Antipodal structure of the intersection of real flag manifolds in a complex flag manifold

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Plan of this talk

1. The intersection of two real forms in a Hermitian symmetric space (Tanaka-Tasaki)
2. Generalized symmetric structures and an antipodal set of a complex flag manifold
3. The intersection of two real flag manifolds in a complex flag manifold
4. Some applications

Antipodal set of a compact symmetric space

$M = G/K$: compact Riemannian symmetric space

Definition (Chen-Nagano)

$A \subset M$: **antipodal set** $\overset{\text{def}}{\Longleftrightarrow}$ $s_x(y) = y$ for all $x, y \in A$

$$\#_2 M := \max\{\#A \mid A : \text{antipodal set of } M\} \quad 2\text{-}\textbf{number}$$

An antipodal set A of M is said to be **great** if $\#A = \#_2 M$.

e.g. $M = \mathbb{R}P^n$

$$A = \{\mathbb{R}e_1, \mathbb{R}e_2, \ldots, \mathbb{R}e_{n+1}\}$$

$$\#_2 \mathbb{R}P^n = n + 1$$

Theorem (Takeuchi)

M : *symmetric R-space* $\Longrightarrow$ $\#_2 M = \dim H^*(M; \mathbb{Z}_2)$

The intersection of real forms in a Herm. symm. space

M : Kähler manifold

σ : anti-holomorphic involutive isometry of M

$$L = \mathrm{Fix}(\sigma, M)_0 \subset M \quad \textbf{real form}$$

Theorem (Tasaki (2010), Tanaka-Tasaki (2012))

$M = G/K$: Hermitian symmetric space of compact type

$L_1, L_2 \subset M$: real forms, L_1 and L_2 intersect transversally

$\Longrightarrow$ $L_1 \cap L_2$ is an antipodal set of L_1 and L_2.

In addition, if L_1 and L_2 are congruent to each other, then $L_1 \cap L_2$ is a great antipodal set of L_1 and L_2.

e.g. $\mathbb{R}P^n \subset \mathbb{C}P^n$, $u \in U(n+1)$ s.t. $\mathbb{R}P^n \pitchfork u\mathbb{R}P^n$

$$\mathbb{R}P^n \cap u\mathbb{R}P^n \cong \{\mathbb{R}e_1, \mathbb{R}e_2, \ldots, \mathbb{R}e_{n+1}\}$$

$$\#(\mathbb{R}P^n \cap u\mathbb{R}P^n) = n + 1 = \#_2\mathbb{R}P^n = \dim H^*(\mathbb{R}P^n, \mathbb{Z}_2)$$

Complex flag manifolds and real flag manifolds

G : connected compact semisimple Lie group

$H(\neq 0) \in \mathfrak{g}$

$$M_{\mathbb{C}} := \mathrm{Ad}(G)H \subset \mathfrak{g} : \textbf{complex flag manifold}$$

$$M_{\mathbb{C}} = \mathrm{Ad}(G)H \cong G/G_H \cong G^{\mathbb{C}}/P$$

G/K : Riemannian symmetric space of compact type

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

$H(\neq 0) \in \mathfrak{p}$

$$M := \mathrm{Ad}(K)H \subset \mathfrak{p} : \textbf{real flag manifold}$$

$$M = \mathrm{Ad}_G(K)H \cong K/K_H$$

$$M = \mathrm{Ad}_G(K)H \subset \mathrm{Ad}(G)H = M_{\mathbb{C}} : \text{real form}$$

Generalized symmetric structure on $M_{\mathbb{C}}$ (1/3)

$\mathfrak{t} \subset \mathfrak{g}$: maximal abelian subalgebra containing H

$\alpha \in \mathfrak{t}^*$

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g}^{\mathbb{C}} \mid [T, X] = \sqrt{-1}\alpha(T)X \text{ for } T \in \mathfrak{t}\}$$

$$\Delta := \{\alpha \in \mathfrak{t}^* - \{0\} \mid \mathfrak{g}_\alpha \neq \{0\}\} : \text{root system of } \mathfrak{g}^{\mathbb{C}} \text{ w.r.t. } \mathfrak{t}$$

