Antipodal sets of compact symmetric spaces

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Introduction and preliminaries

M: a Riemannian symmetric space

$A \subset M$: a subset

A: an **antipodal** set $:\Leftrightarrow \forall x, y \in A,\ s_x(y) = y$,

where s_x: the symmetry at x.

An antipodal set is discrete since x is an isolated fixed point of s_x.

E.g. $M = S^n\,(\subset \mathbb{R}^{n+1})$, $\forall x \in S^n, \{x, -x\}$: an antipodal set.

$M = \mathbb{R}P^n$, $\forall x \in \mathbb{R}P^n$, $\forall y \in \mathbb{R}P^n, y \subset x^{\perp}, \{x, y\}$: an antipodal set.

A: a **maximal** antipodal set $:\Leftrightarrow$

$A' \subset M$: an antipodal set, $A \subset A' \Rightarrow\ A = A'$.

$M = S^n, \{x, -x\}$: maximal

$M = \mathbb{R}P^n, \{x, y\}$: not maximal ($n \geq 2$)

Remark. A_1, A_2: antipodal sets $\Rightarrow A_1 \cap A_2$: an antipodal set, $A_1 \cup A_2$: not necessarily an antipodal set.

If M: connected, $\forall x, y \in A$, $\exists \gamma$: a closed geodesic s.t.

$\gamma(0) = \gamma(2) = x, \gamma(1) = y$, i.e., x, y are antipodal points on γ.

If M: connected, $|A| < \infty$.

$\exists \max\{|A| : A \subset M : \text{antpodal}\} =: \#_2 M$: the **2-number** of M

A: a **great** antipodal set $:\Leftrightarrow |A| = \#_2 M$.

Remark. A great antipodal set $\Rightarrow$ a maximal antipodal set. A maximal antipodal set $\not\Rightarrow$ a great antipodal set.

E.g. $\#_2 S^n = 2$, $\#_2 \mathbb{R}P^n = n+1$, $\#_2 \mathbb{R}^n = 1$, $\#_2 \mathbb{R}H^n = 1$.

$u_1, \ldots, u_{n+1}$: an o.n.b. of $\mathbb{R}^{n+1}$, $\{\mathbb{R}u_1, \ldots, \mathbb{R}u_{n+1}\}$: a great antipodal set of $\mathbb{R}P^n$.

G: a compact Lie group

$r_2(G)$: the **2-rank** of G, i.e., the maximal integer t satisfying $\exists G'$: a subgroup of G, $G' \cong (\mathbb{Z}_2)^t$.

G is a Riemannian symmetric space w.r.t. a bi-invariant metric.

$\#_2 G = 2^{r_2(G)}$.

Chen and Nagano (Trans. Amer. Math. Soc. 1988) studied $\#_2 M$ of a compact Riemannian symmetric space M.

They also studied relations between the Euler characteristic $\chi(M)$ and $\#_2 M$, e.g. $\chi(M) \leq \#_2 M$. $\chi(M) = \#_2 M$ if M is a Hermitian symmetric space of semisimple type.

Takeuchi (Nagoya Math. J. 1989) showed: $\#_2 M =$(the sum of $\mathbb{Z}_2$-Betti numbers of M) if M is a symmetric R-space.

Tasaki and T. (J. Math. Soc. Japan 2012) obtained the following result:

M: a Hermitian symmetric space of compact type

L_1, L_2: real forms of M, intersect transversely

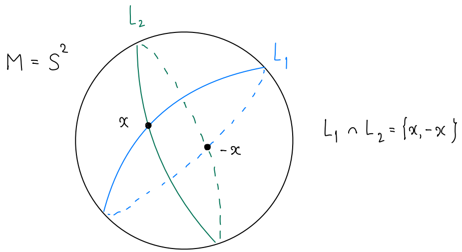
$\Rightarrow L_1 \cap L_2$ is an antipodal set of L_1, L_2. Moreover, if L_1, L_2 are congruent, $L_1 \cap L_2$ is an great antipodal set.

