複素旗多様体内の実形の交叉の対蹠性と Floer ホモロジー

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joint works with 井川治 (京都工繊大), 入江博 (茨城大),
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Main results

- $M = (M, \omega, J)$: a Kähler–Einstein C-space.
- (L_1, L_2) : a pair of real forms of M with ....

Theorem (Main theorem).

For any Hamiltonian diffeomorphism $\phi \in \mathrm{Ham}(M, \omega)$ with $L_1 \pitchfork \phi L_2$ in M,

$$\begin{aligned}\#(L_1 \cap \phi L_2) &\geq \#(\iota(M) \cap \mathfrak{a}_{L_1,L_2}) \\ &= \dim_{\mathbb{Z}_2} H^*(L_1 \cap L_2; \mathbb{Z}_2).\end{aligned}$$

This inequality is sharp.

Main results

- $M = (M, \omega, J)$: a Kähler–Einstein C-space, i.e. a compact 1-connected homogeneous Kähler–Einstein manifold.
- (L_1, L_2) : a pair of real forms of M defined by a commutative pair of anti-holomorphic isometries (τ_1, τ_2) on M.
- Suppose that the minimal Maslov numbers Σ_{L_1}, Σ_{L_2} are both ≥ 3.

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For any $\phi \in \mathrm{Ham}(M, \omega)$ *with* $L_1 \pitchfork \phi L_2$ *in* M,

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Notation: We put the compact semisimple Lie algebra

$$\mathfrak{g} := \mathrm{Lie}\, A_0(M).$$

The canonical embedding of M into $\mathfrak{g}$ will be denoted by

$$\iota : M \hookrightarrow \mathfrak{g}.$$

Then $\iota(M)$ is an adjoint orbit in $\mathfrak{g}$. We take

$$\mathfrak{a}_{L_1,L_2} \subset \mathfrak{g}^{-\tau_1^*} \cap \mathfrak{g}^{-\tau_2^*}$$

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Example: Let

$$M := \mathrm{Flag}^{\mathbb{C}}_{2,2}(\mathbb{C}^6)$$
$$:= \{(V_1, V_2) \mid V_1 \subset V_2 \subset \mathbb{C}^6, \dim_{\mathbb{C}} V_1 = 2, \dim_{\mathbb{C}} V_2 = 4\}$$

with the unique (up to scalar) $SU(6)$-invariant Kähler–Einstein structure. Take real forms (L_1, L_2) of M as

$$L_1 = \mathrm{Flag}^{\mathbb{R}}_{2,2}(\mathbb{R}^6), \quad L_2 = \mathrm{Flag}^{\mathbb{H}}_{1,1}(\mathbb{H}^3).$$

Then the conditions of our main theorem hold and

$$L_1 \cap L_2 = \mathrm{Flag}^{\mathbb{C}}_{1,1}(\mathbb{C}^3).$$

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Example: Let $(M, L_1, L_2) = (\mathrm{Flag}^{\mathbb{C}}_{2,2}(\mathbb{C}^6), \mathrm{Flag}^{\mathbb{R}}_{2,2}(\mathbb{R}^6), \mathrm{Flag}^{\mathbb{H}}_{1,1}(\mathbb{H}^3))$. Then for any Hamiltonian diffeomorphism ϕ on $\mathrm{Flag}^{\mathbb{C}}_{2,2}(\mathbb{C}^6)$ with

$$\mathrm{Flag}^{\mathbb{R}}_{2,2}(\mathbb{R}^6) \pitchfork \phi(\mathrm{Flag}^{\mathbb{H}}_{1,1}(\mathbb{H}^3)) \text{ in } \mathrm{Flag}^{\mathbb{C}}_{2,2}(\mathbb{C}^6),$$

we have

$$\begin{aligned} \#(\mathrm{Flag}^{\mathbb{R}}_{2,2}(\mathbb{R}^6) \cap \phi(\mathrm{Flag}^{\mathbb{H}}_{1,1}(\mathbb{H}^3))) &\geq \iota(M) \cap \mathfrak{a}_{L_1,L_2} \\ &= 6 \\ &= \dim_{\mathbb{Z}_2} H^*(\mathrm{Flag}^{\mathbb{C}}_{1,1}(\mathbb{C}^3); \mathbb{Z}_2). \end{aligned}$$

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Picture of $\iota(M) \cap \mathfrak{a}_{L_1,L_2}$:

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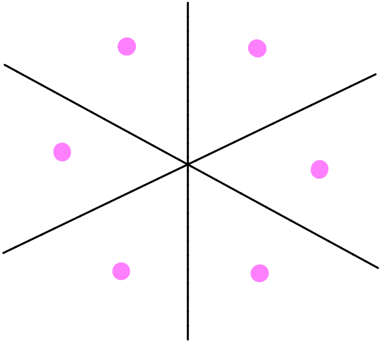

$\dim \mathfrak{a}_{L_1,L_2} = 2$

$W(A_2) \simeq \mathfrak{S}_3 \curvearrowright \mathfrak{a}_{L_1,L_2}$

Remark:

- $M = (M, \omega)$: a compact symplectic manifold.
- L : a connected component of an anti-symplectic involution on M.