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

$$\mathrm{Lie}(G_H) = \mathfrak{g}_H = \{X \in \mathfrak{g} \mid [H, X] = 0\} = \mathfrak{t} + \mathfrak{g} \cap \sum_{\substack{\alpha \in \Delta \\ \alpha(H) = 0}} \mathfrak{g}_\alpha$$

$$T_H(M_{\mathbb{C}}) \cong \mathfrak{g}/\mathfrak{g}_H \cong \mathfrak{g} \cap \sum_{\substack{\alpha \in \Delta \\ \alpha(H) \neq 0}} \mathfrak{g}_\alpha$$

Generalized symmetric structure on $M_{\mathbb{C}}$ (2/3)

$\Delta = \Delta_1 \cup \cdots \cup \Delta_r$: decomposition to irreducible root systems

$\{\alpha_{i,j} \mid 1 \leq i \leq r,\ 1 \leq j \leq p_i\}$: fundamental system of Δ

which satisfies

1. $\alpha_{i,j}(H) > 0$
2. $\{\alpha_{i,j} \mid 1 \leq j \leq p_i\}$ is a fundamental system of $\Delta_i (1 \leq i \leq r)$
3. $\{\alpha_{i,j} \mid 1 \leq i \leq r,\ q_i + 1 \leq j \leq p_i\}$ is a fundamental system of $\{\alpha \in \Delta \mid \alpha(H) = 0\}$

$$\delta_i = \sum_{j=1}^{p_i} m_{i,j}\alpha_{i,j} : \quad \text{highest root of } \Delta_i (1 \leq i \leq r)$$

$$k_0 := \max_{1 \leq i \leq r} \left\{ 1 + \sum_{j=1}^{q_i} m_{i,j} \right\}$$

Generalized symmetric structure on $M_{\mathbb{C}}$ (3/3)

$$\{\alpha_{i,j} \mid 1 \leq i \leq r,\ 1 \leq j \leq p_i\} \overset{\text{dual basis}}{\rightsquigarrow} \{H_{i,j} \mid 1 \leq i \leq r,\ 1 \leq j \leq p_i\}$$

$$Z := \sum_{i=1}^{r} \sum_{j=1}^{q_i} H_{i,j} \in \mathfrak{t}$$

$$k \geq k_0 \qquad g_k := \exp \frac{2\pi}{k} Z \in \exp \mathfrak{t} \subset G_H$$

$$\begin{aligned} \theta_k : G &\longrightarrow G \\ g &\longmapsto g_k g g_k^{-1} \\ \tilde{\theta}_k : M_{\mathbb{C}} &\longrightarrow M_{\mathbb{C}} \\ \mathrm{Ad}(g)H &\longmapsto \mathrm{Ad}(\theta_k(g))H = \mathrm{Ad}(g_k)\mathrm{Ad}(g)H \end{aligned}$$

$\tilde{\theta}_k$ defines a structure of generalized symmetric space on $M_{\mathbb{C}}$ and $\tilde{\theta}_k$ is the k-symmetry of $M_{\mathbb{C}}$ at H.

Antipodal set of a complex flag manifold (1/3)

$s_x^{(k)}$: k-symmetry of $M_{\mathbb{C}}$ at $x \in M_{\mathbb{C}}$

Definition

$A \subset M_{\mathbb{C}}$: **antipodal set** $\overset{\text{def}}{\Longleftrightarrow}$ $s_x^{(k)}(y) = y$ for all $x, y \in A$

$$\#_k M_{\mathbb{C}} := \max\{\#A \mid A : \text{antipodal set of } M_{\mathbb{C}}\}$$

Theorem (Sanchez, Berndt-Console-Fino)

$M_{\mathbb{C}}$: *complex flag manifold*

$$\Longrightarrow \qquad \#_k(M_{\mathbb{C}}) = \dim H^*(M_{\mathbb{C}}, \mathbb{Z}_2)$$

M : *real flag manifold*

$$\Longrightarrow \qquad \#_I(M) = \dim H^*(M, \mathbb{Z}_2)$$

Antipodal set of a complex flag manifold (2/3)

