The intersection of two real forms in Hermitian symmetric spaces of compact type

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The 16th International Workshop on Differential Geometry
Kyungpook National University
November 1–3, 2012

Joint with Hiroyuki Tasaki (University of Tsukuba)

Contents

1. Introduction

Tasaki studied the intersection of two real forms in the complex hyperquadric $Q_n(\mathbb{C})$ which is isometric to the 2-oriented Grassmannian manifold $\tilde{G}_2(\mathbb{R}^{n+2})$. He determined the intersection of two real forms L_1 and L_2 in $Q_n(\mathbb{C})$ (Tohoku Math. J., 2010).

$\Longrightarrow$

· $L_1 \cap L_2$ is an antipodal set, i.e., $s_x(y) = y$ for ${}^\forall x, y \in L_1 \cap L_2$.

· $\#(L_1 \cap L_2) = \min\{\#_2 L_1, \#_2 L_2\}$

where $\#_2 L_i$ is $\max\{\#A \mid A \subset L_i : \text{ antipodal set}\}$.

In order to extend Tasaki's result we studied the intersection of two real forms L_1 and L_2 in a Hermitian symmetric space M of compact type (J. Math. Soc. Japan, 2012).

$\Longrightarrow$

· $L_1 \cap L_2$ is an antipodal set.

· If M is irreducible, the equality $\#(L_1 \cap L_2) = \min\{\#_2 L_1, \#_2 L_2\}$ holds with one exception.

Application of these results to Floer homology (Iriyeh-Sakai-Tasaki)

Tanaka-Tasaki (preprint)

· correction of the proof of the antipodal property of $L_1 \cap L_2$.

· investigation into the case where M is not irreducible.
$\Longrightarrow$ We can reduce the problem to either the case where M is irreducible or the case where L_1 and L_2 are diagonal real forms.

2. Preliminaries

Definitions

M : a Riemannian symmetric space

s_x : the geodesic symmetry at $x \in M$

$S \subset M$: an antipodal set $\overset{\text{def}}{\Longleftrightarrow}$ ${}^\forall x, y \in S,\ s_x(y) = y$

The 2-number $\#_2 M$ of M

$\#_2 M := \sup\{\#S \mid S \subset M$: antipodal set$\}$

An antipodal set S is great if $\#S = \#_2 M$.

Remarks

(1) These notions were introduced by B. -Y. Chen - Nagano (Trans. A. M. S, 1988).

(2) $\#_2 M < \infty$

(3) If M is a symmetric R-space, $\#_2 M = \dim H_*(M, \mathbb{Z}_2)$ (Takeuchi)

Examples

(1) $M = S^n$

$S = \{x, -x\}$: a great antipodal set

$\#_2 S^n = 2$

(2) $M = \mathbb{R}P^n$

$e_1, \ldots, e_{n+1}$: an o.n.b. of $\mathbb{R}^{n+1}$

$S = \{\langle e_1 \rangle, \ldots, \langle e_{n+1} \rangle\}$: a great antipodal set

$\#_2 \mathbb{R}P^n = n + 1$

(3) $M = U(n)$

$S = \{\text{diag}\{\pm 1, \ldots, \pm 1\}\}$: a great antipodal set

$\#_2 U(n) = 2^n$

Definition

M : a compact Riemannian symmetric space

G : the identity component of the isometry group of M

K : the isotropy subgroup at $o \in M$

$$F(s_o, M) = \{x \in M \mid s_o(x) = x\} = \bigcup_{j=0}^{r} M_j^+$$

where M_j^+ is a connected component and we set $M_0^+ = \{o\}$.

Each connected component M_j^+ is called a <u>polar</u> w.r.t. o.
Every polar is a K-orbit.

Examples

(1) $M = S^n$, $F(s_o, M) = \{o, -o\}$

(2) $M = \mathbb{R}P^n$, $o = \langle e_1 \rangle$

$F(s_o, M) = \{o\} \cup \{1 - \text{dim subspace} \subset \langle e_2, \ldots, e_{n+1} \rangle\} (\cong \mathbb{R}P^{n-1})$

Definition

M : a Hermitian symmetric space

τ : an involutive anti-holomorphic isometry of M

$F(\tau, M) = \{x \in M \mid \tau(x) = x\}$, which is a connected totally geodesic Lagrangian submanifold of M, is called a real form in M.

The classification of real forms in irreducible Hermitian symmetric spaces of compact type was given by Leung and Takeuchi.

We will give the classification of real forms in non-irreducible Hermitian symmetric spaces of compact type (Theorem 3.3).