$L_1, L_2 \subset M$ are **congruent** if there exists $f \in Iso(M)_0$ such that $f(A) = B$.

A **real form** of a Hermitian symmetric space M is a connected component of an involutive aniti-holomorphic isometry of M.

A real form of a Herm. sym. sp. of cpt. type $\longleftrightarrow$ A symmetric R-space

E.g. $M = \mathbb{C}P^1 = S^2$, $L_1, L_2 \cong \mathbb{R}P^1 = S^1$ (a great circle)



Antipodal sets are "good" finite subsets of compact Riemannian symmetric spaces. To classify maximal antipodal sets is a fundamental problem for studies of antipodal sets.

Aims: (1) To classify maximal antipodal sets of a compact Riemannian symmetric space up to congruence and give an explicit description of a representative of each congruent class.

(2) To determine the maximum of the cardinalities of maximal antipodal sets as well as to determine antipodal sets whose cardinalities attain the maximum. (Explicit descriptions of maximal antipodal sets make us possible to calculate their cardinalities. This gives an alternative proof of Chen-Nagano's result of the determination of $\#_2 M$.)

Tasaki and T. showed that if M is a symmetric R-space, any antipodal sets of M is included in a great antipodal sets, and furthermore, any two great antipodal sets of M are congruent (Osaka J. Math. 2013).

Under the collaboration with Tasaki we classified maximal antipodal sets of:

・some compact classical Lie groups. ($U(n), SU(n), Sp(n), O(n), SO(n)$, and their quotient groups) (J. Lie Theory 2017).

・some compact classical symmetric spaces.
($G_k(\mathbb{K}^n), \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, Sp(n)/U(n), SO(2n)/U(n)$ and their quotient spaces) (Differ. Geom. Appl. 2020).

・some compact classical symmetric spaces.
($U(n)/O(n), U(2n)/Sp(n), SU(n)/SO(n), SU(2n)/Sp(n)$ and their quotient spaces) (in preparation).

・G_2 and $G_2/SO(4)$ (with Yasukura, Proc. Amer. Math. Soc. 2022)

Other researches:

Tasaki: Maximal antipodal sets of the oriented Grassmann manifolds $\tilde{G}_k(\mathbb{R}^n)$,

Sasaki: Maximal antipodal sets of compact exceptional symmetric spaces F_4, FI, E_6, EI, EII, $EIII$, EIV,

etc.

Maximal antipodal subgroups of compact Lie groups

G: a compact Lie group with a bi-invariant Riemannian metric

$x \in G$, $s_x(y) = xy^{-1}x$ $(y \in G)$.

e: the identity element, $s_e(y) = y^{-1}$ $(y \in G)$.

A: an antipodal set, $e \in A$

$x, y \in A \Rightarrow x^2 = y^2 = e,\ xy = yx$.

If A is maximal, A is a subgroup $\cong (\mathbb{Z}_2)^t$, where $t = r_2(G)$.

We call A a **maximal antipodal subgroup** (MAS).

$\underline{G = O(n), U(n), Sp(n)}$ $\rightsquigarrow$ symmetric R-spaces

$O(n) := \{X \in GL(n, \mathbb{R}) \mid {}^t XX = 1_n\}$

$U(n) := \{X \in GL(n, \mathbb{C}) \mid {}^t \bar{X}X = 1_n\}$

$Sp(n) := \{X \in GL(n, \mathbb{H}) \mid {}^t \bar{X}X = 1_n\}$

$$\Delta_n := \left\{ \begin{bmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{bmatrix} \right\} \subset O(n)$$

Δ_n is a unique MAS of G up to conjugation (simultaneous diagonalization). $\#_2 O(n) = \#_2 U(n) = \#_2 Sp(n) = 2^n$.

$\underline{G = SO(n), SU(n)}$

$\Delta_n^+ := \{d \in \Delta_n \mid \det(d) = 1\}$ is a unique MAS of G up to conjugation. $\#_2 SO(n) = \#_2 SU(n) = 2^{n-1}$.