The eigenspace of $\tilde{\theta}_k = \mathrm{Ad}(g_k) : \mathfrak{g} \longrightarrow \mathfrak{g}$ corresponding to 1 is equal to $\mathfrak{g}_H$. Thus

$$\begin{aligned} F(\tilde{\theta}_k, M_{\mathbb{C}}) &= M_{\mathbb{C}} \cap \mathfrak{g}_H \\ &= \{y \in M_{\mathbb{C}} \mid [H, y] = 0\}. \end{aligned}$$

Proposition

For any $x \in M_{\mathbb{C}}$, the fixed point set of $s_x^{(k)}$ is

$$F(s_x^{(k)}, M_{\mathbb{C}}) = \{y \in M_{\mathbb{C}} \mid [x, y] = 0\}.$$

In particular, $F(s_x^{(k)}, M_{\mathbb{C}})$ is independent of the choice of $k \geq k_0$.

$A \subset M_{\mathbb{C}}$: antipodal set

$$\overset{\text{def}}{\iff} \quad s_x^{(k)}(y) = y \text{ for all } x, y \in A$$

$$\iff \quad [x, y] = 0 \text{ for all } x, y \in A$$

$$\iff \quad A_{\mathbb{R}} := \langle x \mid x \in A \rangle_{\mathbb{R}} \subset \mathfrak{g} \text{ is an abelian subalgebra}$$

Theorem (Iriyeh-S.-Tasaki)

$A \subset M_{\mathbb{C}}$: *maximal antipodal set*

$\Longrightarrow \quad \exists \mathfrak{t}' \subset \mathfrak{g}$: *maximal abelian subalgebra* *s.t.*

$$A = M_{\mathbb{C}} \cap \mathfrak{t}'.$$

Hence A is an orbit of the Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{t}'$.

Maximal antipodal sets of $M_{\mathbb{C}}$ are conjugate to each other by G.

Complex flags and real flags

$$F_{n_1,\dots,n_r}(\mathbb{K}^n) := \left\{ f = (V_1,\dots,V_r) \;\middle|\; \begin{array}{l} V_i \text{ is a } \mathbb{K}\text{-subspace of } \mathbb{K}^n, \\ V_1 \subset V_2 \subset \cdots \subset V_r \subset \mathbb{K}^n, \\ \dim V_i = n_1 + \cdots + n_i \end{array} \right\}$$

$$F_{n_1,\dots,n_r}(\mathbb{C}^n) \cong SU(n)/S(U(n_1)\times\cdots\times U(n_{r+1})) \cong \mathrm{Ad}(SU(n))H$$

$$uf_0 \quad \longleftrightarrow \quad uS(U(n_1)\times\cdots\times U(n_{r+1})) \quad \longleftrightarrow \quad \mathrm{Ad}(u)H$$

$$\begin{aligned} f_0 &:= (\langle e_1,\dots,e_{n_1}\rangle_{\mathbb{C}}, \langle e_1,\dots,e_{n_1+n_2}\rangle_{\mathbb{C}}, \dots, \langle e_1,\dots,e_{n_1+\cdots+n_r}\rangle_{\mathbb{C}}) \\ H &:= \mathrm{diag}(x_1\sqrt{-1}1_{n_1},\dots,x_{n_{r+1}}\sqrt{-1}1_{n_{r+1}}) \in \mathfrak{t} \subset \mathfrak{su}(n) \end{aligned}$$

$(G,K) = (SU(n), SO(n))$: compact symmetric pair

$$F_{n_1,\dots,n_r}(\mathbb{R}^n) \cong SO(n)/S(O(n_1)\times\cdots\times O(n_{r+1})) \cong \mathrm{Ad}_G(SO(n))H$$

$F_{n_1,\dots,n_r}(\mathbb{R}^n) \subset F_{n_1,\dots,n_r}(\mathbb{C}^n)$: real form

lemma 1

For any $u \in U(n)$ there exist $z_i \in U(1)\,(1 \leq i \leq n)$ and positively oriented orthonormal bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ of $\mathbb{R}^n$ which satisfy

$$uw_i = z_i v_i \quad (1 \leq i \leq n), \qquad \det u = z_1 \cdots z_n.$$

$$i \sim j \quad \Longleftrightarrow \quad z_i = \pm z_j$$

$\{1, \ldots, n\} = N_1 \cup \cdots \cup N_s$: equivalent classes w.r.t $\sim$

lemma 2

If $u \in U(n)$, unit vectors $v, w \in \mathbb{R}^n$ and $z \in \mathbb{C}$ satisfy $uw = zv$, then there exists $1 \leq a \leq s$ which satisfy

$$v \in \bigoplus_{i \in N_a} \langle v_i \rangle_{\mathbb{R}}, \quad w \in \bigoplus_{i \in N_a} \langle w_i \rangle_{\mathbb{R}}, \quad z = \pm z_i\,(i \in N_a).$$

