Antipodal sets of compact symmetric spaces

Makiko Sumi Tanaka

Tokyo University of Science

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Antipodal sets - Introduction and overview

M : a Riemannian symmetric space

$\forall x \in M$, $\exists s_x$: an involutive isometry s.t. x is an isolated fixed point of s_x.

$A \subset M$: a subset

A : an **antipodal set** $:\Leftrightarrow \forall x, y \in A,\ s_x(y) = y$

An antipodal set is a discrete subset.

E.g. $M = S^n\,(\subset \mathbb{R}^{n+1})$ $\quad \forall x \in S^n,\ \{x, -x\}$: an antipodal set

$M = \mathbb{R}P^n$ $\quad \forall x, y \in \mathbb{R}P^n$ with $y \subset x^{\perp},\ \{x, y\}$: an antipodal set

A : a **maximal** antipodal set in M $:\Leftrightarrow$

$A' \subset M$: an antipodal set, $A \subset A' \Rightarrow A = A'$

In the examples above, $\{x, -x\}$ is a maximal antipodal set of S^n. $\{x, y\}$ is not a maximal antipodal set of $\mathbb{R}P^n$ if $n \geq 2$.

If M is connected, for $\forall x, y$ in an antipodal set A, $\exists \gamma$: a closed geodesic s.t. x, y are antipodal points on γ.

If M is a Riem. sym. sp. of noncompact type or the Euclidean space, any antipodal set consists of one point.

Assume M is compact and connected.

$A \subset M$: an antipodal set $\Rightarrow |A| < \infty$

$\exists \max\{|A| : A \subset M : \text{antpodal}\} =: \#_2 M$: the **2-number** of M

A : a **great** antipodal set $:\Leftrightarrow |A| = \#_2 M$

Remark. Great antipodal set $\Rightarrow$ Maximal antipodal set.

Maximal antipodal set $\not\Rightarrow$ Great antipodal set.

E.g. $\#_2 S^n = 2$, $\#_2 \mathbb{R}P^n = n+1$, $\#_2 \mathbb{R}^n = 1$, $\#_2 \mathbb{R}H^n = 1$.

$u_1, \dots, u_{n+1}$: an o.n.b. of $\mathbb{R}^{n+1}$, $\{\mathbb{R}u_1, \dots, \mathbb{R}u_{n+1}\}$: a great antipodal set of $\mathbb{R}P^n$.

G: a compact Lie group (with a bi-invariant Riemannian metric)

$r_2(G)$: the **2-rank** of G, i.e., the maximal integer t satisfying $\exists G'$: a subgroup of G with $G' \cong (\mathbb{Z}_2)^t$

$\#_2 G = 2^{r_2(G)}$

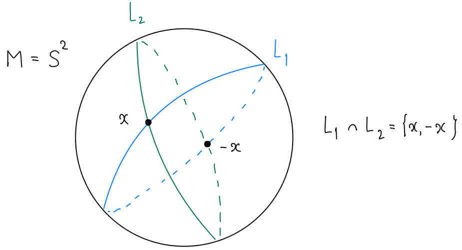
These notions were introduced by Chen and Nagano (Trans. Amer. Math. Soc. 1988). They studied $\#_2 M$ of a compact Riemannian symmetric space M. They also studied relations between the Euler characteristic $\chi(M)$ and $\#_2 M$. E.g. $\chi(M) \leq \#_2 M$. If M is a Hermitian symmetric space of compact type, $\chi(M) = \#_2 M$.

Takeuchi showed that $\#_2 M =$ (the sum of $\mathbb{Z}_2$-Betti numbers of M) if M is a symmetric R-space (Nagoya Math. J. 1989).

Tasaki and T. obtained the following result (J. Math. Soc. Japan 2012):
M : a Hermitian symmetric space of compact type
L_1, L_2 : real forms of M, intersect transversely
$\Rightarrow L_1 \cap L_2$ is an antipodal set of L_1, L_2. Moreover, if L_1, L_2 are congruent, $L_1 \cap L_2$ is a great antipodal set.

Here, $L_1, L_2 \subset M$ are congruent if $\exists f \in I(M)_0$ such that $f(L_1) = L_2$.
A **real form** of a Hermitian symmetric space M is a connected component of an involutive aniti-holomorphic isometry of M. A real form of a Herm. sym. sp. of cpt. type is a symmetric R-space, and vise versa.

E.g. $M = \mathbb{C}P^1 = S^2$, $\quad L_1, L_2 \cong \mathbb{R}P^1 = S^1$ (a great circle)



Tasaki and T. showed that if M is a symmetric R-space, any antipodal sets of M is included in a great antipodal sets, and furthermore, any two great antipodal sets of M are congruent (Osaka J. Math. 2013).

