Maximal antipodal sets of classical compact symmetric spaces

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1. Introduction

M : a compact (Riemannian) symmetric space

s_x : the geodesic symmetry at $x \in M$

i.e., (i) s_x is an isometry of M, (ii) $s_x \circ s_x =$ id, (iii) x is an isolated fixed point of s_x.

$A \subset M$: a subset

A : an **antipodal set** $\overset{\text{def}}{\Longleftrightarrow} \forall x, y \in A,\ s_x(y) = y$

An antipodal set is finite.

The **2-number** $\#_2 M$ of M is defined by

$\#_2 M = \max\{|A| \mid A \subset M :$ an antipodal set$\}$.

A : a **great** antipodal set $\overset{\text{def}}{\Longleftrightarrow} |A| = \#_2 M$

(Chen-Nagano 1982, 1988)

Example 1. $M = S^n \, (\subset \mathbb{R}^{n+1})$ **: the sphere**

$\pi_{\langle x \rangle_{\mathbb{R}}}, \pi_{{\langle x \rangle_{\mathbb{R}}}^{\perp}}$ **: the orthogonal projection onto**

$\langle x \rangle_{\mathbb{R}}, {\langle x \rangle_{\mathbb{R}}}^{\perp} \quad (x \in S^n)$

$s_x(y) = (\pi_{\langle x \rangle_{\mathbb{R}}} - \pi_{{\langle x \rangle_{\mathbb{R}}}^{\perp}})(y) \quad (y \in S^n)$

$s_x(y) = y \Leftrightarrow y = \pm x$

$\{x, -x\}$ **: a great antipodal set**

$\#_2 S^n = 2$

Example 2. $M = \mathbb{R}P^n$ **: the real proj. space**

$x \in \mathbb{R}P^n$, $\quad s_x(y) = (\pi_x - \pi_{x^{\perp}})(y) \quad (y \in \mathbb{R}P^n)$

$s_x(y) = y \Leftrightarrow y = x$ **or** $y \subset x^{\perp}$

$\{e_1, \ldots, e_{n+1}\}$ **: an o.n.b. of** $\mathbb{R}^{n+1}$

$\{\langle e_1 \rangle_{\mathbb{R}}, \ldots, \langle e_{n+1} \rangle_{\mathbb{R}}\}$ **: a great antipodal set**

$\#_2 \mathbb{R}P^n = n + 1$

Fact 1. $N \subset M$ **: totally geodesic**

$\forall x \in N$**,** $s_x(N) = N$

N **is a symmetric space w.r.t. the induced metric.**

If $A \subset N$ **is an antipodal set,** A **is an antipodal set of** M**.**

Hence $\#_2 N \leq \#_2 M$**.**

Fact 2. **(Chen-Nagano 1988)**

M **: a connected compact symmetric space**

$\chi(M)$ **: the Euler characteristic of** M

$$\#_2 M \geq \chi(M)$$

Fact 3. A connected compact symmetric space M **is a symmetric** R**-space if and only if** M **is an orbit of the linear isotropy representation of a symmetric space of compact type (or non-compact type).**

(Takeuchi 1989)

M **: a symmetric** R**-space**

$b_k(M, \mathbb{Z}_2)$ **: the** k**-th Betti number of** M **with coefficients in** $\mathbb{Z}_2$

$$\#_2 M = \sum_{k \geq 0} b_k(M, \mathbb{Z}_2)$$

Fact 4. A symmetric R**-space is a real form** L **of the certain Hermitian symmetric space** M **of compact type, and vice versa.**

$\exists \tau$ **: an involutive anti-holomorphic isometry of** M**;** $L = \{x \in M \mid \tau(x) = x\}$

(T.-Tasaki 2012)

M **: a Herm. sym. sp. of compact type**

L_1, L_2 **: real forms of** M**,** $L_1 \pitchfork L_2$

$\Rightarrow L_1 \cap L_2$ **is an antipodal set of** L_i $(i = 1, 2)$**.**

Moreover, if L_1, L_2 **are** $I_0(M)$**-congruent,** $L_1 \cap L_2$ **is a great antipodal set, that is,** $|L_1 \cap L_2| = \#_2 L_1 = \#_2 L_2$**.**

Application : Determination of Lagrangian Floer homology (Iriyeh-Sakai-Tasaki 2013)

Fact 5. (T.-Tasaki 2013)

M **: a symmetric** R**-space**

(i) Any antipodal set of M **is included in a great antipodal set.**

(ii) Any two great antipodal sets of M **are** $I_0(M)$**-congruent.**

Remark. In general each of (i) and (ii) is not necessarily satisfied.

Chen-Nagano determined $\#_2 M$ of almost all compact symmetric spaces M and referred to great antipodal sets. A great antipodal set is a maximal antipodal set but the converse is not true.
(An antipodal set A is maximal $\Leftrightarrow$ A' : an antipodal set, $A \subset A' \Rightarrow A' = A$)

Aim : To understand maximal antipodal sets of compact symmetric spaces (classifications, properties, etc.).