The intersection of real flag manifolds (1/2)

In the complex Grassmannian manifold $F_k(\mathbb{C}^n)$ the intersection of real Grassmannian manifolds $F_k(\mathbb{R}^n)$ and $F_k(u\mathbb{R}^n)$ is

$$F_k(\mathbb{R}^n) \cap F_k(u\mathbb{R}^n)$$
$$= \bigcup_{\substack{k_1+\cdots+k_s=k \\ 0\leq k_a\leq \# N_a (1\leq a\leq s)}} F_{k_1}\left(\bigoplus_{i_1\in N_1}\langle v_{i_1}\rangle_{\mathbb{R}}\right) \times \cdots \times F_{k_s}\left(\bigoplus_{i_s\in N_s}\langle v_{i_s}\rangle_{\mathbb{R}}\right).$$

In the complex flag manifold $F_{n_1,\dots,n_r}(\mathbb{C}^n)$ the intersection of real flag manifold $F_{n_1,\dots,n_r}(\mathbb{R}^n)$ and $F_{n_1,\dots,n_r}(u\mathbb{R}^n)$ is

$$F_{n_1,\dots,n_r}(\mathbb{R}^n) \cap F_{n_1,\dots,n_r}(u\mathbb{R}^n)$$
$$= \{(V_1,\dots,V_r) \in F_{n_1,\dots,n_r}(\mathbb{C}^n) \mid V_i \in F_{n_1+\cdots+n_i}(\mathbb{R}^n) \cap F_{n_1+\cdots+n_i}(u\mathbb{R}^n)$$

The intersection of real flag manifolds (2/2)

Theorem (Iriyeh-S.-Tasaki)

For $u \in U(n)$*,*

$F_{n_1,\ldots,n_r}(\mathbb{R}^n)$ *and* $F_{n_1,\ldots,n_r}(u\mathbb{R}^n)$ *intersect transversally*

$$\iff \quad z_i \neq \pm z_j \ (i \neq j).$$

$$F_{n_1,\ldots,n_r}(\mathbb{R}^n) \cap F_{n_1,\ldots,n_r}(u\mathbb{R}^n)$$
$$= \{(\langle v_{i_1}, \ldots, v_{i_{n_1}} \rangle_{\mathbb{C}}, \langle v_{i_1}, \ldots, v_{i_{n_1+n_2}} \rangle_{\mathbb{C}}, \ldots, \langle v_{i_1}, \ldots, v_{i_{n_1+\cdots+n_r}} \rangle_{\mathbb{C}})$$
$$\mid 1 \leq i_1 < \cdots < i_{n_1} \leq n,\ 1 \leq i_{n_1+1} < \cdots < i_{n_1+n_2} \leq n, \ldots,$$
$$1 \leq i_{n_1+\cdots+n_{r-1}+1} < \cdots < i_{n_1+\cdots+n_r} \leq n,$$
$$\#\{i_1, \ldots, i_{n_1+\cdots+n_r}\} = n_1 + \cdots + n_r\},$$

that is a maximal antipodal set of $F_{n_1,\ldots,n_r}(\mathbb{C}^n)$*.*

The intersection number

Corollary

If $F_{n_1,\dots,n_r}(\mathbb{R}^n)$ *and* $F_{n_1,\dots,n_r}(u\mathbb{R}^n)$ *intersect transversally, then*

$$\begin{aligned}
&\#(F_{n_1,\dots,n_r}(\mathbb{R}^n) \cap F_{n_1,\dots,n_r}(u\mathbb{R}^n)) \\
&= \#_k(F_{n_1,\dots,n_r}(\mathbb{C}^n)) = \dim H^*(F_{n_1,\dots,n_r}(\mathbb{C}^n), \mathbb{Z}_2) \\
&= \#_I(F_{n_1,\dots,n_r}(\mathbb{R}^n)) = \dim H^*(F_{n_1,\dots,n_r}(\mathbb{R}^n), \mathbb{Z}_2) \\
&= \frac{n!}{n_1! n_2! \cdots n_{r+1}!}
\end{aligned}$$