Conjecture (The Arnold–Givental conjecture).

For any $\phi \in \mathrm{Ham}(M, \omega)$ *with* $L \pitchfork \phi(L)$ *in* M

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Fact (Y.G.Oh (Progr. Math., 1995)).

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Theorem (Main theorem of this talk).

The same results hold for

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- $M = (M, \omega, J)$: *a Kähler–Einstein C-space, and*
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Strategy: Compute the $\mathbb{Z}_2$-Lagrangian Floer homology.

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$$\dim_{\mathbb{Z}_2} \mathrm{HF}(L_1, L_2; \mathbb{Z}_2) = \#(\iota(M) \cap \mathfrak{a}_{L_1,L_2}).$$

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Lemma (Key Lemma).

For any $g \in A_0(M)$,

$$L_1 \pitchfork gL_2 \text{ in } M \Rightarrow L_1 \cap gL_2 \text{ is “antipodal” in } M,$$

where $A_0(M)$ denotes the identity component of the group of holomorphic isometries on M.

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Remark:

Kähler C-spaces M

$\leftrightarrow$ Adjoint orbits $\mathcal{O}$ in compact semisimple Lie algebras $\mathfrak{g}$ with invariant Kähler metrics

$\leftrightarrow$ Generalized complex flag manifolds $F_{\mathbb{C}}$ with invariant Kähler metrics.

Real forms L of M

$\leftrightarrow$ Intersections $\mathcal{O} \cap \mathfrak{g}^{-\theta}$ in $\mathcal{O}$ for involutions θ on $\mathfrak{g}$.

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Plan of this talk:

- Antipodal subsets of Kähler C-spaces.
- Computations of $\mathbb{Z}_2$-Lagrangian Floer homologies.
- Proof of our main theorem.

Antipodal subsets of Kähler C-spaces

- $M = (M, \omega, J)$: a Kähler C-space.
- $A_0(M) \curvearrowright M$: the identity component of the group of all holomorphic isometries on M.

For each $x \in M$, we put $\mathbb{T}_x := Z(A_0(M)^x)$.

Definition .

A pair of points (x, y) of M will be called antipodal in M if

$$t_x y = y \text{ for any } t_x \in \mathbb{T}_x.$$

A finite subset X of M is said to be antipodal if any pair of points (x, y) of X is antipodal in M.

Let M be a Kähler C-space. For each $x \in M$, we put $\mathbb{T}_x := Z(A_0(M)^x)$.

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Let $M = S^2$. Then $\mathbb{T}_x \simeq \mathbb{T}^1$ as the rotations at $x \in S^2$. In particular, (x, y) are antipodal if and only if $x = -y$.

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A finite subset X of a Kähler C-space M will be called antipodal if

$$t_x y = y \text{ for any } x, y \in X \text{ and any } t_x \in \mathbb{T}_x.$$

Remark: Let N be a symmetric space.

Definition (Chen–Nagano (1988)).

A finite subset X of N is called antipodal if

$$s_x y = y \text{ for any } x, y \in X,$$

where s_x denotes the point symmetry at x.

Let M be a Hermitian symmetric space of compact type. Then M is a symmetric space and a Kähler C-space. The two definitions of antipodal sets on M above coincide with each other.

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Example of non-symmetric case:

Let $M = \mathrm{Flag}_{2,2}(\mathbb{C}^6)$.

1. For $x = (V_1, V_2, V_3 = \mathbb{C}^6) \in \mathrm{Flag}^{\mathbb{C}}_{2,2}(\mathbb{C}^6)$,

$$\mathbb{C}^6 = W_1 \oplus W_2 \oplus W_3 \quad \text{(the orthogonal decomposition)}$$

   such that $V_1 = W_1$, $V_2 = W_1 \oplus W_2$. Consider

$$\mathbb{T}_x = S(U(1) \times U(1) \times U(1)) \curvearrowright \mathbb{C}^6 = W_1 \oplus W_2 \oplus W_3.$$

   $\rightsquigarrow \mathbb{T}_x \curvearrowright \mathrm{Flag}_{2,2}(\mathbb{C}^6)$.

2. $\{x, y\} \subset \mathrm{Flag}_{2,2}(\mathbb{C}^6)$ is antipodal if and only if the decompositions corresponding to x and y of $\mathbb{C}^6$ are compatible (i.e. projections are commutative to each other).

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- (L_1, L_2) : real forms of M defined by commutative (τ_1, τ_2).