Example

$M = G_k(\mathbb{C}^n)$: the Grassmannian manifold of k-dim subspaces in $\mathbb{C}^n$

The real forms in M :

$$\begin{cases} G_k(\mathbb{R}^n) \\ G_l(\mathbb{H}^m) \text{ (if } k = 2l \text{ and } n = 2m) \\ U(k) \text{ (if } n = 2k) \end{cases}$$

3. Real forms in a non-irreducible Hermitian symmetric space of compact type

Definition

M : a Hermitian symmetric space

τ : an anti-holomorphic isometry of M

A map $M \times M \ni (x,y) \mapsto (\tau^{-1}(y), \tau(x)) \in M \times M$ is an involutive anti-holomorphic isometry of $M \times M$.

The real form determined by the map is

$$D_\tau(M) := \{(x, \tau(x)) \mid x \in M\}$$

and it is called a diagonal real form in $M \times M$.

$A(M)$: the group of the holomorphic isometries of M

Remark

$(g_1, g_2)D_\tau(M) = D_{g_2 \tau g_1^{-1}}(M) \quad (g_1, g_2 \in A(M))$

$I(M)$: the group of the isometries of M

Proposition 3.1

M : an irreducible Hermitian symmetric space of compact type

$\Longrightarrow$

(1) Every element in $I(M) - A(M)$ is an anti-holomorphic isometry.

(2) The connected components of $I(M) - A(M)$ corresponds to the congruent classes of diagonal real forms bijectively by the correspondence $\tau \mapsto D_\tau(M)$.

Two diagonal real forms are congruent if there exists an element in $A_0(M \times M)$ which maps one to another.

Lemma 3.2 (Murakami, Takeuchi)

M : an irreducible Hermitian symmetric space of compact type

$I_0(M)$: the identity component of $I(M)$

$A_0(M)$: the identity component of $A(M)$

$\Longrightarrow$

(A) If $M = Q_{2m}(\mathbb{C})$ $(m \geq 2)$ or $M = G_m(\mathbb{C}^{2m})$ $(m \geq 2)$,

$$I(M)/I_0(M) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \qquad \text{and} \qquad A(M)/A_0(M) \cong \mathbb{Z}_2.$$

(B) Otherwise,

$$I(M)/I_0(M) \cong \mathbb{Z}_2 \qquad \text{and} \qquad A(M) = A_0(M).$$

Remark

Since M is a Hermitian symmetric space of compact type, $I_0(M) = A_0(M)$. Hence $I(M)/A(M) \cong \mathbb{Z}_2$ in both cases (A) and (B).

In particular, there are two congruent classes of diagonal real forms in $M \times M$ for M in (A) and there is one congruent class of diagonal real forms in $M \times M$ for M in (B).

Theorem 3.3 (Tanaka-Tasaki)

A real form in a Hermitian symmetric space of compact type is a product of real forms in irreducible factors and diagonal real forms.

Theorem 3.4 (Tanaka-Tasaki)

M : a Hermitian symmetric space of compact type

$M = M_1 \times \cdots \times M_m$: the decomposition of M into a product of irreducible factors

L_1, L_2 : real forms in M

$\Longrightarrow$

Each L_i has a decomposition $L_i = L_{i,1} \times \cdots \times L_{i,n}$ $(i = 1, 2)$ and

for each a $(1 \leq a \leq n)$ a pair of $L_{1,a}$ and $L_{2,a}$ is one of the following:

(1) Each $L_{i,a}$ $(i = 1, 2)$ is a real form in an irreducible factor of M.

(2) After renumbering M_j's if we need,

$$L_{1,a} = N_1 \times D_{\tau_2}(M_2) \times D_{\tau_4}(M_4) \times \cdots \times D_{\tau_{2s}}(M_{2s}) \text{ and}$$

$$L_{2,a} = D_{\tau_1}(M_1) \times D_{\tau_3}(M_3) \times \cdots \times D_{\tau_{2s-1}}(M_{2s-1}) \times N_{2s+1}$$

(or the opposite case) where $N_1 \subset M_1$ and $N_{2s+1} \subset M_{2s+1}$ are real forms and $\tau_i : M_i \to M_{i+1}$ $(1 \leq i \leq 2s)$ is an anti-holomorphic isometric map.

(3) After renumbering M_j's if we need,

$$L_{1,a} = N_1 \times D_{\tau_2}(M_2) \times D_{\tau_4}(M_4) \times \cdots \times D_{\tau_{2s-2}}(M_{2s-2}) \times N_{2s} \text{ and}$$

$$L_{2,a} = D_{\tau_1}(M_1) \times D_{\tau_3}(M_3) \times \cdots \times D_{\tau_{2s-3}}(M_{2s-3}) \times D_{\tau_{2s-1}}(M_{2s-1})$$

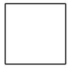
(or the opposite case) where $N_1 \subset M_1$ and $N_{2s} \subset M_{2s}$ are real forms and $\tau_i : M_i \to M_{i+1}$ $(1 \leq i \leq 2s-1)$ is an anti-holomorphic isometric map.

