Classification of maximal antipodal sets of compact symmetric spaces

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Introduction

An **antipodal** set is a subset A of a symmetric space M which satisfies $s_x(y) = y$ for any $x, y \in A$, where s_x denotes the symmetry at x of M. In the joint research with Hiroyuki Tasaki, we have been studying classification of maximal antipodal sets of compact symmetric spaces. Here an antipodal set A of M is called **maximal** if A' is an antipodal set of M satisfying $A \subset A'$, then $A = A'$ holds.

In the last International Workshop on February in 2023 at Osaka Metrop. Univ., I talked in detail the background of this research and some related former results. In this talk, I explain in detail how to classify maximal antipodal sets, in particular, our recent results of the classification of maximal antipodal sets of M where $M = U(n)/O(n)$, $U(2n)/Sp(n)$, $SU(n)/SO(n)$, $SU(2n)/Sp(n)$, and those of the quotient spaces of M.

Our aim is to classify maximal antipodal sets of a compact symmetric space M up to congruence and give an explicit description of a representative of each congruent class. Here subsets $A, B \subset M$ are **congruent** if there exists $f \in \mathrm{Iso}(M)_0$ such that $f(A) = B$.

Remark. Chen-Nagano [1] defined an antipodal set of a symmetric space M and studied the maximal possible cardinality of an antipodal set, called the **2-number** $\#_2 M$. They showed results which indicate $\#_2 M$ is related to the topological structure of M and determined $\#_2 M$ for most M.

Another aim is to determine the maximum of the cardinalities of maximal antipodal sets as well as to determine antipodal sets whose cardinalities attain the maximum. Such antipodal sets are called **great** antipodal sets. This gives an alternative proof of Chen-Nagano's result of the determination of $\#_2 M$. Explicit descriptions of maximal antipodal sets make us possible to calculate the cardinalities of maximal antipodal sets.

Basic principle of classifying maximal antipodal sets

M : a Riemannian symmetric space, i.e.,

M is a Riemannian manifold equipped with an involutive isometry s_x for $\forall x \in M$ where x is an isolated fixed point of s_x.

For $o \in M$, a connected component of $F(s_o, M) := \{x \in M \mid s_o(x) = x\}$ is called a **polar** of M w.r.t. o. A polar is a totally geodesic submanifold, hence a polar is a Riemannian symmetric space w.r.t. the induced metric.

In our basic principle we make use of the realization of M as a polar of a compact Lie group G, It is known that a compact Lie group G admits a bi-invariant Riemannian metric and G is a Riemannian symmetric space with

$s_x(y) = xy^{-1}x \quad (x, y \in G)$.

$s_e(y) = y^{-1} \quad (e :$ the identity element$)$

$F(s_e, G) = \{x \in G \mid s_e(x) = x\} = \{x \in G \mid x^2 = e\}$

For $x, y \in F(s_e, G)$, $s_x(y) = y$ iff $xy = yx$.

A : a maximal antipodal set, $e \in A$

$\rightsquigarrow$ A is a subgroup, $A \cong \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$.

We refer to a maximal antipodal set containing e as a **maximal antipodal subgroup**.

I_g $(g \in G)$: an inner automorphism defined by $I_g(x) = gxg^{-1}$ $(x \in G)$.

G_0 : the identity component of G

We refer to a polar of G w.r.t. e as a polar of G.

Proposition 1

Let M be a polar of G. Then $M = \{I_g(x_0) \mid g \in G_0\}$ for $x_0 \in M$ and $\mathrm{Iso}(M)_0 = \{I_g|_M \mid g \in G_0\}$.

Therefore each polar of G is a conjugate class of involutive elements of G, and vice versa.

Based on these facts, basic principle is stated.

M : a polar of G

A : a maximal antipodal set of M

$\rightsquigarrow$ $A \cup \{e\}$: an antipodal set of G $\quad (\because A \subset M \subset F(s_e, G))$

$\exists \tilde{A}$: a maximal antipodal subgroup of G, $A \cup \{e\} \subset \tilde{A}$

$\rightsquigarrow$ $A = M \cap \tilde{A}$ $\quad (\because A$: maximal)

$[B_0], \ldots, [B_k]$: all G_0-conjugacy classes of maximal antipodal subgroups of G, where $B_0, \ldots, B_k$ are representatives.

