Antipodal sets of compact symmetric spaces

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Antipodal sets and polars of symmetric spaces

Definition 1 (Loos, Nagano)

A C^∞ manifold M with a map $s : M \to \mathrm{Diff}(M), x \mapsto s_x := s(x)$ is called a *symmetric space* if it satisfies the following properties:

(i) $s_x \circ s_x = \mathrm{id}$,

(ii) x is an isolated fixed point of s_x,

(iii) $s_x \circ s_y = s_{s_x(y)} \circ s_x$ for $\forall x, y \in M$,

(iv) the map $M \times M \to M, (x, y) \mapsto s_x(y)$ is C^∞.

・When M is a Riemannian symmetric space, i.e., $\forall x \in M$ is an isolated fixed point of an involutive isometry s_x of a Riemannian manifold M, then M is a symmetric space if M is connected.

・A compact connected symmetric space is a Riemannian symmetric space.

Definition 2 (Chen-Nagano [1])

A subset A of a symmetric space M is called an *antipodal set* if $s_x(y) = y$ holds for $\forall x, y \in A$.

- Since x is an isolated fixed point of s_x, an antipodal set is discrete.
- An antipodal set A of M is called *maximal* if the following holds:

  A' : an antipodal set of M, $A \subset A' \Rightarrow A = A'$

Example. $M = S^n$ $(\subset \mathbb{R}^{n+1})$

$x \in \mathbb{R}^{n+1}$, $\langle x \rangle_{\mathbb{R}} := \{kx \mid k \in \mathbb{R}\}$, $\rho_{\langle x \rangle_{\mathbb{R}}}$: the reflection along $\langle x \rangle_{\mathbb{R}}$

$s_x = \rho_{\langle x \rangle_{\mathbb{R}}}|_{S^n}$ $(x \in S^n)$

$\{x, -x\}$ is a maximal antipodal set for $\forall x \in S^n$.

- If a symmetric space M is compact, an antipodal set of M is finite.

Definition 3 (cf. Chen-Nagano [1])

Let M be a symmetric space. A connected component of $F(s_x, M) := \{y \in M \mid s_x(y) = y\}$ is called a *polar* of M w.r.t. x.

- $\{x\}$ is trivially a polar.

- By (iii), $s_y(F(s_x, M)) \subset F(s_x, M)$ for any $y \in F(s_x, M)$. In particular, if M^+ is a polar, $s_y(M^+) \subset M^+$ for $\forall y \in M^+$. Hence M^+ is a symmetric space with $s_y^{M^+} = s_y|_{M^+}$ $(y \in M^+)$.

- If A is an antipodal set of M, $A \subset F(s_x, M)$ for any $x \in A$. If M^+ is a polar, $A \cap M^+$ is an antipodal set of M^+.

Chen-Nagano and Nagano studied polars of compact symmetric spaces in detail.

Example. The unitary group $U(n)$ is a compact symmetric space with $s_x(y) = xy^{-1}x$ $(x, y \in U(n))$.

1_n : the identity matrix of size n, $\quad s_{1_n}(y) = y^{-1}$ $(y \in U(n))$

$I_j := \mathrm{diag}(\underbrace{-1, \dots, -1}_{j}, \underbrace{1, \dots, 1}_{n-j}\}$

$$F(s_{1_n}, U(n)) = \{\pm 1_n\} \cup \bigcup_{j=1}^{n-1} \{g\, I_j\, g^{-1} \mid g \in U(n)\},$$

where $\{g\, I_j\, g^{-1} \mid g \in U(n)\} \cong U(n)/(U(j) \times U(n-j)) \cong G_j(\mathbb{C}^n)$.

That is, $G_j(\mathbb{C}^n)$ is realized as a polar of $U(n)$.

Remark. A compact connected symmetric space is not necessarily realized as a polar of a certain connected compact Lie group. For example, $U(n)/O(n), U(2n)/Sp(n)$ are realized as polars of certain disconnected compact Lie groups but not compact connected Lie groups.

$\Delta_n := \{\mathrm{diag}\{\underbrace{\pm 1, \ldots, \pm 1}_{n})\}$ is a maximal antipodal set of $U(n)$. Δ_n is an abelian subgroup $\cong \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{n}$.

Δ_n is a unique maximal antipodal subgroup up to conjugation.