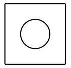
(4) After renumbering M_j's if we need,

$$L_{1,a} = D_{\tau_2}(M_2) \times D_{\tau_4}(M_4) \times \cdots \times D_{\tau_{2s}}(M_{2s}) \quad \text{and}$$

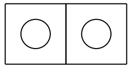
$$L_{2,a} = D_{\tau_1}(M_1) \times D_{\tau_3}(M_3) \times \cdots \times D_{\tau_{2s-1}}(M_{2s-1})$$

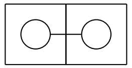
(or the opposite case) where $\tau_i : M_i \to M_{i+1}$ $(1 \leq i \leq 2s-1)$ and $\tau_{2s} : M_{2s} \to M_1$ are anti-holomorphic isometric maps.

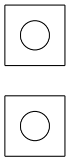
 : an irreducible H.s.s. of compact type

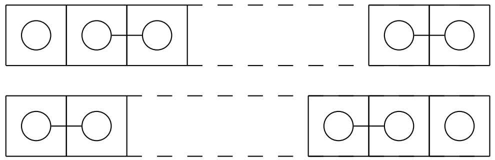
 : a real form in an irreducible H.s.s. of compact type

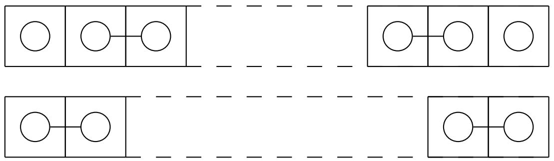
 : a product of two irreducible H.s.s. of compact type

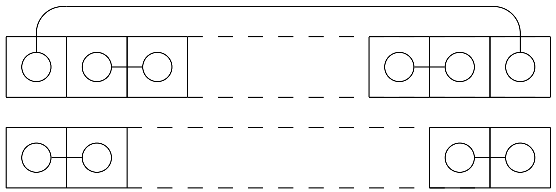
 : a product of real forms in irreducible factors

 : a diagonal real form

(1) 

(2) 

(3) 

(4) 

4. The intersection of two real forms

Theorem 4.1 (Tanaka-Tasaki)

M : a Hermitian symmetric space of compact type

L_1, L_2 : real forms in M

$L_1 \cap L_2$: discrete

$\Longrightarrow$

$L_1 \cap L_2$ is an antipodal set of L_1 and L_2.

Outline of Proof

Since the holomorphic sectional curvatures of $M > 0$,

$L_1 \cap L_2 \neq \emptyset$ (Tasaki).

We may assume that a base point $o \in M$ is contained in $L_1 \cap L_2$.

We will prove $s_o(p) = p$ for ${}^\forall p \in L_1 \cap L_2 - \{o\}$.

Let A_i be a maximal torus of L_i containing o and p $(i = 1, 2)$.

Let $\mathfrak{a}_i$ be a maximal abelian subspace corresponding to A_i $(i = 1, 2)$.

Let A_i' be a maximal torus of M containing A_i $(i = 1, 2)$.

Let $\mathfrak{a}_i'$ be a maximal abelian subspace corresponding A_i' $(i = 1, 2)$.

$\mathrm{Exp}_o tH_2$ $(H_2 \in \mathfrak{a}_2)$: a shortest geodesic segment from o to p in A_2

$\Longrightarrow$ It is also shortest in A_2' (because of the convexity of A_2).

$\Longrightarrow$ We can choose a fundamental root system Π_2 of the restricted root system of M w.r.t. $\mathfrak{a}_2'$ so that the fundamental cell S_2 determined by Π_2 satisfies $H_2 \in \overline{S_2}$.

Since $\overline{S_2} = \bigcup_{\Delta \subset \Pi_2^{\#}} S_2^{\Delta}$ (direct sum), $\exists^1 \Delta_2 \in \Pi_2^{\#}$ s.t. $H_2 \in S_2^{\Delta_2}$.

By using Takeuchi's result we obtain

$$\mathrm{Exp}_o S_2^{\Delta_2} \subset A_1' \cap A_2'$$

Irreducible cases :

Since the root sytem of M is of type C or type BC, we obtain

$$\mathrm{Exp}_o S_2^{\Delta_2} \subset A_1 \cap A_2$$

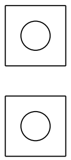
by using Takeuchi's result.

