Lagrangian intersection theory and Hamiltonian volume minimizing problem

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Plan of this talk

1. Antipodal sets of complex flag manifolds
2. The structure of the intersection of real flag manifolds in a complex flag manifold
3. Application to the Hamiltonian volume minimizing problem

M : compact Riemannian symmetric space

s_x : geodesic symmetry at $x \in M$

Definition (Chen-Nagano 1988)

1. $A \subset M$: **antipodal set** $\overset{\text{def}}{\Longleftrightarrow}$ $s_x(y) = y$ for all $x, y \in A$
2. $\#_2 M := \max\{\#A \mid A \subset M : \text{antipodal set}\}$ **2-number**
3. $A \subset M$: **great** antipodal set $\overset{\text{def}}{\Longleftrightarrow}$ $\#A = \#_2 M$

Theorem (Takeuchi 1989)

M : *symmetic R-space* $\Longrightarrow$ $\#_2 M = \dim H_*(M, \mathbb{Z}_2)$

The intersection of real forms in a Herm. symm. space

Theorem (Tanaka-Tasaki 2012)

M : Hermitian symmetric space of compact type

$L_1, L_2 \subset M$: real forms, $L_1 \pitchfork L_2$

$\Longrightarrow$ *$L_1 \cap L_2$ is an antipodal set of L_1 and L_2.*

In addition, if L_1 and L_2 are congruent to each other,

$\Longrightarrow$ *$L_1 \cap L_2$ is a great antipodal set of L_1 and L_2.*

Example

$\mathbb{R}P^n \subset \mathbb{C}P^n$

$A := \{\mathbb{R}e_1, \ldots, \mathbb{R}e_{n+1}\} \subset \mathbb{R}P^n$ great antipodal set

For $u \in U(n+1)$, $\mathbb{R}P^n \pitchfork u\mathbb{R}P^n$ in $\mathbb{C}P^n$

$$\mathbb{R}P^n \cap u\mathbb{R}P^n \cong \{\mathbb{C}e_1, \ldots, \mathbb{C}e_{n+1}\} \subset \mathbb{C}P^n$$

$$\#(\mathbb{R}P^n \cap u\mathbb{R}P^n) = n+1 = \#_2\mathbb{R}P^n = \dim H_*(\mathbb{R}P^n, \mathbb{Z}_2)$$

Complex flag manifolds and real flag manifolds

(G, K) : symmetric pair of compact type

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

$H \in \mathfrak{p}$

$$\begin{array}{cccl} L & := & \mathrm{Ad}(K)H \subset \mathfrak{p} & : \textbf{real flag manifold} \\ \text{real form} \quad \cap & & \cap \qquad \cap & \\ M & := & \mathrm{Ad}(G)H \subset \mathfrak{g} & : \textbf{complex flag manifold} \\ & \cong & G/G_H \cong G^{\mathbb{C}}/P & \end{array}$$

$\mathfrak{t} \subset \mathfrak{g}$: a maximal abelian subalgebra containing H

Δ : root system of $\mathfrak{g}$ w.r.t. $\mathfrak{t}$

$$\mathfrak{g}_H = \{X \in \mathfrak{g} \mid [H, X] = 0\} = \mathfrak{t} + \mathfrak{g} \cap \sum_{\substack{\alpha \in \Delta \\ \alpha(H) = 0}} \mathfrak{g}_\alpha$$

Antipodal set of a complex flag manifold (1/2)

Suppose $\mathfrak{g}$ is simple.

$$\{\alpha_1, \ldots, \alpha_q, \underbrace{\alpha_{q+1}, \ldots, \alpha_p}_{\text{simple roots of } \mathfrak{g}_H}\} : \text{fundamental system of } \Delta$$

$$k_0 := 1 + \sum_{i=1}^{q} m_i \quad \text{where} \quad \delta = \sum_{i=1}^{p} m_i \alpha_i \quad \text{heighest root}$$

$$Z := \sum_{i=1}^{q} \alpha_i^* \in \mathfrak{t} \subset \mathfrak{g}$$

For $k \geq k_0$

$$g_k := \exp \frac{2\pi}{k} Z \in \exp \mathfrak{t} \subset G_H$$

$$s_H^{(k)} : M \to M; \; x \mapsto \mathrm{Ad}(g_k)x$$

Then $s_H^{(k)}$ defines a k-symmetric structure on M, if $k \geq k_0$.

$A \subset M$: **antipodal set** $\overset{\text{def}}{\Longleftrightarrow}$ $s_x^{(k)}(y) = y$ for all $x, y \in A$

Antipodal set of a complex flag manifold (2/2)

Proposition (Iriyeh-Tasaki-S.)

For any $x \in M$, the fixed point set of $s_x^{(k)}$ is

$$F(s_x^{(k)}, M) = \{y \in M \mid [x, y] = 0\}.$$

In particular, $F(s_x^{(k)}, M)$ is independent of the choice of $k \geq k_0$.