$\rightsquigarrow$ $\exists g \in G_0,\ \exists s \in \{0, \ldots, k\}$ s.t. $\tilde{A} = I_g(B_s)$.

$A = M \cap \tilde{A} = M \cap I_g(B_s) = I_g(M \cap B_s) \quad (\because I_g(M) = M)$

i.e., A is $\mathrm{Iso}(M)_0$-congruent to $M \cap B_s$.

$\rightsquigarrow$ Any representative of $\mathrm{Iso}(M)_0$-congruent class of maximal antipodal sets of M is one of $M \cap B_0, \ldots, M \cap B_k$.

Example. $G = U(n)$

1_n : the identity matrix of size n

$I_j := \mathrm{diag}(\underbrace{-1. \dots, -1}_{j}, \underbrace{1, \dots, 1}_{n-j})$

$$F(s_{1_n}, U(n)) = \{\pm 1_n\} \cup \bigcup_{j=1}^{n-1} \{I_g(I_j) \mid g \in U(n)\},$$

where $\{I_g(I_j) \mid g \in U(n)\} \cong U(n)/(U(j) \times U(n-j)) \cong G_j(\mathbb{C}^n)$.

Here $G_j(\mathbb{C}^n)$ denotes the complex Grassmann manifold consisting of j-dimensional complex subspaces of $\mathbb{C}^n$.

Similarly, we obtain

$$F(s_{1_n}, O(n)) = \{\pm 1_n\} \cup \bigcup_{j=1}^{n-1} \{I_g(I_j) \mid g \in O(n)\},$$

where $\{I_g(I_j) \mid g \in O(n)\} \cong O(n)/(O(j) \times O(n-j)) \cong G_j(\mathbb{R}^n)$, and

$$F(s_{1_n}, Sp(n)) = \{\pm 1_n\} \cup \bigcup_{j=1}^{n-1} \{I_g(I_j) \mid g \in Sp(n)\},$$

where $\{I_g(I_j) \mid g \in Sp(n)\} \cong Sp(n)/(Sp(j) \times Sp(n-j)) \cong G_j(\mathbb{H}^n)$.

$\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$

$G_{\mathbb{K}} = O(n)\ (\mathbb{K} = \mathbb{R}),\ U(n)\ (\mathbb{K} = \mathbb{C}),\ Sp(n)\ (\mathbb{K} = \mathbb{H})$

Note that the above embedding $G_j(\mathbb{K}^n)$ into $G_{\mathbb{K}}$ is given by $x \mapsto \pi_{x^\perp} - \pi_x$,

$\pi_{x^\perp}, \pi_x$: the orthogonal projections on $x^\perp, x$.

$\gamma : G_m(\mathbb{K}^{2m}) \to G_m(\mathbb{K}^{2m}),\ \gamma(x) := x^\perp \quad (x \in G_m(\mathbb{K}^{2m}))$

$\langle\gamma\rangle$: the subgroup of $\mathrm{Iso}(M)$ generated by γ, $\ M = G_m(\mathbb{K}^{2m})$

$G_m(\mathbb{K}^{2m})^* := G_m(\mathbb{K}^{2m})/\langle\gamma\rangle$ is a compact Riemannian symmetric space.

Since $\gamma(x) \mapsto \pi_{\gamma(x)^\perp} - \pi_{\gamma(x)} = -(\pi_{x^\perp} - \pi_x)$, we obtain

$G_m(\mathbb{K}^{2m})^* \subset G_{\mathbb{K}}^* := G_{\mathbb{K}}/\{\pm 1_{2m}\}$.

$\pi : G_m(\mathbb{K}^{2m}) \to G_m(\mathbb{K}^{2m})^*$: the natural projection

$\pi \circ s_x = s_{\pi(x)} \circ \pi \quad (x \in G_m(\mathbb{K}^{2m}))$

$\rightsquigarrow$ $G_m(\mathbb{K}^{2m})^*$ is a polar of $G_{\mathbb{K}}^*$.

In [2] we classified maximal antipodal subgroups of $G_{\mathbb{K}}^*$ and using the results we classified maximal antipodal sets of $G_m(\mathbb{K}^{2m})^*$ in [3].