Tasaki and T. classified maximal antipodal sets of:

・some compact classical Lie groups ($U(n), SU(n), Sp(n), O(n), SO(n)$, and their quotient groups) (J. Lie Theory 2017).

・some compact classical symmetric spaces ($G_k(\mathbb{K}^n), \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, Sp(n)/U(n), SO(2n)/U(n)$ and their quotient spaces) (Differ. Geom. Appl. 2020).

・some compact classical outer symmetric spaces ($U(n)/O(n), U(2n)/Sp(n), SU(n)/SO(n), SU(2n)/Sp(n)$ and their quotient spaces) (in preparation).

・G_2 and $G_2/SO(4)$ (with Yasukura, Proc. Amer. Math. Soc. 2022).

Maximal antipodal subgroups of compact Lie groups

G : a compact Lie group with a bi-invariant Riemannian metric

$x \in G$, $s_x(y) = xy^{-1}x$ $(y \in G)$

e : the identity element of G, $s_e(y) = y^{-1}$ $(y \in G)$

A : an antipodal set of G, $e \in A$

$x, y \in A \Rightarrow x^2 = y^2 = e, \ xy = yx$

If A is maximal, A is a subgroup $\cong (\mathbb{Z}_2)^t$, where $t = r_2(G)$

We call A a **maximal antipodal subgroup**.

$\underline{G = O(n), U(n), Sp(n)}$

$O(n) := \{x \in GL(n, \mathbb{R}) \mid {}^t xx = 1_n\}$

$U(n) := \{x \in GL(n, \mathbb{C}) \mid {}^t \bar{x}x = 1_n\}$

$Sp(n) := \{x \in GL(n, \mathbb{H}) \mid {}^t \bar{x}x = 1_n\}$

$$\Delta_n := \left\{ \begin{bmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{bmatrix} \right\} \subset O(n)$$

Δ_n is a maximal antipodal subgroup of G. Any maximal antipodal group of G is conjugate to Δ_n. $\#_2 O(n) = \#_2 U(n) = \#_2 Sp(n) = 2^n$.

<u>$G = SO(n), SU(n)$</u>

$SO(n) := \{x \in O(n) \mid \det(x) = 1\}$

$SU(n) := \{x \in U(n) \mid \det(x) = 1\}$

$\Delta_n^+ := \{d \in \Delta_n \mid \det(d) = 1\}$ is a maximal antipodal subgroup of G. Any maximal antipodal group of G is conjugate to Δ_n^+.

$\#_2 SO(n) = \#_2 SU(n) = 2^{n-1}$.

Covering homomorphisms with odd degree

We show that all of the maximal antipodal subgroups in compact Lie groups do not change through covering homomorphisms with odd degree.

Tanaka and Tasaki, *Maximal antipodal subgroups and covering homomorphisms with odd degree*, Int. Electron. J. Geom. **17**, No.1 (2024), 153–156. (This issue was dedicated to the anniversary of Bang-Yen Chen's 80th birthday.)

Preparation from Group theory

G: a group

e: the unit element of G

$X, Y \subset G, \;\; XY = \{xy \mid x \in X, y \in Y\}$

Lemma 1

Let H, K be subgroups of a group G. The following conditions are equivalent.

(1) *For each $x \in HK$, there is the unique $(h, k) \in H \times K$ such that $x = hk$.*

(2) *If $hk = e$ for $h \in H, k \in K$, then $h = k = e$.*

(3) *$H \cap K = \{e\}$.*

Theorem 2 (Lagrange)

If H is a subgroup of a finite group G, then $|H|$ divides $|G|$.

Corollary 3

Let G be a finite group. For each $g \in G$, $\min\{n \in \mathbb{N} \mid g^n = e\}$ divides $|G|$.

Theorem 4 (Sylow)

Let G be a finite group with $|G| = p^n m$, where p is a prime, and p, m are mutually prime.

(1) *There is a subgroup H of G with $|H| = p^n$, called a p-Sylow subgroup.*

(2) *Any two p-Sylow subgroups are conjugate.*

(3) *If K is a subgroup of G with $|K| = p^k$, there is a p-Sylow subgroup H which satisfies $K \subset H$.*

Lemma 5

Let G, G' be compact Lie groups and let $\pi : G \to G'$ be a covering homomorphism whose covering degree is odd. If A' is an antipodal subgroup of G', then there exists an antipodal subgroup B of G which satisfies the following conditions.