Theorem 1 (Iriyeh-Tasaki-S.)

$A \subset M$: maximal antipodal set

$\implies \exists \mathfrak{t}' \subset \mathfrak{g}$: maximal abelian subalgebra s.t.

$$A = M \cap \mathfrak{t}'.$$

Hence A is an orbit of the Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{t}'$.

Maximal antipodal sets of M are conjugate to each other by G.

The intersection of real flag manifolds

$\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$

$$F^{\mathbb{K}}_{n_1,\dots,n_r}(\mathbb{K}^n) := \left\{ (V_1, \dots, V_r) \;\middle|\; \begin{array}{l} V_1 \subset V_2 \subset \cdots \subset V_r \subset \mathbb{K}^n \\ \dim V_i = n_1 + \cdots + n_i \end{array} \right\}$$

1. $F^{\mathbb{R}}_{n_1,\dots,n_r}(\mathbb{R}^n) \subset F^{\mathbb{C}}_{n_1,\dots,n_r}(\mathbb{C}^n) \quad \leftrightsquigarrow \quad (SU(n), SO(n))$
2. $F^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n) \subset F^{\mathbb{C}}_{2n_1,\dots,2n_r}(\mathbb{C}^{2n}) \quad \leftrightsquigarrow \quad (SU(2n), Sp(n))$

$$SU(2n) = Sp(n)\, T\, Sp(n)$$

For $u \in SU(2n)$, $\exists g_1, g_2 \in Sp(n)$, $\exists a \in T$ s.t. $u = g_1 a g_2$

$$F^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n) \cap u F^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n) = g_1 (F^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n) \cap a F^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n))$$

$\mathbb{H}^n \cong \mathbb{C}^{2n} = W_1 \oplus \cdots \oplus W_s$ eigenspace decomposition of a^2

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The intersection of real flag manifolds

Theorem 2 (Iriyeh-Tasaki-S.)

For $0 < k \leq n$ we have

$$F_k^{\mathbb{H}}(\mathbb{H}^n) \cap a F_k^{\mathbb{H}}(\mathbb{H}^n) = \bigcup_{\substack{k_1+\cdots+k_s=k \\ 0 \leq k_i \leq \dim_{\mathbb{H}}(W_i)(1\leq i\leq s)}} F_{k_1}^{\mathbb{H}}(W_1) \times \cdots \times F_{k_s}^{\mathbb{H}}(W_s)$$

in $F_{2k}^{\mathbb{C}}(\mathbb{C}^{2n})$. The intersection is transverse if and only if $\dim_{\mathbb{H}} W_i = 1$ for all i. In this case

$$F_k^{\mathbb{H}}(\mathbb{H}^n) \cap a F_k^{\mathbb{H}}(\mathbb{H}^n) = \{W_{i_1} \oplus \cdots \oplus W_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\},$$

which is an antipodal set of $F_{2k}^{\mathbb{C}}(\mathbb{C}^{2n})$, and a maximal antipodal set of $F_k^{\mathbb{H}}(\mathbb{H}^n)$.

Theorem 3 (Iriyeh-Tasaki-S.)

For $n_1, \ldots, n_r$ which satisfy $n_1 + \cdots + n_r < n$ we have

$$
\begin{aligned}
&F^{\mathbb{H}}_{n_1,\ldots,n_r}(\mathbb{H}^n) \cap a F^{\mathbb{H}}_{n_1,\ldots,n_r}(\mathbb{H}^n) \\
&= \{(V_1, \ldots, V_r) \in F^{\mathbb{C}}_{2n_1,\ldots,2n_r}(\mathbb{C}^{2n}) \mid \\
&\qquad V_j \in F^{\mathbb{H}}_{n_1+\cdots+n_j}(\mathbb{H}^n) \cap a F^{\mathbb{H}}_{n_1+\cdots+n_j}(\mathbb{H}^n) \ (1 \leq j \leq r)\}
\end{aligned}
$$

in $F^{\mathbb{C}}_{2n_1,\ldots,2n_r}(\mathbb{C}^{2n})$. The intersection is transverse if and only if $\dim_{\mathbb{H}} W_i = 1$ for all i. In this case

$$
\begin{aligned}
&F^{\mathbb{H}}_{n_1,\ldots,n_r}(\mathbb{H}^n) \cap a F^{\mathbb{H}}_{n_1,\ldots,n_r}(\mathbb{H}^n) \\
&= \{(W_{i_1} \oplus \cdots \oplus W_{i_{n_1}}, \ldots, W_{i_1} \oplus \cdots \oplus W_{i_{n_1+\cdots+n_r}}) \\
&\quad \mid 1 \leq i_1 < \cdots < i_{n_1} \leq n, \ \ldots, 1 \leq i_{n_1+\cdots+n_{r-1}+1} < \cdots < i_{n_1+\cdots+n_r} \leq n, \\
&\quad \#\{i_1, \ldots, i_{n_1+\cdots+n_r}\} = n_1 + \cdots + n_r\},
\end{aligned}
$$

which is an antipodal set of $F^{\mathbb{C}}_{2n_1,\ldots,2n_r}(\mathbb{C}^{2n})$.