Remark. A compact connected symmetric space is not necessarily realized as a polar of a certain <u>connected</u> compact Lie group. For example, the so-called outer symmetric spaces such as $U(n)/O(n), U(2n)/Sp(n)$ are realized as polars of certain <u>disconnected</u> compact Lie groups but not connected compact Lie groups.

Classification of maximal antipodal sets

$\sigma_I : U(n) \to U(n),\ \sigma_I(x) = \bar{x},$ the complex conjugation

$\sigma_I \in \mathrm{Aut}(U(n)),\ \sigma_I^2 = 1$

$\langle \sigma_I \rangle = \{1, \sigma_I\}$: the subgroup of $\mathrm{Aut}(U(n))$ generated by σ_I

$G := U(n) \rtimes \langle \sigma_I \rangle = (U(n), 1) \cup (U(n), \sigma_I)$: disconnected

$(x, 1)(y, \alpha) = (xy, \alpha),\ (x, \sigma_I)(y, \alpha) = (x\sigma_I(y), \sigma_I \alpha) \quad (\alpha \in \langle \sigma_I \rangle)$

$F(s_e, G) = (F(s_{1_n}, U(n)), 1) \cup (\{g \in U(n) \mid \sigma_I(g) = g^{-1}\}, \sigma_I)$

$UI(n) := \{g \in U(n) \mid \sigma_I(g) = g^{-1}\} = \rho_{\sigma_I}(U(n))(1_n) \cong U(n)/O(n),$

where $\rho_{\sigma_I}(g)(x) := g\, x\, \sigma_I(g)^{-1}$ $(g, x \in U(n))$.

$UI(n)$ is a compact Riemannian symmetric space.

$(UI(n), \sigma_I)$ is a polar of $G = U(n) \rtimes \langle \sigma_I \rangle$.

In order to classify maximal antipodal sets of $UI(n)$, we classify maximal antipodal subgroups of $U(n) \rtimes \langle \sigma_I \rangle$ first.

$$\Delta_n := \{\mathrm{diag}(\underbrace{\pm 1, \ldots, \pm 1}_{n})\}$$

Theorem 2

Any maximal antipodal subgroup of $U(n) \rtimes \langle \sigma_I \rangle$ is conjugate to $\Delta_n \rtimes \langle \sigma_I \rangle$ by an element of $(U(n), 1)$.

Theorem 3

Any maximal antipodal set of $UI(n)$ is congruent to Δ_n.

Corollary 4

Δ_n is a unique great antipodal set of $UI(n)$ up to $U(n)$-congruence. We have $\#_2 UI(n) = 2^n$.

The quotient space $UI(n)/\mathbb{Z}_\mu$

$\mathbb{Z}_\mu = \{z1_n \mid z^\mu = 1\}$, μ : a natural number

$\rightsquigarrow$ $\mathbb{Z}_\mu$: a normal subgroup of $U(n)$, $U(n)/\mathbb{Z}_\mu$: the quotient group

$\mathbb{Z}_\mu$ preserves $UI(n)$, $\mathbb{Z}_\mu \subset Z(U(n))$: the center

$\rightsquigarrow$ the quotient space $UI(n)/\mathbb{Z}_\mu$ is defined.

$\sigma_I(\mathbb{Z}_\mu) = \mathbb{Z}_\mu$

$\rightsquigarrow$ σ_I induces an involutive autom. of $U(n)/\mathbb{Z}_\mu$, also denoted by σ_I.

$UI(n)/\mathbb{Z}_\mu \subset M := \{x \in U(n)/\mathbb{Z}_\mu \mid \sigma_I(x) = x^{-1}\}$

Note. M is not necessarily connected and $UI(n)/\mathbb{Z}_\mu$ is the connected component containing the identity element.

$(UI(n)/\mathbb{Z}_\mu, \sigma_I)$ is a polar of $U(n)/\mathbb{Z}_\mu \rtimes \langle \sigma_I \rangle$.

$(\mathbb{Z}_\mu, 1)$ is a normal subgroup of $U(n) \rtimes \langle \sigma_I \rangle$.

We denote the quotient group $(U(n) \rtimes \langle \sigma_I \rangle)/(\mathbb{Z}_\mu, 1)$ by $(U(n) \rtimes \langle \sigma_I \rangle)/\mathbb{Z}_\mu$.