(1) *B is a 2-Sylow subgroup of $\pi^{-1}(A')$ such that $|B| = |A'|$.*

(2) *The restriction of π to B is an isomorphism from B onto A'.*

Proof. Since $A' \cong (\mathbb{Z}_2)^r$ for some r, $|A'| = 2^r$. Set $|\ker(\pi)| = k$, where k is odd. Since $\pi^{-1}(A')$ is a subgroup of G and $|\pi^{-1}(A')| = |A'||\ker(\pi)|$ $= 2^r k$, there is a 2-Sylow subgroup B by Thm.4, where $|B| = 2^r = |A'|$. Since $|\ker(\pi)|$ is odd, $B \cap \ker(\pi) = \{e\}$. Hence π is injective on B.

AS: antipodal subgroup(s), MAS: maximal antipodal subgroup(s)

Theorem 6

G, G' : cpt. Lie gr., $\pi : G \to G'$: a covering homo. with odd degree

G_0, G'_0 : the identity comp. of G, G'

(1) *A : AS (resp. MAS) of G $\Rightarrow$ $\pi(A)$: AS (resp. MAS) of G'.*

MAS $A_1, A_2 \subset G$ are G-conjugate (resp. G_0-conjugate) $\Rightarrow$ MAS $\pi(A_1), \pi(A_2) \subset G'$ are G'-conjugate (resp. G'_0-conjugate).

(2) *A' : AS (resp. MAS) of G' $\Rightarrow$ $\exists A$: AS (resp. MAS) of G s.t. $\pi|_A : A \to A'$is an isom.*

MAS $A'_1, A'_2 \subset G'$: G'-conjugate $\Rightarrow$ MAS $A_1, A_2 \subset G$: G-conjugate, where $\pi|_{A_i} : A_i \to A'_i$: isom. $(i = 1, 2)$.

Furthermore, if G_0 contains $\ker \pi$, we can replace G'-conjugate (resp. G-conjugate) to G'_0-conjugate (resp. G_0-conjugate) in the above.

Proof. (1) It is easy to see that $\pi(A)$ is an AS of G' if A is an AS of G, since π is a homomorphism. Assume A is a MAS of G. Set $Z' = \ker(\pi)$. In order to show $\pi(A)$ is a MAS of G', let A' be an AS of G' with $\pi(A) \subset A'$. By Lem.5, $\exists$ a 2-Sylow subgroup B of $\tilde{A} := \pi^{-1}(A')$ such that $\pi|_B : B \to A'$ is an isomorphism. Note that B is an AS of G. Since $B, Z' \subset \tilde{A}$, we have $BZ' \subset \tilde{A}$. Since $B \cap Z' = \{e\}$, $\forall x \in BZ'$ is uniquely described as $x = bz$ for $b \in B, z \in Z'$. Hence $|BZ'| = |B||Z'| = |A'||Z'|$ $= |\tilde{A}|$, thus $BZ' = \tilde{A}$. Since A is a subgroup of $\tilde{A}$, $\exists g \in \tilde{A}$ such that $gAg^{-1} \subset B$ by Thm.4. Since A is a MAS of G, $gAg^{-1} = B$ holds. Thus $|\pi(A)| = |\pi(gAg^{-1})| = |\pi(B)| = |A'|$, which implies $\pi(A) = A'$. Therefore, $\pi(A)$ is a MAS of G'.

If MAS $A_1, A_2 \subset G$ are conjugate by $g \in G$, i.e., $A_2 = gA_1g^{-1}$, then $\pi(A_2) = \pi(g)\pi(A_1)\pi(g)^{-1}$, hence $\pi(A_1)$ and $\pi(A_2)$ are G'-conjugate. If $g \in G_0$, $\pi(A_1)$ and $\pi(A_2)$ are G_0'-conjugate since $\pi(g) \in \pi(G_0) = G_0'$.

(2) If A' is an AS of G', by Lem.5 there is a 2-Sylow subgroup A of $\pi^{-1}(A')$ such that $\pi|_A : A \to A'$ is an isomorphism, where A is an AS of G. Assume A' is a MAS of G'. We show this A is a MAS of G. In order that, let C be an AS of G with $A \subset C$. Since $\ker(\pi) \cap C = \{e\}$, $\pi|_C : C \to \pi(C)$ is injective, thus it is an isomorphism. Hence $\pi(C)$ is an AS of G'. Since $A' = \pi(A) \subset \pi(C)$, we obtain $A' = \pi(A) = \pi(C)$ by the maximality of A'. Since π is injective on C, we obtain $A = C$. Therefore A is a MAS of G.

If MAS $A_1', A_2' \subset G'$ are G'-conjugate, $\exists g' \in G'$ s.t. $A_2' = g'A_1'(g')^{-1}$. Then $\pi^{-1}(A_2') = \pi^{-1}(g'A_1'(g')^{-1})$.