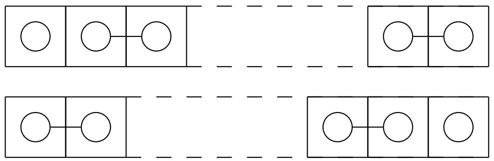
Since the intersection $L_1 \cap L_2$ is discrete by the assumption, $S_2^{\Delta_2}$ is a vertex of $\overline{S_2}$.

$\Longrightarrow$ $$s_o(p) = p$$

Non-irreducible cases :

It is sufficient to consider (1)-(4) in Theorem 3.4.

(1)  $\Longrightarrow$ irreducible case

(2) 

$$N_1 \times D_{\tau_2}(M_2) \times D_{\tau_4}(M_4) \times \cdots \times D_{\tau_{2s}}(M_{2s})$$

$$D_{\tau_1}(M_1) \times D_{\tau_3}(M_3) \times \cdots \times D_{\tau_{2s-1}}(M_{2s-1}) \times N_{2s+1}$$

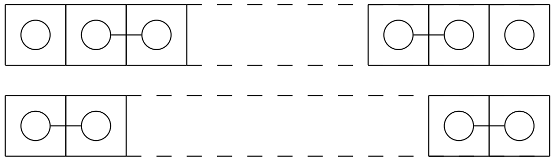
The intersection is

$$\{(x, \tau_1(x), \tau_2\tau_1(x), \ldots, \tau_{2s}\tau_{2s-1}\cdots\tau_1(x)) \\ \mid x \in N_1 \cap (\tau_{2s}\tau_{2s-1}\cdots\tau_1)^{-1}(N_{2s+1})\}$$

$\Longrightarrow$ This case is reduced to an irreducible case.

Remark

$(\tau_{2s}\tau_{2s-1}\cdots\tau_1)^{-1}(N_{2s+1})$ is a real form in M_1.

(3) 

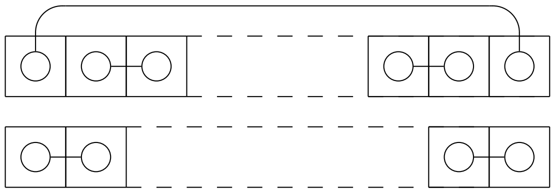
$$N_1 \times D_{\tau_2}(M_2) \times D_{\tau_4}(M_4) \times \cdots \times D_{\tau_{2s-2}}(M_{2s-2}) \times N_{2s}$$

$$D_{\tau_1}(M_1) \times D_{\tau_3}(M_3) \times \cdots \times D_{\tau_{2s-3}}(M_{2s-3}) \times D_{\tau_{2s-1}}(M_{2s-1})$$

The intersection is

$$\{(x, \tau_1(x), \tau_2\tau_1(x), \ldots, \tau_{2s-1}\cdots\tau_1(x)) \mid x \in N_1 \cap (\tau_{2s-1}\cdots\tau_1)^{-1}(N_{2s})\}.$$

$\Longrightarrow$ This case is reduced to an irreducible case.

(4) 

$$D_{\tau_2}(M_2) \times D_{\tau_4}(M_4) \times \cdots \times D_{\tau_{2s}}(M_{2s})$$

$$D_{\tau_1}(M_1) \times D_{\tau_3}(M_3) \times \cdots \times D_{\tau_{2s-1}}(M_{2s-1})$$

The intersection is

$$\{(x, \tau_1(x), \tau_2\tau_1(x), \ldots, \tau_{2s-1}\cdots\tau_1(x)) \\ \mid (x, \tau_{2s}^{-1}(x)) \in D_{\tau_{2s-1}\cdots\tau_1}(M_1) \cap D_{\tau_{2s}^{-1}}(M_1)\}.$$

$\Longrightarrow$ This case is reduced to the case of two diagonal real forms which is the simplest case of (4).

To summarize, in the non-irreducible cases we only need to investigate the intersection of two diagonal real forms. And by the similar argument to the irreducible cases we obtain $s_o(p) = p$.

Definition

M : a Hermitian symmetric space of compact type

L_1, L_2 : real forms in M

L_1 and L_2 are congruent if ${}^\exists g \in A_0(M)$ s.t. $g(L_1) = L_2$.

Theorem 4.2 (Tanaka-Tasaki)

M : a Hermitian symmetric space of compact type

L_1, L_2 : real forms in M, congruent

$L_1 \cap L_2$: discrete

$\Longrightarrow$

$L_1 \cap L_2$ is a great antipodal set of L_1 and L_2,

i.e., $\#(L_1 \cap L_2) = \#_2 L_1 = \#_2 L_2$.