By the equality

$$(U(n) \rtimes \langle \sigma_I \rangle)/\mathbb{Z}_\mu \ \ni \ (g, \alpha)(\mathbb{Z}_\mu, 1) = (g\mathbb{Z}_\mu, \alpha) \ \in \ U(n)/\mathbb{Z}_\mu \rtimes \langle \sigma_I \rangle,$$

we identify $U(n)/\mathbb{Z}_\mu \rtimes \langle \sigma_I \rangle$ with $(U(n) \rtimes \langle \sigma_I \rangle)/\mathbb{Z}_\mu$.

$$I_1 := \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad J_1 := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad K_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$D[4] := \{\pm 1_2, \pm I_1, \pm J_1, \pm K_1\}$$

$n = 2^k \cdot l, \ \ l :$ an odd number

$s \in \{0, \ldots, k\}$

$$D(s,n) := \{d_1 \otimes \cdots \otimes d_s \otimes d_0 \mid d_i \in D[4]\,(1 \leq i \leq s), d_0 \in \Delta_{n/2^s}\} \subset O(n)$$

e.g., $I_1 \otimes J_1 = \begin{bmatrix} -J_1 & 0 \\ 0 & J_1 \end{bmatrix}, \ J_1 \otimes I_1 = \begin{bmatrix} 0 & -I_1 \\ I_1 & 0 \end{bmatrix} \in O(4)$

Note : $D(0, n) = \Delta_n$

Theorem 5

$\pi_n : U(n) \rtimes \langle \sigma_I \rangle \to (U(n) \rtimes \langle \sigma_I \rangle)/\mathbb{Z}_\mu$: *the natural projection*

θ : *a primitive* 2μ*-th root of* 1

Any maximal antipodal subgroup of $(U(n) \rtimes \langle \sigma_I \rangle)/\mathbb{Z}_\mu$ *is* $\pi_n((U(n), 1))$*-conjugate to one of the following.*

1. *If* μ *is odd,* $\pi_n(\Delta_n \rtimes \langle \sigma_I \rangle)$.
2. *If* μ *is even,* $\pi_n(\{1, \theta\} D(s, n) \rtimes \langle \sigma_I \rangle)$ $(s \in \{0, \ldots, k\})$, *where the case of* $(s, n) = (k - 1, 2^k)$ *is excluded.*

The case of $(s, n) = (k-1, 2^k)$ is excluded because $\Delta_2 \subsetneq D[4]$ implies $D(k-1, 2^k) \subsetneq D(k, 2^k)$ hence $\pi_n(\{1, \theta\}D(k-1, 2^k) \rtimes \langle \sigma_I \rangle)$ is not maximal. In general, the following holds.

Proposition 6

G, G' : compact Lie groups

$\pi : G \to G'$: a covering homomorphism whose covering degree is odd

1. *If A is a maximal antipodal subgroup of G, $\pi(A)$ is a maximal antipodal subgroup of G' and $\pi(A)$ is isomorphic to A by π.*
2. *If A' is a maximal antipodal subgroup of G', there exists a maximal antipodal subgroup A of G such that A is isomorphic to A' by π.*

Note : $\pi_n(\{1,\theta\}D(s,n))$ $(s \in \{0,\ldots,k\})$ is a maximal antipodal subgroup of $U(n)/\mathbb{Z}_\mu$.

Theorem 7

Any maximal antipodal subgroup of $U(n)/\mathbb{Z}_\mu$ is conjugate to one of the following.

1. *If n or μ is odd, $\pi_n(\{1,\theta\}\Delta_n)$.*
2. *If both n and μ are even, $\pi_n(\{1,\theta\}D(s,n))$ $(s \in \{0,\ldots,k\})$, where the case of $(s,n) = (k-1, 2^k)$ is excluded.*

Sketch of Proof.

$\pi_n : U(n) \to U(n)/\mathbb{Z}_\mu$: the natural projection

A : a maximal antipodal subgroup of $U(n)/\mathbb{Z}_\mu$

$B := \pi_n^{-1}(A)$

B : commutative $\rightsquigarrow$ $A \overset{\text{conj}}{\sim} \pi_n(\{1,\theta\}\Delta_n) = \pi_n(\{1,\theta\}D(0,n))$

B : not commutative, i.e., $\exists a, b \in B$ s.t. $ab \neq ba$.