Application to Lagrangian Floer homology

(M, ω) : closed symplectic manifold

$L_1, L_2 \subset M$: closed Lagrangian submanifolds, $L_1 \pitchfork L_2$

$p, q \in L_1 \cap L_2$

$u : \mathbb{R} \times [0,1] \longrightarrow M$: J-holomorphic strip from p to q

$$CF(L_1, L_2) := \bigoplus_{p \in L_1 \cap L_2} \mathbb{Z}_2\, p$$

$\partial : CF(L_1, L_2) \longrightarrow CF(L_1, L_2)$

$$\partial(p) = \sum_{q \in L_1 \cap L_2} n(p, q) \cdot q$$

$n(p, q) := \#\{\text{isolated } J\text{-holomorphic strips from } p \text{ to } q\} \pmod 2$

$$\partial \circ \partial = 0 \quad \Longrightarrow \quad HF(L_1, L_2 : \mathbb{Z}_2) := \ker \partial / \mathrm{im} \partial$$

Application to Lagrangian Floer homology

Theorem (Iriyeh-Tasaki-S.)

(M, J_0, ω) : monotone Hermitian symmetric space of compact type

L_1, L_2 : real forms, $L_1 \pitchfork L_2$, $\Sigma_{L_1}, \Sigma_{L_2} \geq 3$

$\Longrightarrow$

$$HF(L_1, L_2 : \mathbb{Z}_2) \cong \bigoplus_{p \in L_1 \cap L_2} \mathbb{Z}_2[p]$$

1. (M, J_0, ω) is monotone if and only if it is Kähler-Einstein.
2. If M is irreducible, then the assumptions are satisfied except for the case $\mathbb{R}P^1 \subset \mathbb{C}P^1$.

Generalized Arnold-Givental inequality

Corollary

M : irreducible Hermitian symmetric space of compact type

(L_1, L_2) : real forms of M

$\Longrightarrow$ for any $\phi \in \mathrm{Ham}(M, \omega)$, $L_1 \pitchfork \phi L_2$

1. $(M, L_1, L_2) \cong (G^{\mathbb{C}}_{2m}(\mathbb{C}^{4m}),\ G^{\mathbb{H}}_m(\mathbb{H}^{2m}),\ U(2m)) \quad (m \geq 2)$

$$\#(L_1 \cap \phi L_2) \geq 2^m$$

2. (M, L_1, L_2) : otherwise

$$\#(L_1 \cap \phi L_2) \geq \min\{\dim H^*(L_1, \mathbb{Z}_2), \dim H^*(L_2, \mathbb{Z}_2)\}$$

Application to Hamiltonian volume minimizing problem

$$
\begin{aligned}
Q_n(\mathbb{C}) &= \{[z] \in \mathbb{C}P^{n+1} \mid z_1^2 + z_2^2 + \cdots + z_{n+2}^2 = 0\} \\
S^{k,n-k} &= \{[x] \in \mathbb{R}P^{n+1} \mid x_1^2 + \cdots + x_{k+1}^2 - x_{k+2}^2 - \cdots - x_{n+2}^2 = 0\} \\
&\cong (S^k \times S^{n-k})/\mathbb{Z}_2
\end{aligned}
$$

Theorem (Iriyeh-Tasaki-S.)

$\mathrm{vol}(\phi S^{k,n-k}) \geq \mathrm{vol}(S^n)$ for $\forall \phi \in \mathrm{Ham}(Q_n(\mathbb{C}), \omega)$

Corollary

$S^{0,n} \subset Q_n(\mathbb{C})$ is Hamiltonian volume minimizing.

$S^{0,n} \subset Q_n(\mathbb{C}) \cong \widetilde{G_2}(\mathbb{R}^{n+2})$ is volume minimizing in its homology class when n is even (Gluck-Morgan-Ziller). When n is odd, $S^{0,n} \subset Q_n(\mathbb{C})$ can not be homologically volume minimizing.