Quotient groups of $U(n)$

"The center of $U(n)$" $= \{\alpha 1_n \mid \alpha \in \mathbb{C}, |\alpha| = 1\} \supset \mathbb{Z}_\mu = \{\alpha 1_n \mid \alpha^\mu = 1\}$

(μ: a natural number)

$\pi_n : U(n) \to U(n)/\mathbb{Z}_\mu$: the natural projection

$$I_1 := \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad J_1 := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad K_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$D[4] := \{\pm 1_2, \pm I_1, \pm J_1, \pm K_1\}$

$n = 2^k \cdot l$, l: an odd number

$$s \in \{0, \ldots, k\},\ D(s,n) := \underbrace{D[4] \otimes \cdots \otimes D[4]}_{s} \otimes \Delta_{n/2^s} \subset O(n)$$

$D(0,n) = \Delta_n$

$D(s,n) = \{d_1 \otimes \cdots \otimes d_s \otimes d_0 \mid d_1, \ldots, d_s \in D[4], d_0 \in \Delta_{n/2^s}\}$ $(1 \le s \le k)$

$$A = [a_{ij}], \quad A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix}$$

E.g. $I_1 \otimes J_1 = \begin{bmatrix} -J_1 & 0 \\ 0 & J_1 \end{bmatrix}$, $J_1 \otimes I_1 = \begin{bmatrix} 0 & -I_1 \\ I_1 & 0 \end{bmatrix}$

Theorem 1 ([2])

$\pi_n : U(n) \to U(n)/\mathbb{Z}_\mu$: the natural projection

θ: a primitive 2μ-th root of 1

$n = 2^k \cdot l$, l: odd

Any MAS of $U(n)/\mathbb{Z}_\mu$ is conjugate to one of the following:

(1) μ: odd $\Rightarrow$ $\pi_n(\{1, \theta\}\Delta_n)$

(2) μ: even

(2-1) $k = 0$ $\Rightarrow$ $\pi_n(\{1, \theta\}\Delta_n)$

(2-2) $k \geq 1$ $\Rightarrow$ $\pi_n(\{1, \theta\}D(s, n))$ $(0 \leq s \leq k)$, where the case of $s = k - 1, n = 2^k$ is excluded.

Remark. $\Delta_2 \subsetneq D[4]$ implies $D(k-1, 2^k) \subsetneq D(k, 2^k)$.

Corollary 2 ([2])

Great antipodal subgroups of $U(n)/\mathbb{Z}_\mu$ and their cardinalities are as follows:

(1) μ: odd $\Rightarrow$ $\pi_n(\{1,\theta\}\Delta_n)$ is a unique great antipodal subgroup up to conjugation and its cardinality is 2^n, and hence $\#_2 U(n)/\mathbb{Z}_\mu = 2^n$.

(2) μ: even,

(2-1) $k = 0$ $\Rightarrow$ $\pi_n(\{1,\theta\}\Delta_n)$, $\#_2 U(n)/\mathbb{Z}_\mu = 2^n$.

(2-2) $k \geq 1$

$n = 2$ $\Rightarrow$ $\pi_2(\{1,\theta\}D[4])$, $\#_2 U(2)/\mathbb{Z}_\mu = 2^3 = 8$.

$n = 4$ $\Rightarrow$ $\pi_4(\{1,\theta\}D(2,4))$, $\#_2 U(4)/\mathbb{Z}_\mu = 2^5 = 32$.

$n \neq 2,4$ $\Rightarrow$ $\pi_n(\{1,\theta\}\Delta_n)$ $\#_2 U(n)/\mathbb{Z}_\mu = 2^n$.

E.g. $U(4)/\mathbb{Z}_2$ ($n = 4, \mu = 2, k = 2, l = 1, \theta = i$)

A MAS of $U(4)/\mathbb{Z}_2$ is conjugate to one of the following:

$\pi_4(\{1, i\}D(0, 4)) = \pi_4(\{1, i\}\Delta_4)$,

$\pi_4(\{1, i\}D(2, 4)) = \pi_4(\{1, i\}(D[4] \otimes D[4])$.