$\rightsquigarrow$ $ab = \pm ba$

$\rightsquigarrow$
- $\mathrm{tr}(a) = \mathrm{tr}(b) = 0$
- n, μ : even
- $a, b \sim$ an element of $\{1, \theta, \theta^2, \ldots, \theta^{2\mu-1}\}I_{n'}$, $n' = n/2$.

$$I_{n'} = \begin{bmatrix} -1_{n'} & 0 \\ 0 & 1_{n'} \end{bmatrix} = I_1 \otimes 1_{n'}.$$

B : not commutative $\rightsquigarrow$ $B \overset{\text{conj}}{\sim}$ a subgroup of $D[4] \otimes U(n')$

$A \overset{\text{conj}}{\sim}$ a subgroup of $\pi_n(D[4] \otimes U(n'))$

Furthermore, $\exists A'$: a maximal antipodal subgroup of $U(n')/\mathbb{Z}_\mu$ s.t. $A \overset{\text{conj}}{\sim} \pi_n(D[4] \otimes \pi_{n'}^{-1}(A'))$. Conversely, if C is a maximal antipodal subgroup of $U(n')/\mathbb{Z}_\mu$, $\pi_n(D[4] \otimes \pi_{n'}^{-1}(C))$ is a maximal antipodal subgroup of $U(n)/\mathbb{Z}_\mu$.

Induction on k $\rightsquigarrow$ Theorem 7

$d \in D(s,n) \rightsquigarrow d^2 = \pm 1_n$

$PD(s,n) := \{d \in D(s,n) \mid d^2 = 1_n\}$

$ND(s,n) := \{d \in D(s,n) \mid d^2 = -1_n\}$

Theorem 8

$\pi_n : U(n) \to U(n)/\mathbb{Z}_\mu$: the natural projection

Any maximal antipodal set of $UI(n)/\mathbb{Z}_\mu$ is $U(n)/\mathbb{Z}_\mu$-congruent to one of the following.

1. *If μ is odd,* $\pi_n(\Delta_n)$.
2. *If μ is even,* $\pi_n(\{1, \theta\}PD(s,n))$ $(s \in \{0, \dots, k\})$, *where the case of* $(s,n) = (k-1, 2^k)$ *is excluded.*

Outline of Proof.

e : the identity element of $(U(n) \rtimes \langle \sigma_I \rangle)/\mathbb{Z}_\mu$

A : a maximal antipodal set of $UI(n)/\mathbb{Z}_\mu$

$\rightsquigarrow$ $(A, \sigma_I) \subset (UI(n)/\mathbb{Z}_\mu, \sigma_I)$: a polar of $(U(n) \rtimes \langle \sigma_I \rangle)/\mathbb{Z}_\mu$

$\rightsquigarrow$ (A, σ_I) : an antipodal set of $(U(n) \rtimes \langle \sigma_I \rangle)/\mathbb{Z}_\mu$

$\exists \tilde{A}$: a maximal antipodal subgroup of $(U(n) \rtimes \langle \sigma_I \rangle)/\mathbb{Z}_\mu$

s.t. $\{e\} \cup (A, \sigma_I) \subset \tilde{A}$

By Theorem 5,

(1) In the case μ is odd, A is congruent to $\pi_n(\Delta_n)$.

(2) In the case μ is even, $\exists s \in \{0, \ldots, k\}$, $\exists g \in U(n)$ s.t.

$$\begin{aligned}\tilde{A} &= \pi_n((g, 1)(\{1, \theta\}D(s, n) \rtimes \langle \sigma_I \rangle)(g, 1)^{-1}) \\ &= \pi_n(\{1, \theta\}I_g(D(s, n)), 1) \cup \pi_n(\{1, \theta\}\rho_{\sigma_I}(g)(D(s, n)), \sigma_I)\end{aligned}$$

$$\begin{aligned}(A, \sigma_I) &\subset \tilde{A} \cap (UI(n)/\mathbb{Z}_\mu, \sigma_I) \\ &= (\rho_{\sigma_I}(g)(\pi_n(\{1, \theta\}D(s, n)) \cap \pi_n(UI(n)), \sigma_I)\end{aligned}$$

By the maximality of A, $A = \rho_{\sigma_I}(g)(\pi_n(\{1, \theta\}D(s, n)) \cap \pi_n(UI(n))$.

