Maximal antipodal subgroups of the automorphism groups of compact Lie algebras

Makiko Sumi Tanaka
Tokyo University of Science

The 20th International Workshop on Hermitian Symmetric Spaces and Submanifolds
Kyungpook National University
July 26–30, 2016

Joint work with Hiroyuki Tasaki

Contents

1. Antipodal sets

Chen-Nagano (1988)

M : a Riemannian symmetric space

s_x : the geodesic symmetry at $x \in M$

If M is connected,

$s_x(y) = y \Leftrightarrow x, y$ are antipodal on a closed geod.

$S \subset M$: antipodal set $\Leftrightarrow {}^\forall x, y \in S \quad s_x(y) = y$

$|S|$: the cardinality of S

$\#_2 M$: 2-number of M

$:= \sup\{|S| \mid S$: antipodal set $\subset M\}$

S : great antip. set $\Leftrightarrow \#_2 M = |S|$

Ex. $M = S^n$

$x \in S^n$ $\quad \{x, -x\}$: great antip. set

$\#_2 S^n = 2$

Ex. $M = \mathbb{R}P^n$

$e_1, \ldots, e_{n+1}$: o.n.b. of $\mathbb{R}^{n+1}$

$\{\mathbb{R}e_1, \ldots, \mathbb{R}e_{n+1}\}$: great antip. set

$\#_2 \mathbb{R}P^n = n + 1$

$\#_2 M$ depends on the topology of M

$f : M \to N$: totally geodesic embedding

S : antip. set of M $\Rightarrow$ S : antip. set of N

$\#_2 M \leq \#_2 N$

Chen-Nagano determined $\#_2 M$ for most compact Riem. sym. sp. M.

$\#_2 \tilde{G}_k(\mathbb{R}^n)$ $(2k \leq n)$ is not completely known for $k \geq 5$.

T. - Tasaki (2012)

M : Hermitian sym. sp. of cpt type

L_1, L_2 : real forms of M $\quad L_1 \pitchfork L_2$

$\Rightarrow$ $L_1 \cap L_2$: antip. set of L_1, L_2

${}^\exists g \in I_0(M)$ s.t. $g(L_1) = L_2$

$\Rightarrow$ $|L_1 \cap L_2| = \#_2 L_1 = \#_2 L_2$

Ex. $M = S^2$ $\quad L_1, L_2$: great circles

We are interested not only in $|S|$ of an antip. set S but also in S itself.

$\rightsquigarrow$ Classification of maximal antp. set

2. Maximal antip. subgr. of cpt Lie gr.

G : compact Lie gr.

G is a compact Riem. sym. sp. w.r.t. a bi-inv. Riem. metric.

$s_x(y) = xy^{-1}x \qquad (x, y \in G)$

e : the identity elem. of G

S : antipodal set of $G \qquad e \in S$

${}^\forall x \in S \qquad x^2 = e$

${}^\forall x, y \in S \qquad xy = yx$

S : maximal $\Rightarrow$ S : subgr. $\cong \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$

$$\Delta_n := \left\{ \begin{bmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{bmatrix} \right\} \subset O(n)$$

$$\Delta_n^+ := \{ x \in \Delta_n \mid \det(x) = 1 \}$$

Δ_n is a unique maximal antipodal subgroup of $U(n)$, $O(n)$, $Sp(n)$ up to conjugation.
Δ_n^+ is a unique maximal antipodal subgroup of $SU(n)$, $SO(n)$ up to conjugation.

$$D[4] := \left\{ \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}, \begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix} \right\} \subset O(2)$$

$n = 2^k \cdot l \qquad l$: odd

$0 \leq s \leq k$

$$D(s,n) := \underbrace{D[4] \otimes \cdots \otimes D[4]}_{s} \otimes \Delta_{n/2^s} \subset O(n)$$

$A = [a_{ij}] \in O(n),\ B \in O(m)$

$$A \otimes B := \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix} \in O(nm)$$

$G_1 \subset O(n),\ G_2 \subset O(m)$

$$G_1 \otimes G_2 := \{g_1 \otimes g_2 \mid g_i \in G_i\}$$

$$D(0,2) = \Delta_2 \subsetneq D[4] = D(1,2)$$

$$D(k-1, 2^k) = \underbrace{D[4] \otimes \cdots \otimes D[4]}_{k-1} \otimes \Delta_2$$

$$\subsetneq \underbrace{D[4] \otimes \cdots \otimes D[4] \otimes D[4]}_{k} = D(k, 2^k)$$

$\pi_2 : O(2) \to O(2)/\{\pm 1_2\}$: proj.

$D[4]$: not antip. subgr. of $O(2)$

$\pi_2(D[4])$: antip. subgr. of $O(2)/\{\pm 1_2\}$

$$Q[8] := \{\pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}\}$$

1, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$: the standard basis of $\mathbb{H}$

G : compact classical Lie gr.