Remark. $\pi_4(\{1, i\}D(1, 4)) = \pi_4(\{1, i\}D[4] \otimes \Delta_2) \subsetneq \pi_4(\{1, i\}D(2, 4))$.

$$\{1, i\}\Delta_4 = \left\{ \begin{bmatrix} \pm 1 & & & \\ & \pm 1 & & \\ & & \pm 1 & \\ & & & \pm 1 \end{bmatrix}, \begin{bmatrix} \pm i & & & \\ & \pm i & & \\ & & \pm i & \\ & & & \pm i \end{bmatrix} \right\}.$$

$|\pi_4(\{1, i\}\Delta_4)| = (2^4 + 2^4)/2 = 2^4$.

$\{1, i\}D[4] \otimes D[4] =$

$\{\pm\{1_2, I_1, J_1, K_1\} \otimes \{1_2, I_1, J_1, K_1\}, \pm i\{1_2, I_1, J_1, K_1\} \otimes \{1_2, I_1, J_1, K_1\}\}$.

$|\pi_4(\{1, i\}D[4] \otimes D[4])| = (2 \cdot 4^2 + 2 \cdot 4^2)/2 = 2^5$.

$\pi_4(\{1, i\}D[4] \otimes D[4])$ is a unique great antipodal subgroup up to conjugation. $\#_2 U(4)/\mathbb{Z}_2 = 2^5 = 32$.

Remark. In $SU(8)/\mathbb{Z}_\mu$ $(\mu = 2, 4, 8)$, there are two great antipodal subgroups which are not conjugate:

$\pi_8(\{1, \theta\}\Delta_8^+)$, $\pi_8(\{1, \theta\}D(3, 8))$, their cardinalities $= 2^7$.

Sketch of the proof of Theorem 1

A : a maximal antipodal subgroup of $U(n)/\mathbb{Z}_\mu$

$B := \pi_n^{-1}(A)$

B : commutative $\rightsquigarrow$ $A \overset{\text{conj}}{\sim} \pi_n(\{1, \theta\}\Delta_n)$

B : not commutative, i.e., $\exists a, b \in B$ s.t. $ab \neq ba$.

$\rightsquigarrow$ $ab = -ba$

$\rightsquigarrow$ • $\mathrm{tr}(a) = \mathrm{tr}(b) = 0$

• n, μ : even

• $\{a, b\} \overset{\text{conj}}{\sim} \{I_1 \otimes 1_{n'},\ K_1 \otimes 1_{n'}\}$ $(n' = n/2)$

$\rightsquigarrow$ $\langle a, b \rangle \cong D[4] \otimes 1_{n'}$

$\rightsquigarrow$ $B \overset{\text{conj}}{\sim}$ a subgroup of $D[4] \otimes U(n')$

$A = \pi_n(B) \overset{\text{conj}}{\sim}$ a subgroup of $\pi_n(D[4] \otimes U(n'))$

Furthermore, $\exists A'$: a maximal antipodal subgroup of $U(n')/\mathbb{Z}_\mu$ s.t. $A \overset{\text{conj}}{\sim} \pi_n(D[4] \otimes \pi_{n'}^{-1}(A'))$. Conversely, if C is a maximal antipodal subgroup of $U(n')/\mathbb{Z}_\mu$, $\pi_n(D[4] \otimes \pi_{n'}^{-1}(C))$ is a maximal antipodal subgroup of $U(n)/\mathbb{Z}_\mu$.

Induction on k $\rightsquigarrow$ Theorem 1.

In [2] we classified MAS of the quotient groups of $O(n)$, $Sp(n)$, $SO(n)$ in similar ways and determined great antipodal subgroups and their cardinalities. To classify MAS of the quotient groups of $SU(n)$ we used the following: $\forall A$: a MAS of $SU(n)/\mathbb{Z}_\mu$, $\exists \tilde{A}$: a MAS of $U(n)/\mathbb{Z}_\mu$ satisfying $A = \tilde{A} \cap SU(n)/\mathbb{Z}_\mu$, and vise versa.

