Sequences of maximal antipodal sets of oriented real Grassmann manifolds II

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1 Introduction

Antipodal sets in Riemannian symmetric spaces

: introduced by Chen-Nagano

M : a compact Riemannian symmetric space

s_x : the geodesic symmetry at $x \in M$

$A \subset M$: an antipodal set

$\Leftrightarrow s_x(y) = y \quad (x, y \in A)$

$\#_2 M = \max\{|A| \mid A \subset M : \text{antipodal}\}$

the 2-number of M a geometric invariant

$\boldsymbol{A} \subset \boldsymbol{M}$: a great antipodal set

$\Leftrightarrow \boldsymbol{A}$: antipodal, $|\boldsymbol{A}| = \#_2 \boldsymbol{M}$

$\boldsymbol{A_1}, \boldsymbol{A_2} \subset \boldsymbol{M}$: congruent

$\Leftrightarrow \exists \boldsymbol{g} \in \boldsymbol{I_0}(\boldsymbol{M})\ \boldsymbol{A_2} = \boldsymbol{g}\boldsymbol{A_1}$

In a symmetric $\boldsymbol{R}$-space

great antipodal set $=$ maximal antipodal set

In general

great antipodal set $\not\Leftarrow$ maximal antipodal set

All congruent classes of maximal antipodal sets

2 Oriented real Grassmann mfds

$G_k(\mathbb{R}^n) = \{k\text{-dim subspaces in } \mathbb{R}^n\}$

$\tilde{G}_k(\mathbb{R}^n) = \{\text{oriented } k\text{-dim subspaces in } \mathbb{R}^n\}$

All of $G_k(\mathbb{R}^n)$: symmetric R-spaces

$\min\{k, n-k\} \leq 2$

$\Rightarrow \tilde{G}_k(\mathbb{R}^n)$: a symmetric R-space

$\min\{k, n-k\} > 2$

$\Rightarrow \tilde{G}_k(\mathbb{R}^n)$: not a symmetric R-space

All congruent classes of maximal antipodal sets are not known.

$\binom{X}{k} = \{\alpha \subset X \mid |\alpha| = k\}$ for a set X

$[n] = \{1, 2, \ldots, n\}$ for a natural number n

$\alpha \backslash \beta = \{i \in \alpha \mid i \notin \beta\}$ for $\alpha, \beta \in \binom{[n]}{k}$

α, β : antipodal $\Leftrightarrow$ $|\alpha \backslash \beta|$: even

$A \subset \binom{[n]}{k}$: antipodal

$\Leftrightarrow$ α, β : antipodal for any $\alpha, \beta \in A$

$A_1, A_2 \subset \binom{[n]}{k}$: congruent

$\Leftrightarrow \exists g \in \mathrm{Sym}(n)\ A_2 = gA_1$

$\boldsymbol{v} = (\boldsymbol{v_1}, \ldots, \boldsymbol{v_n})$: an orthonormal basis of $\mathbb{R}^{\boldsymbol{n}}$

$\binom{[\boldsymbol{n}]}{\boldsymbol{k}} \ni \boldsymbol{\alpha} = \{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha_k}\}$

$(\boldsymbol{\alpha_1} < \cdots < \boldsymbol{\alpha_k})$

$\boldsymbol{A} \subset \binom{[\boldsymbol{n}]}{\boldsymbol{k}}$

$\boldsymbol{v}(\boldsymbol{A}) = \{\pm \mathbf{span}\{\boldsymbol{v}_{\boldsymbol{\alpha}_1}, \ldots, \boldsymbol{v}_{\boldsymbol{\alpha_k}}\} \mid \boldsymbol{\alpha} \in \boldsymbol{A}\}$

Theorem (T.)

"$\boldsymbol{A} \mapsto \boldsymbol{v}(\boldsymbol{A})$" induces a bijection from

$\left\{\text{cong. classes of MAS in } \binom{[\boldsymbol{n}]}{\boldsymbol{k}}\right\}$ to

$\left\{\text{cong. classes of MAS in } \tilde{\boldsymbol{G}}_{\boldsymbol{k}}(\mathbb{R}^{\boldsymbol{n}})\right\}$

3 Antipodal subsets of $\binom{[n]}{k}$

In the case $k = 1$

$\{\{1\}\}$: MAS in $\binom{[n]}{1}$

$\{\pm v\}$: MAS in $\tilde{G}_1(\mathbb{R}^{n+1}) = S^n$

In the case $k = 2$

$A(2, 2l) = \{\{1, 2\}, \{3, 4\}, \ldots, \{2l - 1, 2l\}\}$

: MAS in $\binom{[2l]}{2}, \binom{[2l+1]}{2}$

In the case $\boldsymbol{k = 3}$

All congruent classes of MAS in $\binom{[\boldsymbol{n}]}{\boldsymbol{3}}$ are classified. (T.)