$M_0^+ := \{1_n\}, \quad M_n^+ := \{-1_n\}$

$M_j^+ := \{g\, I_j\, g^{-1} \mid g \in U(n)\} \quad (1 \leq j \leq n-1)$

$M_0^+, \ldots, M_n^+$: the polars of $U(n)$

$\Delta_n \cap M_j^+ = \{\mathrm{diag}\{\epsilon_1, \ldots, \epsilon_n) \in \Delta_n \mid \#\{k \mid \epsilon_k = -1\} = j\} \quad (0 \leq j \leq n)$

V_x : the (-1)-eigenspace of $x \in \Delta_n \cap M_j^+$

Under the correspondence $\Delta_n \cap M_j^+ \ni x \mapsto V_x \in G_j(\mathbb{C}^n)$, $\Delta_n \cap M_j^+$ gives an antipodal set of $G_j(\mathbb{C}^n)$.

Known results related to antipodal sets

Hereafter we will deal with compact symmetric spaces equipped with a Riemannian metric invariant under every symmetry.

M : a compact symmetric space

A : an antipodal set of M $\rightsquigarrow$ $\#A < \infty$

$\{\#A \mid A \subset M, A : \textit{antipodal}\} < \infty$ (cf. [3])

$\#_2 M := \max\{\#A \mid A \subset M, A : \textit{antipodal}\}$, called the *2-number* of M ([1])

Remark. $\#_2 M$ is related to the 2-rank $r_2(M)$ when M is a compact connected Lie group, where $r_2(M)$ is the maximal possible rank of the elemntary 2-subgroup $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$. In fact, $\#_2 M = 2^{r_2(M)}$ holds.

Chen-Nagano determined $\#_2 M$ in [1] but there are some exceptions such as the oriented Grassmann manifold $\tilde{G}_j(\mathbb{R}^n)$ $(j \geq 3)$.

Remark. Tasaki determined $\#_2 \tilde{G}_j(\mathbb{R}^n)$ $(j = 3, 4)$. $\#_2 \tilde{G}_j(\mathbb{R}^n)$ $(j \geq 5)$ is still unknown.

If an antipodal set A of M satisfies $\#A = \#_2 M$, A is called *great* ([1]).
great $\Rightarrow$ maximal, great $\not\Leftarrow$ maximal.

Theorem 2.1 ([1])

$\chi(M) \leq \#_2 M$, where $\chi(M)$ denotes the Euler characteristic of M.

$$o \in M, \quad F(s_o, M) = \bigcup_{j=0}^{r} M_j^+, \quad M_j^+ : \text{a polar}, \quad M_0 := \{o\}.$$

It is easy to see $\#_2 M \leq \sum_{j=0}^{r} \#_2 M_j^+$.

A symmetric space M is called a *Hermitian symmetric space* if M is a Hermitian manifold whose Hermitian metric is invariant under every symmetry. Hermitian symmetric space is called *of compact type* if $HI_0(M)$ is a compact semisimple Lie group.

Theorem 2.2 ([1])

If M is a Hermitian symmetric space of compact type, then

(i) $\#_2 M = \chi(M)$, *and* (ii) $\#_2 M = \sum_{j=0}^{r} \#_2 M_j^+$.

Takeuchi generalized (ii) to the case where M is a symmetric R-space.

Takeuchi also proved that if M is a symmetric R-space, $\#_2 M$ coincides with the sum of $\mathbb{Z}_2$-Betti numbers of M.

M : a Hermitian symmetric space

σ : an involutive aniti-holomorphic isometry of M

A connected component $F(\sigma, M)$ is called a *real form* of M.

a real form of a Herm. sym. sp. of cpt. type $\longleftrightarrow$ a symmetric R-space

Theorem 2.3 ([2])

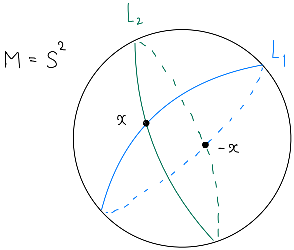
M : a Hermitian symmetric space of compact type

L_1, L_2 : real forms of M, intersect transversely

$\Rightarrow L_1 \cap L_2$ is an antipodal set of L_1, L_2.

Moreover, if L_1, L_2 are congruent, $L_1 \cap L_2$ is great.

Example. $M = \mathbb{C}P^1 = S^2$, $L_1, L_2 \cong \mathbb{R}P^1 = S^1$



$L_1 \cap L_2 = \{x, -x\}$

Antipodal sets are “good” finite subsets of compact symmetric spaces.

Classification of maximal antipodal sets is a fundamental problem.

Classification of maximal antipodal sets

The case where M is a symmetric R-space.

Theorem 3.1 ([3])

M : a symmetric R-space

(i) *Any antipodal set of M is included in a great antipodal set. In particular, every maximal antipodal set is great.*

(ii) *Any two great antipodal sets of M are congruent.*

Hence, a maximal antipodal set A of a symmetric R-space M is unique up to congruence and $\#A = \#_2 M$.

The case where M is a compact Lie group G.

$\exists$ a bi-invariant Riemannian metric on G.

