbol{P_4(2l)}$$

$$\boldsymbol{Ev_8 = \{\{\alpha_1, \ldots, \alpha_4\} \mid \alpha_i \in \{2i-1, 2i\}}$$
$$\boldsymbol{(1 \leq i \leq 4)\ \#(\text{even numbers } \alpha_i) : \text{even}\}}$$
$$\subset \boldsymbol{P_4(8)}$$

$$\boldsymbol{A(2k,2l) = \{\alpha_1 \cup \cdots \cup \alpha_k \mid \alpha_i \in A(2,2l), \alpha_i \neq \alpha_j\}}$$
$$\boldsymbol{\quad \subset P_{2k}(2l)}$$
$$\boldsymbol{A(2k+1,2l+1) = \{\alpha \cup \{2l+1\} \mid \alpha \in A(2k,2l)\}}$$
$$\boldsymbol{\quad \subset P_{2k+1}(2l+1)}$$

Theorem (T.)

$\boldsymbol{A(2k,2l), A(2k+1,2l+1)}$: antipodal

$\boldsymbol{l \geq 3k+1 \Rightarrow}$

$\boldsymbol{A(2k,2l)}$: MAS in $\boldsymbol{P_{2k}(2l)}$, $\boldsymbol{P_{2k}(2l+1)}$

$\boldsymbol{A(2k+1,2l+1)}$: MAS in $\boldsymbol{P_{2k+1}(2l+1)}$,

$\boldsymbol{P_{2k+1}(2l+2)}$

$$\boldsymbol{Ev_{2m} = \{\{\alpha_1, \ldots, \alpha_m\} \mid \alpha_i \in \{2i-1, 2i\}}$$
$$\boldsymbol{(1 \le i \le m)\ \#(\text{even numbers } \alpha_i) : \text{even}\}}$$
$$\boldsymbol{\subset P_m(2m)}$$

Theorem (T.)

(0) $\boldsymbol{m \ge 1}$

$\boldsymbol{Ev^{+}_{8m} = Ev_{8m} \cup A(4m, 8m)}$: MAS in $\boldsymbol{P_{4m}(8m)}$

(1) $\boldsymbol{m \ge 1}$

$\boldsymbol{Ev_{8m+2}}$: MAS in $\boldsymbol{P_{4m+1}(8m+2)}$

(2) $\boldsymbol{m \ge 0}$

$\boldsymbol{Ev_{8m+4}}$: MAS in $\boldsymbol{P_{4m+2}(8m+4)}$

(3) $\boldsymbol{m \ge 0}$

$\boldsymbol{Ev_{8m+6}}$: MAS in $\boldsymbol{P_{4m+3}(8m+6)}$

4 Estimates of antipodal subsets

$a(k,n) = \max\{\#A \mid A : \text{antipodal in } P_k(n)\}$

$a(1,n) = 1 \quad a(2,n) = \left[\frac{n}{2}\right]$

We have

$$a(2k,n) \geq \#A\left(2k, 2\left[\frac{n}{2}\right]\right) = \binom{[n/2]}{k},$$

$$a(2k+1,n) \geq \#A\left(2k+1, 2\left[\frac{n-1}{2}\right]+1\right)$$

$$= \binom{[(n-1)/2]}{k}.$$

Observation

$$\boldsymbol{n \geq 17 \Rightarrow a(3,n) = \left[\frac{n-1}{2}\right]}$$

$$\boldsymbol{A \subset P_3(n)} \text{ : antipodal, } \boldsymbol{\#A = \left[\frac{n-1}{2}\right]}$$

$$\Rightarrow \boldsymbol{A} \text{ : congruent with } \boldsymbol{A\left(3, 2\left[\frac{n-1}{2}\right]+1\right)}$$

$$\boldsymbol{n \geq 12 \Rightarrow a(4,n) = \binom{[n/2]}{2}}$$

$$\boldsymbol{A \subset P_4(n)} \text{ : antipodal, } \boldsymbol{\#A = \binom{[n/2]}{2}}$$

$$\Rightarrow \boldsymbol{A} \text{ : congruent with } \boldsymbol{A\left(4, 2\left[\frac{n}{2}\right]\right)}$$

Theorem (T.)

$$\boldsymbol{n \geq 57 \Rightarrow a(5, n) = \binom{\left[\frac{n-1}{2}\right]}{2}}$$

$$\boldsymbol{A \subset P_5(n)} \text{ : antipodal, } \boldsymbol{\# A = \binom{\left[\frac{n-1}{2}\right]}{2}}$$

$$\Rightarrow \boldsymbol{A} \text{ : congruent with } \boldsymbol{A\left(5, 2\left[\frac{n-1}{2}\right] + 1\right)}$$