$$\boldsymbol{A(3, 2l+1) = \{\alpha \cup \{2l+1\} \mid \alpha \in A(2, 2l)\}}$$
$$\boldsymbol{\subset \binom{[2l+1]}{3}}$$

$$\boldsymbol{Ev_6 = \{\{\alpha_1, \ldots, \alpha_3\} \mid \alpha_i \in \{2i-1, 2i\}}$$
$$\boldsymbol{(1 \leq i \leq 3)\, |\{\text{even numbers } \alpha_i\}| : \text{even}\}}$$
$$\boldsymbol{\subset \binom{[6]}{3}}$$

In the case $\boldsymbol{k = 4}$

All congruent classes of MAS in $\binom{[\boldsymbol{n}]}{\boldsymbol{k}}$ are classified. (T.)

$$\boldsymbol{A(4, 2l) = \{\alpha \cup \beta \mid \alpha, \beta \in A(2, 2l), \alpha \neq \beta\}}$$
$$\subset \binom{[\boldsymbol{2l}]}{\boldsymbol{4}}$$

$$\boldsymbol{Ev_8 = \{\{\alpha_1, \ldots, \alpha_4\} \mid \alpha_i \in \{2i-1, 2i\}}$$
$$\boldsymbol{(1 \leq i \leq 4)\, |\{\text{even numbers } \alpha_i\}| : \text{even}\}}$$
$$\subset \binom{[\boldsymbol{8}]}{\boldsymbol{4}}$$

We can generalize $\boldsymbol{A(3, 2l+1)}$ and $\boldsymbol{A(4, 2l)}$.
Using these we can estimate

$$\boldsymbol{a(k,n) = \max \left\{ |A| \;\middle|\; A \text{ is antipodal in } \binom{[n]}{k} \right\}}$$

for sufficiently large $\boldsymbol{n}$ with respect to $\boldsymbol{k}$.

The case where $\boldsymbol{k = 5}$: Tasaki 2015

The general case : Frankl-Tokushige 2016

4 Sequences of MAS

$$Ev_{2m} = \{\{\alpha_1, \ldots, \alpha_m\} \mid \alpha_i \in \{2i-1, 2i\}$$
$$(1 \leq i \leq m) \,|\{\text{even numbers } \alpha_i\}| : \text{even}\} \subset \binom{[2m]}{m}$$

Theorem (T.)

(0) $m \geq 1$ Ev_{8m} : not MAS in $\binom{[8m]}{4m}$

(1) $m \geq 1$ Ev_{8m+2} : MAS in $\binom{[8m+2]}{4m+1}$

(2) $m \geq 0$ Ev_{8m+4} : MAS in $\binom{[8m+4]}{4m+2}$

(3) $m \geq 0$ Ev_{8m+6} : MAS in $\binom{[8m+6]}{4m+3}$

Theorem (T.)

(0) $m \geq 1$ $Ev^{+}_{8m} = Ev_{8m} \cup A(4m, 8m)$

: MAS in $\binom{[8m+i]}{4m}$ $(i = 0, 1, 2, 3)$

(1) $m \geq 1$ Ev_{8m+2} : MAS in $\binom{[8m+i]}{4m+1}$ $(i = 2, 3, 4)$

Ev^{+}_{8m+2} : MAS in $\binom{[8m+5]}{4m+1}$

(2) $m \geq 0$ Ev_{8m+4} : MAS in $\binom{[8m+i]}{4m+2}$ $(i = 4, 5)$

Ev^{+}_{8m+4} : MAS in $\binom{[8m+6]}{4m+2}$

(3) $m \geq 0$ Ev_{8m+6} : MAS in $\binom{[8m+6]}{4m+3}$

Ev^{+}_{8m+6} : MAS in $\binom{[8m+7]}{4m+3}$

$Ev_{8m+2} \subsetneq Ev^{+}_{8m+2}$, $Ev_{8m+4} \subsetneq Ev^{+}_{8m+4}$
$Ev_{8m+6} \subsetneq Ev^{+}_{8m+6}$.

$$\boldsymbol{a(k,n) = \max\left\{ |A| \;\middle|\; A \text{ is antipodal in } \binom{[n]}{k} \right\} = \#_2\tilde{G}_k(\mathbb{R}^n)/2}$$

(0) $\boldsymbol{m \geq 1}$ and $\boldsymbol{i = 0, 1, 2, 3}$

$$\boldsymbol{2^{4m-1} + \binom{4m}{2m} \leq a(4m, 8m+i).}$$

(1) $\boldsymbol{m \geq 1}$ and $\boldsymbol{i = 2, 3, 4}$

$$\boldsymbol{2^{4m} \leq a(4m+1, 8m+i),}$$

$$\boldsymbol{2^{4m} + \binom{4m+1}{2m-1} \leq a(4m+1, 8m+5).}$$

(2) $\boldsymbol{m \geq 0}$ and $\boldsymbol{i = 4, 5}$

$$\boldsymbol{2^{4m+1} \leq a(4m+2, 8m+i),}$$

$$\boldsymbol{2^{4m+1} + \binom{4m+2}{2m} \leq a(4m+2.8m+6).}$$

(3) $\boldsymbol{m \geq 0}$

$$\boldsymbol{2^{4m+2} \leq a(4m+3, 8m+6),}$$

$$\boldsymbol{2^{4m+2} + \binom{4m+3}{2m+1} \leq a(4m+3, 8m+7).}$$