2. Maximal antipodal subgroups of compact Lie groups

G : a compact Lie group

$\exists g$: a bi-invariant Riemannian metric

G is a symmetric space w.r.t. g.

$x \in G$, $\quad s_x(y) = xy^{-1}x \quad (y \in G)$

e : the identity element, $\quad s_e(y) = y^{-1}$

$s_e(y) = y \Leftrightarrow y^2 = e$

If $x^2 = y^2 = e$, $s_x(y) = y \Leftrightarrow xy = yx$.

$A \subset G$ **: an antipodal set,** $e \in A$

$\Rightarrow$ **(i)** $\forall x \in A$**,** $x^2 = e$**, (ii)** $\forall x, y \in A$**,** $xy = yx$

If A **is maximal,** A **is a subgroup.**

$A \cong \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{r}$**,** $|A| = 2^r$

$r \geq \mathrm{rank}(G)$

E.g. $U(n)$ **: the unitary group**

A **: a maximal antipodal subgroup (MAS)**

By (i) each eigenvalue of $\forall x \in A$ **is** 1 **or** -1**.**

By (ii) A **is simultaneously diagonalizable.**

A **is conjugate to** $\{\mathrm{diag}(\pm 1, \ldots, \pm 1)\} \subset U(n)$**, which is a maximal (great) antipodal subgroup of** $U(n)$**.** $\#_2 U(n) = 2^n$**.**

$$\Delta_n := \{\mathrm{diag}(\pm 1, \ldots, \pm 1)\} \subset O(n)$$

$$\Delta_n^+ := \{g \in \Delta_n \mid \det g = 1\}$$

Generally we obtain the following.

<u>Theorem.</u> A MAS of $O(n)$, $U(n)$, $Sp(n)$ **is conjugate to** Δ_n**. A MAS of** $SO(n)$, $SU(n)$ **is conjugate to** Δ_n^+**.**

$\#_2 O(n) = \#_2 U(n) = \#_2 Sp(n) = 2^n$

$\#_2 SO(n) = \#_2 SU(n) = 2^{n-1}$

Griess (1991) and Yu (2013) classified conjugate classes of elementary abelian p-subgr. ($\cong \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$) of algebraic groups for prime p by algebraic methods. We classified conjugate classes of maximal antipodal subgroups of the quotient groups of the classical compact Lie groups and gave explicit expressions of their representatives, where we used "polars" and "centrosomes" introduced by Chen-Nagano (T.-Tasaki, J. Lie Theory 2017).

3. Basic principle of classifying maximal antipodal sets of compact symmetric spaces

G : a compact Lie group

e : the identity element of G

G_0 : the identity component of G

Each connected component of $F(s_e, G) := \{g \in G \mid s_e(g) = g\}$ is called a **polar** of G. $\{e\}$ is a trivial polar. A polar is a totally geodesic submanifold. It is a compact symmetric space.

E.g. $G = U(n)$

$$F(s_{1_n}, U(n)) = \bigcup_{j=0}^{n} \{g\, I_j\, g^{-1} \mid g \in U(n)\}$$

$$I_j = \mathrm{diag}(\underbrace{-1, \ldots, -1}_{j}, \underbrace{1, \ldots, 1}_{n-j})$$

$g \in G$, $I_g(h) := ghg^{-1}$ $(h \in G)$

<u>Lemma.</u> Let M be a polar of G and let $x \in M$. Then $M = \{I_g(x) \mid g \in G_0\}$ and $I_0(M) = \{I_g|_M \mid g \in G_0\}$.

A : a maximal antipodal set of a polar M

$A \cup \{e\}$ is an antipo. set of G by $A \subset F(s_e, G)$.

$\exists \tilde{A}$ **: a MAS of** G**;** $A \cup \{e\} \subset \tilde{A}$

$A = M \cap \tilde{A}$ **by the maximality of** A

$B_1, \ldots, B_k$ **: representatives of each** G_0**-conj. class of MAS of** G

$1 \leq \exists s \leq k, \ \exists g \in G_0$**;** $\quad \tilde{A} = I_g(B_s)$

$A = M \cap \tilde{A} = I_g(M \cap B_s)$ **since** M **is invariant by** I_g**.**

A representative of an $I_0(M)$**-congruence class of maximal antipodal sets of** M **is one of** $M \cap B_1, \ldots, M \cap B_k$**.**

4. Classification of maximal antipodal sets of classical compact symmetric spaces

We use an appropriate totally geodesic embedding of each classical compact symmetric space $M = G/K$ into G and the classification of MAS of G.

$\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$

$O(n, \mathbb{K}) := O(n), U(n), Sp(n)$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, resp.)

