Maximal antipodal sets of classical compact symmetric spaces

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1. Introduction

M: a compact Riemannian symmetric space

s_x: the geodesic symmetry at x

i.e., (i) s_x is an isometry of M, (ii) ${s_x}^2 = \mathrm{id}$,

(iii) x is an isolated fixed point of s_x

$S \subset M$: a subset

S: an antipodal set $\overset{\text{def}}{\iff} \forall x, y \in S,\ s_x(y) = y$

The 2-number $\#_2 M$ of M

$\#_2 M := \max\{|S| \mid S \subset M \text{ antipodal set}\}$

S: great $\overset{\text{def}}{\iff} |S| = \#_2 M$

(Chen-Nagano 1988)

Examples. **(1)** $M = S^n \, (\subset \mathbb{R}^{n+1})$

$\{x, -x\}$: **a great antipodal set for any** $x \in S^n$

(2) $M = \mathbb{R}P^n$

$e_1, \ldots, e_{n+1}$: **an o.n.b. of** $\mathbb{R}^{n+1}$

$\{\langle e_1 \rangle_{\mathbb{R}}, \ldots, \langle e_{n+1} \rangle_{\mathbb{R}}\}$: **a great antipodal set**

(3) $M = U(n) \qquad s_x(y) = xy^{-1}x$

$s_{1_n}(x) = x \iff x^2 = 1_n$ **(**1_n**: the unit matrix)**

$x^2 = y^2 = 1_n \;\Rightarrow\; s_x(y) = y$ **iff** $xy = yx$

$$\left\{ \begin{bmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{bmatrix} \right\} \subset U(n)$$: **a great antipodal set**

$M \subset N$: **totally geodesic**

$S \subset M$: **an antip. set** $\Rightarrow S \subset N$: **an antip. set**

$\rightsquigarrow \#_2 M \leq \#_2 N$

(Chen-Nagano) M: **cpt. conn. sym. sp.**

$\#_2 M \geq \chi(M)$, $\chi(M)$: **the Euler number**

"=" **if** M: **a Herm. sym. sp. of compact type**

(Takeuchi) M: **a symmetric** R**-space**

$$\Rightarrow \#_2 M = \sum_{k=0}^{\dim M} b_k(M; \mathbb{Z}_2)$$

b_k: **the k-th Betti number**

A symmetric R-sp. is a real form L of some Herm. sym. sp. M of cpt. type, i.e., $\exists\tau$: an involutive anti-holomorphic isometry of M; $L = \{x \in M \mid \tau(x) = x\}$, which is connected.

(T.-Tasaki 2012)

M: a Herm. sym. sp. of compact type

L_1, L_2: real forms of M, $L_1 \pitchfork L_2$

$\Rightarrow L_1 \cap L_2$ is an antipodal set of L_i $(i = 1, 2)$.

Moreover, if L_1, L_2 are congruent, $L_1 \cap L_2$ is great.

Remark. S^n, $\mathbb{R}P^n$, and $U(n)$ are sym. R-sp.

Remark. There exists a maximal antipodal set which is not great in general.

(T.-Tasaki 2013) For a sym. R-sp. M (i) any antipodal set of M is included in a great antipodal set, (ii) any two great antipodal sets of M are $I(M)_0$-congruent, (iii) a great antipodal set of M is an orbit of the Weyl group.

Chen-Nagano determined $\#_2 M$ of almost all compact symmetric spaces M. On the other hand, we are interested in the structures of antipodal sets of M.

Our goal: Classify maximal antipodal sets of compact symmetric spaces.

2. Maximal antipodal subgroups of compact Lie groups

G: **a cpt. Lie gr. with bi-invariant metric**

$s_x(y) = xy^{-1}x \quad (x, y \in G)$

1 : **the unit element of** G

$s_1(y) = y \Leftrightarrow y^2 = 1$

If $x^2 = y^2 = 1,\ s_x(y) = y \Leftrightarrow xy = yx$

$1 \in S$**:max. antipodal set of** $G \Rightarrow S$**: subgroup**

$S \cong \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{r} \qquad |S| = 2^r$

$r \geq \mathrm{rank}(G)$ **(**$r > \mathrm{rank}(G)$ **can happen)**

Classification of max. antip. subgroups (MAS)

$$\Delta_n := \left\{ \begin{bmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{bmatrix} \right\} \subset O(n)$$

$$\Delta_n^+ := \{ g \in \Delta_n \mid \det g = 1 \}$$

A MAS of $O(n)$, $U(n)$, $Sp(n)$ is conjugate to Δ_n. A MAS of $SO(n)$, $SU(n)$ is conjugate to Δ_n^+.