Maximal antipodal subsets of compact symmetric spaces

M : a compact connected Riemannian symmetric space, not a Lie group

Strategy: To Use the realization of M as a totally geodesic submanifold, called a polar, of a compact Lie group G and to apply classification results of MAS of G.

$o \in M$

$F(s_o, M) := \{x \in M \mid s_o(x) = x\}$

A connected component of $F(s_o, M)$ is called a **polar** of M w.r.t. o.

$\{o\}$: a trivial polar

A polar M^+ $(\dim M^+ > 0)$ is a totally geodesic submanifold.

$\rightsquigarrow \forall x \in M^+,\ s_x(M^+) = M^+$, hence M^+ is a Riemannian symmetric space w.r.t. the induced metric.

E.g. $M = U(n)$

$s_{1_n}(x) = x^{-1}$ $(x \in U(n))$

$F(s_{1_n}, U(n)) = \{x \in U(n) \mid x^2 = 1_n\}$

$$x^2 = 1 \ \rightsquigarrow \ x \overset{\text{conj}}{\sim} I_k, \ \ I_k := \mathrm{diag}(\underbrace{-1. \dots, -1}_{k}, \underbrace{1, \dots, 1}_{n-k})$$

$$F(s_{1_n}, U(n)) = \{\pm 1_n\} \cup \bigcup_{k=1}^{n-1} \{g\, I_k\, g^{-1} \mid g \in U(n)\}$$

$$\{g\, I_k\, g^{-1} \mid g \in U(n)\} \cong U(n)/(U(k) \times U(n-k)) \cong G_k(\mathbb{C}^n)$$

$G_k(\mathbb{C}^n)$ is realized as a polar of $U(n)$.

<u>Polars of a compact Lie group</u>

G: a compact Lie group

e: the identity element of G

G_0: the identity component of G

We simply refer to a polar of G w.r.t. e as a polar of G.

$g \in G$, $\tau_g : G \to G$, $\tau_g(x) := gxg^{-1}$ $(x \in G)$ (an inner automorphism)

Proposition 3 ([3])

Let M be a polar of G. Then $M = \{\tau_g(x_0) \mid g \in G_0\}$ for $x_0 \in M$ and $\mathrm{Iso}(M)_0 = \{\tau_g|_M \mid g \in G_0\}$.

Basic principle:

M: a polar of G

A: a maximal antipodal set of $M \rightsquigarrow A \cup \{e\}$: an antipodal set of G

$(\because A \subset M \subset F(s_e, G))$

$\exists \tilde{A}$: a maximal antipodal subgroup of G satisfying $A \cup \{e\} \subset \tilde{A}$

$\rightsquigarrow A = M \cap \tilde{A}$ $(\because A$: maximal$)$

$[B_0], \dots, [B_k]$: all G_0-conjugacy classes of maximal antipodal subgroups of G, where $B_0, \dots, B_k$ denotes representatives.

$\rightsquigarrow\ \exists g \in G_0,\ \exists s \in \{0, \ldots, k\}$ s.t. $\tilde{A} = \tau_g(B_s)$.

$A = M \cap \tilde{A} = M \cap \tau_g(B_s) = \tau_g(M \cap B_s) \quad (\because\ \tau_g(M) = M)$

i.e., A is $\mathrm{Iso}(M)_0$-congruent to $M \cap B_s$.

$\rightsquigarrow$ Any representative of $\mathrm{Iso}(M)_0$-congruent class of maximal antipodal sets of M is one of $M \cap B_0, \ldots, M \cap B_k$.

Maximal antipodal sets of Grassmann manifolds

$\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$

$G_{\mathbb{K}} := O(n)\ (\mathbb{K} = \mathbb{R}),\ U(n)\ (\mathbb{K} = \mathbb{C}),\ Sp(n)\ (\mathbb{K} = \mathbb{H})$

$G_k(\mathbb{K}^n)$ is regarded as a polar of $G_{\mathbb{K}}$ by the correspondence $x \mapsto \pi_{x^\perp} - \pi_x$.