A is congruent to $\pi_n(\{1,\theta\}D(s,n)) \cap \pi_n(UI(n)) = \pi_n(\{1,\theta\}PD(s,n))$.

Theorem 9

1. *When μ is odd,*

   $\pi_n(\Delta_n)$ *: a unique great antipodal set,* $|\pi_n(\Delta_n)| = 2^n$.

2. *When μ is even,* $|\pi_n(\{1,\theta\}PD(s,n))| = (2^s+1)\cdot 2^{s-1+2^{k-s}\cdot l}$

   (1) *if* $n=2$, $\pi_2(\{1,\theta\}PD(1,2)) = \pi_2(\{1,\theta\}\{1_2, I_1, K_1\})$ *: a unique great antipodal set,* $|\pi_2(\{1,\theta\}PD(1,2))| = 6$.

   (2) *if* $n=4$, $\pi_4(\{1,\theta\}PD(2,4))$ *: a unique great antipodal set,* $|\pi_4(\{1,\theta\}PD(2,4))| = 20$.

   (3) *Otherwise,* $\pi_n(\{1,\theta\}\Delta_n)$ *: a unique great antipodal set,* $|\pi_n(\{1,\theta\}\Delta_n)| = 2^n$.

Corollary 10 ([1])

$\#_2 UI(n)/\mathbb{Z}_\mu$ is as follows. If $n = 2$ and μ is even, $\#_2 UI(2)/\mathbb{Z}_\mu = 6$. If $n = 4$ and μ is even, $\#_2 UI(4)/\mathbb{Z}_\mu = 20$. Otherwise, $\#_2 UI(n)/\mathbb{Z}_\mu = 2^n$.

<u>$UII(n), AI(n), AII(n)$</u>

$$\sigma_{II} : U(2n) \to U(2n),\ \sigma_{II}(x) = J_n \bar{x} J_n^{-1},\ \ J_n = \begin{bmatrix} 0 & -1_n \\ 1_n & 0 \end{bmatrix}$$

$\sigma_{II} \in \mathrm{Aut}(U(2n)),\ \sigma_{II}^2 = 1$

$G := U(2n) \rtimes \langle \sigma_{II} \rangle = (U(2n), 1) \cup (U(2n), \sigma_{II})$

$UII(n) := \{g \in U(2n) \mid \sigma_{II}(g) = g^{-1}\} = \rho_{\sigma_{II}}(U(2n))(1_{2n}) \cong U(2n)/Sp(n)$

$UII(n)$ is a compact Riemannian symmetric space.

$(UII(n), \sigma_{II})$ is a polar of $G = U(2n) \rtimes \langle \sigma_{II} \rangle$.

$AI(n) := UI(n) \cap SU(n) = \rho_{\sigma_I}(SU(n))1_n \cong SU(n)/SO(n)$

$(AI(n), \sigma_I)$ is a polar of $SU(n) \rtimes \langle \sigma_I \rangle$.

$UII(n) \cap SU(2n) = \rho_{\sigma_{II}}(SU(2n))1_{2n} \cup \rho_{\sigma_{II}}(U^-(2n))1_{2n}$

$AII(n) := \rho_{\sigma_{II}}(SU(2n))1_{2n} \cong SU(2n)/Sp(n)$.

$(AII(n), \sigma_{II})$ is a polar of $SU(2n) \rtimes \langle \sigma_{II} \rangle$.

By the similar method we classified maximal antipodal sets of the quotient spaces of $UII(n), AI(n), AII(n)$ and determined the maximums of the cardinalities of maximal antipodal sets. More complicated arguments are needed in those cases.

[1] B.-Y. Chen and T. Nagano, *A Riemannian geometric invariant and its applications to a problem of Borel and Serre*, Trans. Amer. Math. Soc. **308** (1988), 273–297.

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[4] M. S. Tanaka and H. Tasaki, *Polars of disconnected compact Lie groups*, Contemporary Mathematics **777** (2022), 211–225.