Furthermore, $\pi^{-1}(g'A'_1(g')^{-1}) = g\pi^{-1}(A'_1)g^{-1}$ holds for $g \in \pi^{-1}(g')$. By the argument above, there exist MAS $A_1, A_2 \subset G$ such that $\pi|_{A_i} : A_i \to A'_i$ is an isomorphism ($i = 1, 2$). Note that A_i is a 2-Sylow subgroup of $\pi^{-1}(A'_i)$ ($i = 1, 2$). Since gA_1g^{-1} is a 2-Sylow subgroup of $g\pi^{-1}(A'_1)g^{-1} = \pi^{-1}(A'_2)$, gA_1g^{-1} is conjugate to A_2 by an element of $\pi^{-1}(A'_2)$ by Thm.4. In particular, A_1 is conjugate to A_2 by an element of G.

Now we assume that G_0 contains $\ker(\pi)$. We can take g mentioned above as $g \in G_0$ if $g' \in G'_0$. As mentioned before, gA_1g^{-1} is conjugate to A_2 by an element of $\pi^{-1}(A'_2)$. Hence $\exists x \in \pi^{-1}(A'_2)$ such that $xgA_1g^{-1}x^{-1} = A_2$. Note that $\pi^{-1}(A'_2) = A_2\ker(\pi)$. Since $A_2 \cap \ker(\pi) = \{e\}$, $x \in \pi^{-1}(A'_2)$ is uniquely described as $x = az$ for $a \in A_2, z \in \ker(\pi)$. Then $azgA_1g^{-1}(az)^{-1} = A_2$.

Thus we have $zgA_1g^{-1}z^{-1} = a^{-1}A_2a = A_2$. Hence, A_1 and A_2 are G_0-conjugate since $zg \in G_0$.

Remark. Thm.6 is a refinement of the following result by Chen-Nagano in the case of compact Lie groups.

Proposition 7 (Chen-Nagano 1988)

One has $\#_2 M' = \#_2 M$, if there exists a k-fold covering morphism $f : M' \to M$ between compact Riemannian symmetric spaces and k is odd.

Maximal antipodal subgroups of $U(n)/\mathbb{Z}_\mu$

$$\Delta_n := \left\{ \begin{bmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{bmatrix} \right\} \subset O(n) \subset U(n)$$

Proposition 8

Every maximal antipodal subgroup of $U(n)$ is conjugate to Δ_n. Δ_n is a great antipodal subgroup. $\#_2 U(n) = 2^n$.

Proof. Δ_n is a MAS of $U(n)$. Since a MAS A of $U(n)$ is abelian, A is simultaneously diagonalizable. Since $\forall a \in A, a^2 = 1_n$, $\exists g \in U(n)$ s.t. $gAg^{-1} \subset \Delta_n$. By the maximality of A, $gAg^{-1} = \Delta_n$.

$\#_2 U(n) = |\Delta_n| = 2^n$.

μ: a natural number

$\mathbb{Z}_\mu := \{\alpha 1_n \mid \alpha^\mu = 1\}$

$\mathbb{Z}_\mu \subset \{\alpha 1_n \mid \alpha \in \mathbb{C}, |\alpha| = 1\}$: the center of $U(n)$

$U(n)/\mathbb{Z}_\mu$ is a compact Lie group locally isomorphic to $U(n)$.

$\pi_n : U(n) \to U(n)/\mathbb{Z}_\mu$: the natural projection

By Thm.6 we obtain the following.

Theorem 9

When μ is odd, every maximal antipodal subgroup of $U(n)/\mathbb{Z}_\mu$ is conjugate to $\pi_n(\Delta_n)$. $\pi_n(\Delta_n)$ is a great antipodal subgroup of $U(n)/\mathbb{Z}_\mu$. $\#_2 U(n)/\mathbb{Z}_\mu = 2^n$.

To state the result when μ is even we prepare some notation.

$$I_1 := \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad J_1 := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad K_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$D[4] := \{\pm 1_2, \pm I_1, \pm J_1, \pm K_1\}$

$n = 2^k \cdot m, \ \ m : \text{ odd}$

$s \in \{0, \ldots, k\}$

$$D(s, n) := \underbrace{D[4] \otimes \cdots \otimes D[4]}_{s} \otimes \Delta_{n/2^s} \subset O(n)$$

i.e., $D(0, n) = \Delta_n$,

$D(s, n) = \{d_1 \otimes \cdots \otimes d_s \otimes d_0 \mid d_1, \ldots, d_s \in D[4], d_0 \in \Delta_{n/2^s}\} \ (1 \leq s \leq k)$

$$A = [a_{ij}], \quad A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix}$$

e.g. $I_1 \otimes J_1 = \begin{bmatrix} -J_1 & 0 \\ 0 & J_1 \end{bmatrix}, \quad J_1 \otimes I_1 = \begin{bmatrix} 0 & -I_1 \\ I_1 & 0 \end{bmatrix}$