G is a Riemannian symmetric space with $s_x(y) = xy^{-1}x$ $(x, y \in G)$.

A maximal antipodal set containing the unit element is a subgroup.

We classify maximal antipodal subgroups (MAS) of G up to conjugation.

$G = O(n), U(n), Sp(n) \quad \rightsquigarrow \quad$ symmetric R-spaces

$\Delta_n = \{\mathrm{diag}(\pm 1, \dots, \pm 1)\}$ is a unique MAS of G up to conjugation.

$\#_2 O(n) = \#_2 U(n) = \#_2 Sp(n) = 2^n$.

$G = SO(n), SU(n)$

$\Delta_n^+ := \{d \in \Delta_n \mid \det(d) = 1\}$ is a unique MAS of G up to conjugation.

$\#_2 SO(n) = \#_2 SU(n) = 2^{n-1}$.

In [6] we classified conjugacy classes of MAS of the quotient groups of $U(n), SU(n), O(n), SO(n)$ and $Sp(n)$, and gave explicit descriptions of their representatives.

By using the results, in [5] we classified conjugacy classes of MAS of $\mathrm{Aut}(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{su}(n), \mathfrak{so}(n), \mathfrak{sp}(n)$.

In [10] we studied the case of $G = G_2$.

$F(s_e, G_2) = \{e\} \cup M_1^+, \quad M_1^+ \cong G_2/SO(4)$

$F(s_x, M_1^+) = \{x\} \cup M_{1,1}^+$ for $x \in M_1^+, \quad M_{1,1}^+ \cong (S^2 \times S^2)/\mathbb{Z}_2$

Hence, if A is a maximal antipodal set of $M_{1,1}^+$, then $\{x\} \cup A$ is a maximal antipodal set of M_1^+, and $\{e\} \cup \{x\} \cup A$ is a MAS of G_2.

MAS of G_2 is unique up to conjugation. $\quad \#_2 G_2 = 2^3$.

We gave an explicit description of MAS of G_2 by using the identification $G_2 = \mathrm{Aut}(\mathbb{O})$, where $\mathbb{O}$ denotes the octonions.

$G = F_4, E_6 \quad \rightsquigarrow \quad$ Sasaki (JLT 2022, DGA 2022)

Remark. Later we will refer to the case where G is a semi-direct product.

The case where M is a compact symmetric space, not a Lie group.

In [7] we gave a basic principle of classifying maximal antipodal sets of M which is a polar of a compact Lie group G.

G_0 : the identity component of G

$g \in G,\ I_g(h) := ghg^{-1}\ (h \in G)$

If M is a polar of G, then $M = \{I_g(x) \mid g \in G_0\}$ for $x \in M$ and

$I_0(M) = \{I_g|_M \mid g \in G_0\}$.

A : a maximal antipodal set of M

$\{e\} \cup A$ is an antipodal set of G, since M is a polar.

$\exists \tilde{A}$: MAS of G s. t. $\{e\} \cup A \subset \tilde{A}$

$A = \tilde{A} \cap M$

Classif. of max. antp. subgr. of G $\rightsquigarrow$ Classif. of max. antp. subset of M

In [7] we studied the following M's:

$M = G_j(\mathbb{R}^n), G_j(\mathbb{C}^n), G_j(\mathbb{H}^n)$: a polar of $O(n), U(n), Sp(n)$, respectively,
$M = Sp(n)/U(n) \cong \{x \in Sp(n) \mid x^2 = -1_n\}$,
$M = SO(2n)/U(n) \cong$ a conn. comp. of $\{x \in SO(2n) \mid x^2 = -1_{2n}\}$.

We classified congruence classes of maximal antipodal sets of the above M, and those of the quotient spaces of $G_m(\mathbb{K}^{2m})$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$), $Sp(n)/U(n)$ and $SO(4m)/U(2m)$.

Each of these quotient spaces is realized as a polar, for example, $(Sp(n)/U(n))/\mathbb{Z}_2$ is a polar of $Sp(n)/\mathbb{Z}_2$.

We gave explicit descriptions of their representatives.

$M = U(n)/O(n),\ U(2n)/Sp(n),\ SU(n)/SO(n),\ SU(2n)/Sp(n)$

They are not realized as polars of connected compact Lie groups, but they are realized as polars of disconnected compact Lie groups.

$\sigma_I : U(n) \to U(n),\ \sigma_I(x) = \bar{x},$ the complex conjugation

$\sigma_I \in \mathrm{Aut}(U(n)),\ \sigma_I^2 = 1$

$G := U(n) \rtimes \langle \sigma_I \rangle = (U(n), 1) \cup (U(n), \sigma_I)$

G is a disconnected compact Lie group.