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For any $g \in A_0(M)$ with $L_1 \pitchfork gL_2$ in M,

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Computations of $\mathbb{Z}_2$-Lagrangian Floer homologies

- $M = (M, \omega, J)$: a Kähler–Einstein C-space.
- (L_1, L_2) : real forms of M with $\Sigma_{L_1}, \Sigma_{L_2} \geq 3$.

Fact .

Take $g \in \mathrm{Ham}(M, \omega)$ such that

1. $L_1 \pitchfork gL_2$ *in* M
2. J *is regular for* g.
3. *The number of isolated points of* $\{J$*-holomorphic strips for* $(p, q; L_1, gL_2)\}/\sim$ *is even for any* $p, q \in L_1 \cap gL_2$.

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- (L_1, L_2) : real forms of M with $\Sigma_{L_1}, \Sigma_{L_2} \geq 3$.
- $g \in \mathrm{Ham}(M, \omega)$ with $L_1 \pitchfork gL_2$.

For each $p, q \in L_1 \cap gL_2$, we put

$$\widehat{\mathcal{M}}_J(p, q; L_1, gL_2)^0$$

$$:= \text{the set of all isolated points of}$$

$$\{J\text{-holo. strips for } (p, q; L_1, gL_2)\}/\text{parameter shifts.}$$

Definition of J-holomorphic strips:

Let $\Omega := \{s + it \mid s \in \mathbb{R}, 0 \leq t \leq 1\} \subset \mathbb{C}$.
A J-holomorphic map $u : \Omega \to M$ is said to be a strip for $(p, q; L_1, gL_2)$ if

- $u(s) \in L_1$ and $u(s + i) \in gL_2$.
- $\lim_{s \to -\infty} u(s + it) = p$ and $\lim_{s \to \infty} u(s + it) = q$.
- $\int \|\frac{\partial u}{\partial s}\|^2 ds dt < \infty$.
- $u(s + i\,\cdot)$ is not constant for each $s \in \mathbb{R}$.

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Definition .

J is called **regular** for (L_1, gL_2) if

$$E_u = D\bar{\partial}_J : T_u\mathcal{P}_u \to \mathcal{L}_u$$

is surjective for any $u \in \mathcal{M}_J(L_1, gL_2)$
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Put

$$CF_g(L_1, L_2) := \bigoplus_{p \in L_1 \cap gL_2} \mathbb{Z}_2[p].$$

Suppose that J is regular for (L_1, gL_2). We put

$$n_g(p, q; J) := \#\widehat{\mathcal{M}}_J(p, q; L_1, gL_2)^0 < \infty$$

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Fact .

Suppose that $g \in \mathrm{Ham}(M, \omega)$ satisfies that

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Problem: Find such $g \in \mathrm{Ham}(M, \omega)$.

Proof of our main theorem

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For each $x \in M$, there exists a sequence of finite abelian subgroups

$$\{\mathrm{id}\} = \Gamma_x^0 \subset \Gamma_x^1 \subset \cdots \subset \Gamma_x^N$$

of $\mathbb{T}_x = Z(A_0(M)^x)$ such that

- *$\#\Gamma_x^l$ is a power of 2 for any l.*
- *x is isolated in $\mathrm{Fix}(M; \Gamma_x^N)$.*
- *For any real form L with $x \in L$ and any $l = 1, \ldots, N$, $L \cap \mathrm{Fix}(M; \Gamma_x^{l-1})$ is stable by Γ_x^l.*

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Let N be a symmetric space. Then for each $x \in N$,

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For any $p, q \in L_1 \cap gL_2$, there exists a sequence

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Theorem (Main theorem).

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Furture works

- Classifications of (M, L_1, L_2) in our setting.
- Does the main theorem hold even if (τ_1, τ_2) is not commutative?
- Does the main theorem hold even if $\Sigma_{L_1} = 2$ or $\Sigma_{L_2} = 2$?
- Applications for Hamilton volume minimizing problems?

Fact (Iriyeh–Sakai–Tasaki (J. Math. Soc. Japan 2013)).

The real form S^n of $Q_n(\mathbb{C})$ has the Hamilton volume minimizing property.

Thank you for your attention!

Outline of the proof of the regularlity of J

Step 1: Find $g \in A_0(M)$ such that $L_1 \pitchfork gL_2$ and J is regular for (L_1, gL_2).

1. First, we take $g \in A_0(M)$ as $L_1 \pitchfork gL_2$ and
$$(\tau_1 \circ (g \circ \tau_2 \circ g^{-1}))^{2m} = \mathrm{id}_M$$
for some $m \in \mathbb{Z}$ (we can find such g by using symmetric triads).
2. For each $u \in \mathcal{M}(p, q, L_1, gL_2)$, we find a non-constant holomorphic map
$$f_u : \mathbb{C}P^1 \to M.$$
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