$\pi_{x^\perp}, \pi_x$: the orthogonal projections on $x^\perp, x$

Theorem 4 ([3])

A maximal antipodal set of $G_k(\mathbb{K}^n)$ is $G_{\mathbb{K}}$-congruent to $\{\langle e_{i_1}, \dots, e_{i_k}\rangle_{\mathbb{K}} \mid 1 \leq i_1 < \cdots < i_k \leq n\}$, where $e_1, \dots, e_n$ is the standard orthonormal basis of $\mathbb{K}^n$. This is a great antipodal set and $\#_2 G_k(\mathbb{K}^n) = \binom{n}{k}$.

The quotient space of $G_m(\mathbb{K}^{2m})$

$\gamma : G_m(\mathbb{K}^{2m}) \to G_m(\mathbb{K}^{2m}),\ \gamma(x) := x^{\perp} \quad (x \in G_m(\mathbb{K}^{2m}))$

$\langle\gamma\rangle$: the subgroup of $\mathrm{Iso}(G_m(\mathbb{K}^{2m}))$ generated by γ

$G_m(\mathbb{K}^{2m})^* := G_m(\mathbb{K}^{2m})/\langle\gamma\rangle$ is a compact Riemannian symmetric space, doubly covered by $G_m(\mathbb{K}^{2m})$.

We regard $G_m(\mathbb{K}^{2m})$ as a polar of $G_{\mathbb{K}}$ under the correspondence

$x \mapsto \pi_{x^\perp} - \pi_x$.

Since $\gamma(x) \mapsto \pi_{\gamma(x)^\perp} - \pi_{\gamma(x)} = -(\pi_{x^\perp} - \pi_x)$, γ is regarded as $-\mathrm{id}$.

$\rightsquigarrow\;\; G_m(\mathbb{K}^{2m})^* \subset G_{\mathbb{K}}^* := G_{\mathbb{K}}/\{\pm 1_{2m}\}$.

$\pi : G_{\mathbb{K}} \to G_{\mathbb{K}}^*$: the natural projection

$\pi \circ s_x = s_{\pi(x)} \circ \pi \quad (x \in G_{\mathbb{K}})$

$x \in G_m(\mathbb{K}^{2m}),\;\; s_{\pi(e)}(\pi(x)) = \pi(s_e(x)) = \pi(x)$

$\rightsquigarrow\;\; G_m(\mathbb{K}^{2m})^*$ is a polar of $G_{\mathbb{K}}^*$

By using the classification of maximal antipodal subgroups of $G_{\mathbb{K}}^*$ we obtained the classification of maximal antipodal sets of $G_m(\mathbb{K}^{2m})^*$.

Remark. A compact connected Riemannina symmetric space is not necessarily realized as a polar of a <u>connected</u> compact Lie group. For example, $U(n)/O(n), U(2n)/Sp(n)$, so-called outer symmetric spaces, are realized as polars of <u>disconnected</u> compact Lie groups. They are not realized as polars of connected compact Lie groups, since if so, they should be inner symmetric spaces.

Maximal antipodal sets of $UI(n) = U(n)/O(n)$

$\sigma_I : U(n) \to U(n),\ \sigma_I(x) := \bar{x}$ (the complex conjugation)

$\sigma_I \in \mathrm{Aut}(U(n)),\ \sigma_I^2 = 1$

$\langle \sigma_I \rangle = \{1, \sigma_I\}$: the subgroup of $\mathrm{Aut}(U(n))$ generated by σ_I

$G := U(n) \rtimes \langle \sigma_I \rangle$: the semidirect product

$G = (U(n), 1) \cup (U(n), \sigma_I)$: the direct sum of the connected components

G is a disconnected compact Lie group.

e: the identity element of G

$F(s_e, G)$

$= \{(g, 1) \in (U(n), 1) \mid s_e(g, 1) = (g, 1)\} \cup \{(g, \sigma_I) \in (U(n), \sigma_I) \mid$