Application to the Hamiltonian volume minimizing problem

Lagrangian Floer homology

(M, ω) : closed symplectic manifold

$J = \{J_t\}_{0 \le t \le 1}$: family of ω-compatible almost complex structures

L_1, L_2 : closed Lagrangian submanifolds, $L_1 \pitchfork L_2$

$$CF(L_1, L_2) := \bigoplus_{p \in L_1 \cap L_2} \mathbb{Z}_2 \, p$$

$\partial : CF(L_1, L_2) \longrightarrow CF(L_1, L_2)$

$$\partial(p) = \sum_{q \in L_1 \cap L_2} n(p, q) \cdot q$$

$n(p, q) := \#\{$isolated J-holomorphic strips from p to $q\}$ (mod 2)

$$\partial \circ \partial = 0 \quad \Longrightarrow \quad HF(L_1, L_2 : \mathbb{Z}_2) := \ker \partial / \mathrm{im} \partial$$

- $HF(\phi L_1, \psi L_2 : \mathbb{Z}_2) \cong HF(L_1, L_2 : \mathbb{Z}_2) \ \forall \phi, \psi \in \mathrm{Ham}(M, \omega)$

Floer homology for a pair of real forms

Theorem 4 (Iriyeh-Tasaki-S. 2013)

(M, J_0, ω) : monotone Hermitian symmetric space of compact type

L_1, L_2 : real forms, $L_1 \pitchfork L_2$, $\Sigma_{L_1}, \Sigma_{L_2} \geq 3$

$\Longrightarrow$

$$HF(L_1, L_2 : \mathbb{Z}_2) \cong \bigoplus_{p \in L_1 \cap L_2} \mathbb{Z}_2[p]$$

1. (M, J_0, ω) is monotone if and only if it is Kähler-Einstein.
2. If M is irreducible, then the assumptions are satisfied except for the case $\mathbb{R}P^1 \subset \mathbb{C}P^1$.

Generalized Arnold-Givental inequality

Corollary (Iriyeh-Tasaki-S. 2013)

M : irreducible Hermitian symmetric space of compact type

(L_1, L_2) : real forms of M

$\Longrightarrow$ for any $\phi \in \mathrm{Ham}(M, \omega)$, $L_1 \pitchfork \phi L_2$

1. $(M, L_1, L_2) \cong (G_{2n}^{\mathbb{C}}(\mathbb{C}^{4n}),\ G_n^{\mathbb{H}}(\mathbb{H}^{2n}),\ U(2n)) \quad (n \geq 2)$

$$\#(L_1 \cap \phi L_2) \geq 2^n$$

2. (M, L_1, L_2) : otherwise

$$\#(L_1 \cap \phi L_2) \geq \min\{\dim H_*(L_1, \mathbb{Z}_2), \dim H_*(L_2, \mathbb{Z}_2)\}$$

Volume estimate under Hamiltonian deformations

$$
\begin{aligned}
Q_n(\mathbb{C}) &= \{[z] \in \mathbb{C}P^{n+1} \mid z_1^2 + z_2^2 + \cdots + z_{n+2}^2 = 0\} \cong \widetilde{G}_2(\mathbb{R}^{n+2}) \\
\cup \\
S^{k,n-k} &= \{[x] \in \mathbb{R}P^{n+1} \mid x_1^2 + \cdots + x_{k+1}^2 - x_{k+2}^2 - \cdots - x_{n+2}^2 = 0\} \\
&\cong (S^k \times S^{n-k})/\mathbb{Z}_2
\end{aligned}
$$

Theorem 5 (Iriyeh-Tasaki-S. 2013)

1. $\mathrm{vol}(\phi S^{k,n-k}) \geq \mathrm{vol}(S^n)$ for $\forall \phi \in \mathrm{Ham}(Q_n(\mathbb{C}), \omega)$
2. $S^{0,n} \subset Q_n(\mathbb{C})$ is Hamiltonian volume minimizing.

Theorem (Gluck-Morgan-Ziller)

$S^{0,n} \subset Q_n(\mathbb{C}) \cong \widetilde{G}_n(\mathbb{R}^{n+2})$ is volume minimizing in its homology class when n is even.

Further problems

1. Describe the intersection of two real flag manifolds in a complex flag manifold.
2. Calculate Lagrangian Floer homologies of pairs of real flag manifolds in complex flag manifolds.
3. Determine Hamiltonian volume minimizing properties of all real forms in irreducible Hermitian symmetric spaces and complex flag manifolds.

Thank you very much for your attention

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Thank you very much for your attention