$G_k(\mathbb{K}^n)$ **: the Grassman manifold of the k-dimensional $\mathbb{K}$-subpsaces in $\mathbb{K}^n$**

$G_k(\mathbb{K}^n) \cong O(n,\mathbb{K})/O(k,\mathbb{K}) \times O(n-k,\mathbb{K})$

$\iota : G_k(\mathbb{K}^n) \ni x \mapsto \pi_x - \pi_{x^\perp} \in O(n,\mathbb{K})$ **: embedding, the image is a polar.**

$$\iota(G_k(\mathbb{K}^n)) \cap \Delta_n = \{\mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_n) \in \Delta_n \,| \\ |\{i|\varepsilon_i = 1\}| = k,\ |\{i|\varepsilon_i = -1\}| = n-k\}$$

By taking the inverse image, we obtain:

<u>Theorem.</u> Any maximal antipodal set of $G_k(\mathbb{K}^n)$ **is** $O(n,\mathbb{K})$**-congruent to** $\{\langle e_{i_1}, \ldots, e_{i_k}\rangle_{\mathbb{K}} \mid 1 \leq i_1 < \cdots < i_k \leq n\}$, **where** $\{e_1, \ldots, e_n\}$ **is the standard o.n.b. of** $\mathbb{K}^n$**.** $\#_2 G_k(\mathbb{K}^n) = \binom{n}{k}$**.**

$\gamma : G_m(\mathbb{K}^{2m}) \ni x \mapsto x^\perp \in G_m(\mathbb{K}^{2m})$ **: an isometry**

$G_m(\mathbb{K}^{2m})^* := G_m(\mathbb{K}^{2m})/\{\mathbf{id}, \gamma\}$

$s_{[x]}([y]) = [s_x(y)] \quad ([x], [y] \in G_m(\mathbb{K}^{2m})^*)$

$O(2m, \mathbb{K})^* := O(2m, \mathbb{K})/\{\pm 1_{2m}\}$

$\pi_{2m} : O(2m, \mathbb{K}) \to O(2m, \mathbb{K})^*$ **: the projection**

$\iota \circ \gamma(x) = \iota(x^\perp) = -(\pi_x - \pi_{x^\perp}) \ (x \in G_m(\mathbb{K}^{2m}))$

$G_m(\mathbb{K}^{2m})^* \underset{\mathbf{id}}{=} \iota(G_m(\mathbb{K}^{2m}))/\{\pm 1_{2m}\} \subset O(2m, \mathbb{K})^*$

: a polar

Since MAS of $O(2m, \mathbb{K})^*$ **are determined in [T.-Tasaki 2017], we can determine maximal antipodal sets of** $G_m(\mathbb{K}^{2m})^*$**.**

$CI(n) := \{x \in Sp(n) \mid x^2 = -1_n\} \cong Sp(n)/U(n)$

$1, \mathrm{i}, \mathrm{j}, \mathrm{k}$ **: the standard basis of** $\mathbb{H}$

Theorem. **Any maximal antipodal set of** $CI(n)$ **is** $Sp(n)$**-congruent to** $\mathrm{i}\Delta_n$**.** $\#_2 CI(n) = 2^n$**.**

$Sp(n)^* := Sp(n)/\{\pm 1_n\}$

$\pi_n : Sp(n) \to Sp(n)^*$ **: the projection**

$CI(n)^* := \pi_n(CI(n)) = CI(n)/\{\pm 1_n\}$

$CI(n)^* \subset \{x \in Sp(n)^* \mid x^2 = \pi_n(1_n)\}$ **: a polar**

Since MAS of $Sp(n)^*$ are determined in [T.-Tasaki 2017], we can determine maximal antipodal sets of $CI(n)^*$.

$DIII(n) := \{x \in SO(2n) \mid x^2 = -1_{2n}, \mathrm{Pf}(x) = 1\} \cong SO(2n)/U(n)$

$\mathrm{Pf}(x)$ **: the Pfaffian of** x

<u>**Theorem.**</u> **Any maximal antipodal set of $DIII(n)$ is $SO(2n)$-congruent to**

$$\left\{ \begin{bmatrix} \epsilon_1 J & & \\ & \ddots & \\ & & \epsilon_n J \end{bmatrix} \in SO(2n) \middle| \epsilon_i = \pm 1, \epsilon_1 \cdots \epsilon_n = 1 \right\},$$

where $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$**.**

Assume n **is even.**

$SO(2n)^* := SO(2n)/\{\pm 1_{2n}\}$

$\pi_{2n} : SO(2n) \to SO(2n)^*$ **: the projection**

$DIII(n)^* := \pi_{2n}(DIII(n)) = DIII(n)/\{\pm 1_{2n}\}$

$DIII(n)^* \subset \{x \in SO(2n)^* \mid x^2 = \pi_{2n}(1_{2n})\}$ **: a polar**

Since MAS of $SO(2n)^*$ **are determined in [T.-Tasaki 2017], we can determine maximal antipodal sets of** $DIII(n)^*$**.**

Thank you for your kind attention.