$s_e(g, \sigma_I) = (g, \sigma_I)\}$

$= \{(g,1) \in (U(n),1) \mid (g,1)^{-1} = (g,1)\} \cup \{(g,\sigma_I) \in (U(n),\sigma_I) \mid$
$(g,\sigma_I)^{-1} = (g,\sigma_I)\}$
$= \{(g,1) \in (U(n),1) \mid (g^{-1},1) = (g,1)\} \cup \{(g,\sigma_I) \in (U(n),\sigma_I) \mid$
$(\sigma_I(g)^{-1},\sigma_I) = (g,\sigma_I)\}$

Hence $F(s_e, G) = (F(s_{1_n}, U(n)), 1) \cup (\{g \in U(n) \mid \sigma_I(g) = g^{-1}\}, \sigma_I)$.

$UI(n) := \{g \in U(n) \mid \sigma_I(g) = g^{-1}\} = \rho_{\sigma_I}(U(n))(1_n) \cong U(n)/O(n)$,

where $\rho_{\sigma_I}(g)(x) := g\, x\, \sigma_I(g)^{-1} \quad (g, x \in U(n))$.

$UI(n)$ is a compact connected Riemannian symmetric space.

$(UI(n), \sigma_I)$ is a polar of $G = U(n) \rtimes \langle \sigma_I \rangle$.

$UI(n)$ is realized as a polar of the disconnected Lie group $U(n) \rtimes \langle \sigma_I \rangle$.

In order to classify maximal antipodal sets of $UI(n)$, we classify maximal antipodal subgroups of $U(n) \rtimes \langle \sigma_I \rangle$ first.

Theorem 5

Any maximal antipodal subgroup of $U(n) \rtimes \langle \sigma_I \rangle$ is conjugate to $\Delta_n \rtimes \langle \sigma_I \rangle$ by an element of $(U(n), 1)$.

By this, we obtain the following:

Theorem 6

Any maximal antipodal set of $UI(n)$ is congruent to Δ_n.

Corollary 7

Δ_n is a unique great antipodal set of $UI(n)$ up to $U(n)$-congruence. We have $\#_2 UI(n) = 2^n$.

The quotient spaces of $UI(n)$

$\mathbb{Z}_\mu := \{z1_n \mid z^\mu = 1\}$, μ : a natural number

$\mathbb{Z}_\mu \subset$ the center of $U(n)$

$\mathbb{Z}_\mu UI(n) \subset UI(n)$, where $UI(n) = \{g \in U(n) \mid \sigma_I(g) = g^{-1}\}$.

$\rightsquigarrow$ the quotient space $UI(n)/\mathbb{Z}_\mu$ is defined.

$\sigma_I(\mathbb{Z}_\mu) = \mathbb{Z}_\mu$

$\rightsquigarrow$ σ_I induces an involutive autom. of $U(n)/\mathbb{Z}_\mu$, also denoted by σ_I.

$UI(n)/\mathbb{Z}_\mu \subset M := \{x \in U(n)/\mathbb{Z}_\mu \mid \sigma_I(x) = x^{-1}\}$

Note. M is not necessarily connected and $UI(n)/\mathbb{Z}_\mu$ is the connected component containing the identity element.

$(UI(n)/\mathbb{Z}_\mu, \sigma_I)$ is a polar of $U(n)/\mathbb{Z}_\mu \rtimes \langle \sigma_I \rangle$.

$(\mathbb{Z}_\mu, 1)$ is a normal subgroup of $U(n) \rtimes \langle \sigma_I \rangle$.

We denote the quotient group $(U(n) \rtimes \langle \sigma_I \rangle)/(\mathbb{Z}_\mu, 1)$ by $(U(n) \rtimes \langle \sigma_I \rangle)/\mathbb{Z}_\mu$.