θ: a primitive 2μ-th root of 1

Theorem 10

When μ is even, any maximal antipodal subgroup of $U(n)/\mathbb{Z}_\mu$ is conjugate to one of the following:

(1) *If n is odd, $\pi_n(\{1, \theta\}\Delta_n)$.*

(2) *If n is even, $\pi_n(\{1, \theta\}D(s, n))$ $(0 \leq s \leq k)$, where the case of $s = k - 1, n = 2^k$ is excluded.*

Since

$$\Delta_2 = \{\pm 1_2, \pm I_1\} \subsetneq D[4] = \{\pm 1_2, \pm I_1, \pm J_1, \pm K_1\},$$

induces

$$D(k-1, 2^k) = \underbrace{D[4] \otimes \cdots \otimes D[4]}_{k-1} \otimes \Delta_2$$

$$\subsetneq D(k, 2^k) = \underbrace{D[4] \otimes \cdots \otimes D[4]}_{k},$$

$D(k-1, 2^k)$ is not maximal.

Corollary 11

When μ is even, great antipodal subgroups of $U(n)/\mathbb{Z}_\mu$ and their cardinalities are as follows:

(1) *If n is odd,* $\pi_n(\{1,\theta\}\Delta_n)$, $\#_2 U(n)/\mathbb{Z}_\mu = 2^n$.

(2) *If n is even,*

$n = 2 \ \Rightarrow \ \pi_2(\{1,\theta\}D[4])$, $\#_2 U(2)/\mathbb{Z}_\mu = 2^3 = 8$.

$n = 4 \ \Rightarrow \ \pi_4(\{1,\theta\}D(2,4))$, $\#_2 U(4)/\mathbb{Z}_\mu = 2^5 = 32$.

$n \neq 2,4 \ \Rightarrow \ \pi_n(\{1,\theta\}\Delta_n)$ $\#_2 U(n)/\mathbb{Z}_\mu = 2^n$.

Remark. By Thm.9 and Cor.11, a great antipodal subgroup of $U(n)/\mathbb{Z}_\mu$ is unique up to conjugation. Generally, a great antipodal subgroup is not necessarily unique up to conjugation. For example, in $SU(8)/\mathbb{Z}_\mu$ with $\mu = 2, 4, 8$, there are two great antipodal subgroups which are not conjugate.

Sketch of Proof

(In the following proof (J. Lie Theory 2017), we proved Thm.9 and Thm.10 together without using Thm.6.)

A : a maximal antipodal subgroup of $U(n)/\mathbb{Z}_\mu$

$B := \pi_n^{-1}(A)$

B : commutative $\rightsquigarrow$ $A \overset{\text{conj}}{\sim} \pi_n(\{1, \theta\}\Delta_n)$

B : not commutative, i.e., $\exists a, b \in B$ s.t. $ab \neq ba$.

$\rightsquigarrow$ $ab = -ba$

$\rightsquigarrow$ • $\mathrm{tr}(a) = \mathrm{tr}(b) = 0$

- n, μ : even
- $\{a, b\} \overset{\mathrm{conj}}{\sim} \{I_1 \otimes 1_{n'},\ K_1 \otimes 1_{n'}\}$ $(n' = n/2)$

$\rightsquigarrow$ $\langle a, b\rangle \cong D[4] \otimes 1_{n'}$

$\rightsquigarrow$ $B \overset{\mathrm{conj}}{\sim}$ a subgroup of $D[4] \otimes U(n')$

$A = \pi_n(B) \overset{\mathrm{conj}}{\sim}$ a subgroup of $\pi_n(D[4] \otimes U(n'))$

Furthermore, $\exists A'$: a maximal antipodal subgroup of $U(n')/\mathbb{Z}_\mu$ s.t. $A \overset{\mathrm{conj}}{\sim} \pi_n(D[4] \otimes \pi_{n'}^{-1}(A'))$. Conversely, if C is a maximal antipodal subgroup of $U(n')/\mathbb{Z}_\mu$, $\pi_n(D[4] \otimes \pi_{n'}^{-1}(C))$ is a maximal antipodal subgroup of $U(n)/\mathbb{Z}_\mu$.

By induction on k, we get the conclusion.

Thank you for your kind attention.