Theorem 4.3 (Tanaka-Tasaki)

M : an irreducible Hermitian symmetric space of compact type

L_1, L_2 : real forms in M, $\#_2 L_1 \leq \#_2 L_2$

$L_1 \cap L_2$: discrete

$\Longrightarrow$

(1) If $M = G_{2m}(\mathbb{C}^{4m})$ $(m \geq 2)$ and L_1 is congruent to $G_m(\mathbb{H}^{2m})$ and L_2 is congruent to $U(2m)$,

$$\#(L_1 \cap L_2) = 2^m < \binom{2m}{m} = \#_2 L_1 < 2^{2m} = \#_2 L_2.$$

(2) Otherwise,

$$\#(L_1 \cap L_2) = \#_2 L_1 \ (\leq \#_2 L_2).$$

We prove Theorem 4.2 and Theorem 4.3 by induction on polars which is based on the next lemma.

Lemma 4.4

M : a Hermitian symmetric space of compact type

$o \in M$

$F(s_o, M) = \bigcup_{j=0}^{r} M_j^+$, M_j^+ : a polar

(1) L : a real form in M, $o \in L$

$\Longrightarrow$

(i) If $\dim M_j^+ > 0$, M_j^+ is a Hermitian symmetric space of compact type. Moreover, if $L \cap M_j^+ \neq \emptyset$, $L \cap M_j^+$ is a real form in M_j^+.

(ii) $$F(s_o, L) = \bigcup_{j=0}^{r} L \cap M_j^+$$

$$\#_2 L = \sum_{j=0}^{r} \#_2(L \cap M_j^+)$$

(2) L_1, L_2 : real forms in M, $o \in L_1 \cap L_2$

$L_1 \cap L_2$: discrete

$\Longrightarrow$

$$L_1 \cap L_2 = \bigcup_{j=0}^{r} \left\{ (L_1 \cap M_j^+) \cap (L_2 \cap M_j^+) \right\}$$

$$\#(L_1 \cap L_2) = \sum_{j=0}^{r} \# \left\{ (L_1 \cap M_j^+) \cap (L_2 \cap M_j^+) \right\}$$

Theorem 4.5 (Tanaka-Tasaki)

M : an irreducible Hermitian symmetric space of compact type

τ_1, τ_2 : anti-holomorphic isometries of M

$D_{\tau_1}(M),\ D_{\tau_2^{-1}}(M) \subset M \times M$: diagonal real forms

$D_{\tau_1}(M) \cap D_{\tau_2^{-1}}(M)$: discrete

$\Longrightarrow$

(1) If $M = Q_{2m}(\mathbb{C})$ $(m \geq 2)$ and $\tau_2\tau_1 \notin A_0(M)$,

$$\#(D_{\tau_1}(M) \cap D_{\tau_2^{-1}}(M)) = 2m < 2m + 2 = \#_2 M.$$

(2) If $M = G_m(\mathbb{C}^{2m})$ $(m \geq 2)$ and $\tau_2\tau_1 \notin A_0(M)$,

$$\#(D_{\tau_1}(M) \cap D_{\tau_2^{-1}}(M)) = 2^m < \begin{pmatrix} 2m \\ m \end{pmatrix} = \#_2 M.$$

(3) Otherwise,

$$\#(D_{\tau_1}(M) \cap D_{\tau_2^{-1}}(M)) = \#_2 M.$$

Summing-up

To investigate the intersection number of two real forms L_1 and L_2 in a Hermitian symmetric space M of compact type is reduced to the following two cases:

(1) both L_1 and L_2 are real forms in a irreducible factor of M,

(2) both L_1 and L_2 are diagonal real forms.

For the case (1) we conclude that the intersection number of L_1 and L_2 coincides with $\min\{\#_2 L_1, \#_2 L_2\}$ except for the case where

$$M = G_{2m}(\mathbb{C}^{4m}) \quad \text{and} \quad \{L_1, L_2\} = \{G_m(\mathbb{H}^{2m}), U(2m)\}.$$

For the case (2) we conclude that the intersection number of $L_1 = D_{\tau_1}(M)$ and $L_2 = D_{\tau_2^{-1}}(M)$ coincides with $\#_2 M$ except for the cases where

$$M = Q_{2m}(\mathbb{C}) \quad \text{with} \quad \tau_2\tau_1 \notin A_0(M) \quad \text{and}$$
$$M = G_m(\mathbb{C}^{2m}) \quad \text{with} \quad \tau_2\tau_1 \notin A_0(M).$$