By the equality

$$(U(n) \rtimes \langle \sigma_I \rangle)/\mathbb{Z}_\mu \ \ni \ (g, \alpha)(\mathbb{Z}_\mu, 1) = (g\mathbb{Z}_\mu, \alpha) \ \in \ U(n)/\mathbb{Z}_\mu \rtimes \langle \sigma_I \rangle,$$

we identify $U(n)/\mathbb{Z}_\mu \rtimes \langle \sigma_I \rangle$ with $(U(n) \rtimes \langle \sigma_I \rangle)/\mathbb{Z}_\mu$.

Theorem 8

$\pi_n : U(n) \rtimes \langle \sigma_I \rangle \to (U(n) \rtimes \langle \sigma_I \rangle)/\mathbb{Z}_\mu$: *the natural projection*

θ: *a primitive* 2μ*-th root of* 1

Any maximal antipodal subgroup of $(U(n) \rtimes \langle \sigma_I \rangle)/\mathbb{Z}_\mu$ *is*

$\pi_n((U(n), 1))$*-conjugate to one of the following.*

(1) If μ *is odd,* $\quad \pi_n(\Delta_n \rtimes \langle \sigma_I \rangle)$.

(2) If μ *is even,* $\quad \pi_n(\{1, \theta\} D(s, n) \rtimes \langle \sigma_I \rangle)$ $(s \in \{0, \dots, k\})$,

where the case of $(s, n) = (k-1, 2^k)$ *is excluded.*

$d \in D(s,n) \rightsquigarrow d^2 = \pm 1_n$

$PD(s,n) := \{d \in D(s,n) \mid d^2 = 1_n\}$

Theorem 10

$\pi_n : U(n) \to U(n)/\mathbb{Z}_\mu$: *the natural projection*

Any maximal antipodal set of $UI(n)/\mathbb{Z}_\mu$ *is* $U(n)/\mathbb{Z}_\mu$*-congruent to one of the following.*

(1) If μ *is odd,* $\pi_n(\Delta_n)$.

(2) If μ *is even,* $\pi_n(\{1,\theta\}PD(s,n))$ $(s \in \{0,\dots,k\})$,

where the case of $(s,n) = (k-1, 2^k)$ *is excluded.*

$|\pi_n(\Delta_n)| = 2^n$

$|\pi_n(\{1, \theta\}PD(s, n))| = (2^s + 1) \cdot 2^{s-1+2^{k-s} \cdot l}$

Theorem 11

Great antipodal sets of $UI(n)/\mathbb{Z}_\mu$ and their cardinalities are as follows:

(1) μ: odd $\Rightarrow \pi_n(\Delta_n)$*: a unique great antipodal set,* $|\pi_n(\Delta_n)| = 2^n$.

(2) μ: even

(2-1) $n = 2 \ \Rightarrow \ \pi_2(\{1, \theta\}PD(1, 2)) = \pi_2(\{1, \theta\}\{1_2, I_1, K_1\})$*: a unique great antipodal set,* $|\pi_2(\{1, \theta\}PD(1, 2))| = 6$,

(2-2) $n = 4 \ \Rightarrow \ \pi_4(\{1, \theta\}PD(2, 4))$*: a unique great antipodal set,* $|\pi_4(\{1, \theta\}PD(2, 4))| = 20$,

(2-3) $n \neq 2, 4 \ \Rightarrow \ \pi_n(\{1, \theta\}\Delta_n)$*: a unique great antipodal set,* $|\pi_n(\{1, \theta\}\Delta_n)| = 2^n$.

Corollary 12 (cf. [1])

$\#_2 UI(n)/\mathbb{Z}_\mu$ is as follows.

(1) μ:odd $\Rightarrow$ $\#_2 UI(n)/\mathbb{Z}_\mu = 2^n$.

(2) μ: even $\Rightarrow$ $\#_2 UI(2)/\mathbb{Z}_\mu = 6$, $\#_2 UI(4)/\mathbb{Z}_\mu = 20$,

$\#_2 UI(n)/\mathbb{Z}_\mu = 2^n$ $(n \neq 2, 4)$.

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