$F(s_e, G) = (F(s_{1_n}, U(n)), 1) \cup (\{g \in U(n) \mid \sigma_I(g) = g^{-1}\}, \sigma_I)$

$UI(n) := \{g \in U(n) \mid \sigma_I(g) = g^{-1}\} = \rho_{\sigma_I}(U(n))(1_n) \cong U(n)/O(n),$

where $\rho_{\sigma_I}(g)(x) := g\, x\, \sigma_I(g)^{-1}$ $(g, x \in U(n))$.

$(UI(n), \sigma_I)$ is a polar of G.

In order to classify maximal antipodal sets of $UI(n)$, we classify MAS of $U(n) \rtimes \langle \sigma_I \rangle$ first.

To classify MAS of $U(n) \rtimes \langle \sigma_I \rangle$, the canonical forms of $(U(n), \sigma_I)$ is useful.

T : a maximal torus of $U(n)$

T_I : a maximal torus of $F(\sigma_I, U(n)) = O(n)$

$$(U(n), 1) = \bigcup_{g \in U(n)} (g, 1)(T, 1)(g, 1)^{-1} = \bigcup_{g \in U(n)} (gTg^{-1}, 1)$$

$$(U(n), \sigma_I) = \bigcup_{g \in U(n)} (g, 1)(T_I, \sigma_I)(g, 1)^{-1} = \bigcup_{g \in U(n)} (gT\sigma_I(g)^{-1}, \sigma_I)$$

Theorem 3.2 (cf. [5])

Any MAS of $U(n) \rtimes \langle \sigma_I \rangle$ is conjugate to $\Delta_n \rtimes \langle \sigma_I \rangle$ by an element of $(U(n), 1)$.

Theorem 3.3

Any maximal antipodal set of $UI(n)$ is congruent to Δ_n.

$$\sigma_{II} : U(2n) \to U(2n),\ \sigma_{II}(x) = J_n \bar{x} J_n^{-1},\ \ J_n = \begin{bmatrix} 0 & -1_n \\ 1_n & 0 \end{bmatrix}$$

$\sigma_{II} \in \mathrm{Aut}(U(2n)),\ \sigma_{II}^2 = 1$

$G := U(2n) \rtimes \langle \sigma_{II} \rangle = (U(2n), 1) \cup (U(2n), \sigma_{II})$

G is a disconnected compact Lie group.

$UII(n) := \{g \in U(2n) \mid \sigma_{II}(g) = g^{-1}\} = \rho_{\sigma_{II}}(U(2n))(1_{2n}) \cong U(2n)/Sp(n)$

$(UII(n), \sigma_{II})$ is a polar of G.

Theorem 3.4

Any MAS of $U(2n) \rtimes \langle \sigma_{II} \rangle$ *is conjugate to* $(\Delta_{2n}, 1)$ *or* $(1_2 \otimes \Delta_n) \rtimes \langle \sigma_{II} \rangle$ *by an element of* $(U(2n), 1)$.

Here, $1_2 \otimes \Delta_n = \left\{ \begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix} \middle| d \in \Delta_n \right\}.$

Theorem 3.5

Any maximal antipodal set of $UII(n)$ *is congruent to* $1_2 \otimes \Delta_n$.

Remark. $UI(n), UII(n)$ are symmetric R-spaces.

$AI(n) := UI(n) \cap SU(n) = \rho_{\sigma_I}(SU(n))1_n \cong SU(n)/SO(n)$

$(AI(n), \sigma_I)$ is a polar of $SU(n) \rtimes \langle \sigma_I \rangle$.

$UII(n) \cap SU(2n)$ is NOT connected.

$UII(n) \cap SU(2n) = \rho_{\sigma_{II}}(SU(2n))1_{2n} \cup \rho_{\sigma_{II}}(U^-(2n))1_{2n}$

$AII(n) := \rho_{\sigma_{II}}(SU(2n))1_{2n} \cong SU(2n)/Sp(n)$.

$(AII(n), \sigma_{II})$ is a polar of $SU(2n) \rtimes \langle \sigma_{II} \rangle$.

Theorem 3.6

Any maximal antipodal set of $AI(n)$ is congruent to Δ_n^+.

Theorem 3.7

Any maximal antipodal set of $AII(n)$ is congruent to $1_2 \otimes \Delta_n^+$.

・In [9] we studied fundamental properties of polars of disconnected compact Lie groups.

・We classified maximal antipodal sets of the quotient spaces of $UI(n), UII(n), AI(n)$. In order that, we classified MAS of the quotient groups of $U(n) \rtimes \langle \sigma_I \rangle, U(2n) \rtimes \langle \sigma_{II} \rangle$ and $SU(n) \rtimes \langle \sigma_I \rangle$. Classification of maximal antipodal sets of the quotient space of $AII(n)$ is not clear yet.

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