Ad : $G \to GL(\mathfrak{g})$ $\quad$ Ad$(g)(X) = gXg^{-1}$

Theorem (T. - Tasaki)

(I) A max. antip. subgr. of Ad$(SU(n))$ is conjugate to

$$\text{Ad}(D(s,n)) \quad (0 \leq s \leq k),$$

where the case $(s,n) = (k-1, 2^k)$ is excluded.

(II) A max. antip. subgr. of Ad$(O(n))$ is conjugate to

$$\text{Ad}(D(s,n)) \quad (0 \leq s \leq k),$$

where the case $(s,n) = (k-1, 2^k)$ is excluded.

(III) A max. antip. subgr. of Ad$(Sp(n))$ is conjugate to

$$\text{Ad}(Q[8] \cdot D(s, n)) \quad (0 \le s \le k),$$

where the case $(s, n) = (k-1, 2^k)$ is excluded.

Remark. $D(k-1, 2^k)$ is not maximal because $D(k-1, 2^k) \subsetneq D(k, 2^k)$.

Since Ad$(G) \cong G/Z$ (Z : the center of G), we classify maximal antip. subgr. of G/Z.

3. Maximal antip. subgr. of Aut($\mathfrak{g}$)

Aut($\mathfrak{g}$) : the group of autom. of a Lie alg. $\mathfrak{g}$

Int($\mathfrak{g}$) : the identity comp. of Aut($\mathfrak{g}$)

G : cpt connected Lie gr. Lie(G) $= \mathfrak{g}$

$\rightsquigarrow$ $\mathrm{Ad}(G) = \mathrm{Int}(\mathfrak{g}) \cong G/Z$ Z : the center of G

$S = \{\sigma_1, \ldots, \sigma_k\}$: max. antip. subgr. of Aut($\mathfrak{g}$)

$\rightsquigarrow$ $\sigma_i^2 = e \quad \sigma_i\sigma_j = \sigma_j\sigma_i$

$\mathrm{Aut}(\mathfrak{sp}(n)) = \mathrm{Ad}(Sp(n)) \cong Sp(n)/\{\pm 1_n\}$

$\mathrm{Aut}(\mathfrak{o}(n)) = \mathrm{Ad}(O(n)) \cong O(n)/\{\pm 1_n\}$ if $n \neq 8$

$\mathrm{Aut}(\mathfrak{o}(8))/\mathrm{Int}(\mathfrak{o}(8)) \cong S_3$

$\tau : \mathfrak{su}(n) \to \mathfrak{su}(n),\ \tau(X) = \bar{X}$

$\mathrm{Aut}(\mathfrak{su}(n)) \cong \{1_n, \tau\} SU(n)/\mathbb{Z}_n$

Theorem (T. - Tasaki)

(I) A max. antip. subgr. of $\mathrm{Aut}(\mathfrak{su}(n))$ is conjugate to

$$\{e, \tau\}\mathrm{Ad}(D(s,n)) \quad (0 \le s \le k),$$

where the case $(s,n) = (k-1, 2^k)$ is excluded.

(II) A max. antip. subgr. of $\mathrm{Aut}(\mathfrak{o}(n))$ is conjugate to

$$\mathrm{Ad}(D(s,n)) \quad (0 \le s \le k),$$

where the case $(s,n) = (k-1, 2^k)$ is excluded.

(III) A max. antip. subgr. of $\mathrm{Aut}(\mathfrak{sp}(n))$ is conjugate to

$$\mathrm{Ad}(Q[8] \cdot D(s,n)) \quad (0 \leq s \leq k),$$

where the case $(s,n) = (k-1, 2^k)$ is excluded.

For (I) we use :

T : a maximal torus of $F(\tau, U(n)) = O(n)$

$$\tau U(n) = \bigcup_{g \in U(n)} g\,(\tau T)\,g^{-1}$$

Max. antip. subgr. of $\mathrm{Aut}(\mathfrak{g}_2)$

$\mathrm{Aut}(\mathfrak{g}_2) \cong G_2$

$M^+ := F(s_e, G_2) \cong G_2/SO(4) \qquad o \in M^+$

$F(s_o, M^+) \cong (S^2 \times S^2)/\mathbb{Z}_2$

$\pi : S^2 \times S^2 \to (S^2 \times S^2)/\mathbb{Z}_2$: proj.

$\pi((x,y)) = \pi((-x,-y))$

$\{e_1, e_2, e_3\}, \{f_1, f_2, f_3\}$: o.n.b. of $\mathbb{R}^3$

$\{\pi((e_1, \pm f_1)), \pi((e_2, \pm f_2)), \pi((e_3, \pm f_3))\}$ is a unique max. antip. subset of $(S^2 \times S^2)/\mathbb{Z}_2$ up to congruence.

Theorem (T. - Tasaki-Yasukura)

A max. antip. subgr. of G_2 is conjugate to

$$\{e, o, \pi((e_1, \pm f_1)), \pi((e_2, \pm f_2)), \pi((e_3, \pm f_3))\}.$$

We can explicitly express the max. antip. subgr. by using the identification $G_2 = \mathrm{Aut}(\mathbb{O})$.