$\#_2 O(n) = \#_2 U(n) = \#_2 Sp(n) = 2^n$

$\#_2 SO(n) = \#_2 SU(n) = 2^{n-1}$

$$D[4] := \left\{ \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}, \begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix} \right\} \subset O(2)$$

$n = 2^k \cdot l, \ \ l$: **odd**

$0 \leq s \leq k$

$$D(s,n) := \{ d_1 \otimes \cdots \otimes d_s \otimes d_0 \mid d_1, \ldots, d_s \in D[4], d_0 \in \Delta_{n/2^s} \} = \underbrace{D[4] \otimes \cdots \otimes D[4]}_{s} \otimes \Delta_{n/2^s} \subset O(n)$$

$$Q[8] := \{ \pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k} \}$$

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1,$$

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \ \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \ \mathbf{ki} = -\mathbf{ik} = \mathbf{j}$$

Theorem 1 (**T.-Tasaki**)

$\tilde{G} = U(n)$, $O(n)$, $Sp(n)$

$G = U(n)/\{\pm 1_n\}$, $O(n)/\{\pm 1_n\}$, $Sp(n)/\{\pm 1_n\}$

$\pi_n : \tilde{G} \to G$ **: the projection**

$n = 2^k \cdot l, \quad l$ **: odd**

(1) MAS of $G = O(n)/\{\pm 1_n\}$ **is conjugate to**

$$\pi_n(D(s,n)) \quad (0 \leq s \leq k)$$

where $(s,n) = (k-1, 2^k)$ **is excluded.**

(2) MAS of $G = U(n)/\{\pm 1_n\}$ **is conjugate to**

$$\pi_n(\{1, \sqrt{-1}\}D(s,n)) \quad (0 \leq s \leq k)$$

where $(s,n) = (k-1, 2^k)$ **is excluded.**

(3) MAS of $G = Sp(n)/\{\pm 1_n\}$ **is conjugate to**

$$\pi_n(Q[8] \cdot D(s,n)) \qquad (0 \leq s \leq k)$$

where $(s,n) = (k-1, 2^k)$ **is excluded.**

Remark. $\Delta_2 \subsetneq D[4]$.

$$D(k-1, 2^k) = \underbrace{D[4] \otimes \cdots \otimes D[4]}_{s} \otimes \Delta_2$$

$$\subsetneq \underbrace{D[4] \otimes \cdots \otimes D[4]}_{s} \otimes D[4] = D(k, 2^k)$$

Griess (1991) and Yu (2013) classified conjugate classes of elementary abelian p**-subgr. of algebraic groups by algebraic methods.**

3. Maximal antipodal sets of classical compact symmetric spaces

We use an appropriate totally geodesic embedding of classical compact sym. sp. $M = G/K$ into G and the classification of MAS of G.

$\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ $\quad O(n, \mathbb{K}) := O(n)(\mathbb{K} = \mathbb{R}), U(n)(\mathbb{K} = \mathbb{C}), Sp(n)(\mathbb{K} = \mathbb{H})$

$$\{g\,\mathrm{diag}(\underbrace{1, \ldots, 1}_{k}, \underbrace{-1, \ldots, -1}_{n-k})\, g^{-1} \mid g \in O(n, \mathbb{K})\}$$
$$\cong O(n, \mathbb{K})/O(k, \mathbb{K}) \times O(n-k, \mathbb{K}) \cong G_k(\mathbb{K}^n)$$

$CI(n) := \{x \in Sp(n) \mid x^2 = -1_n\} \cong Sp(n)/U(n)$

$DIII(n) := \{x \in SO(2n) \mid x^2 = -1_{2n}, \mathrm{Pf}(x) = 1\} \cong SO(2n)/U(n)$

Maximal antip. sets of $CI(n)$ and $CI(n)/\{\pm 1_n\}$

$\mathrm{i}\Delta_n$: **unique max. antip. set of $CI(n)$ up to congruence** $\qquad \#_2 CI(n) = 2^n$

$Sp(n)^* := Sp(n)/\{\pm 1_n\}$

$\pi_n : Sp(n) \to Sp(n)^*$ **the projection**

$CI(n)^* := \pi_n(CI(n)) = CI(n)/\{\pm 1_n\}$

$CI(n)^* \subset \{x \in Sp(n)^* \mid x^2 = \pi_n(1_n)\}$

Let $S \subset CI(n)^*$**be a max. antipodal set. Then** $\{\pi_n(1_n)\} \cup S$ **is an antipo. set of** $Sp(n)^*$**. There is a max. antipo. subgr.** $\tilde{S}$ **of** $Sp(n)^*$ **such that** $\{\pi_n(1_n)\} \cup S \subset \tilde{S}$**. By Theorem 1,**

$$\tilde{S} = \pi_n(g)\pi_n(Q[8] \cdot D(s,n))\pi_n(g^{-1})$$

for some $g \in Sp(n)$**. Hence**

$$\{\pi_n(1_n)\} \cup \pi_n(g)^{-1} S \pi_n(g) \subset \pi_n(Q[8] \cdot D(s,n)).$$

By the maximality of S **we obtain**

$$\pi_n(g)^{-1} S \pi_n(g) = \pi_n(Q[8] \cdot D(s,n)) \cap CI(n)^*.$$

RHS$= \pi_n(\{x \in Q[8] \cdot D(s,n) \mid x^2 = -1_n\})$.

$$PD(s,n) := \{d \in D(s,n) \mid d^2 = 1_n\}$$
$$ND(s,n) := \{d \in D(s,n) \mid d^2 = -1_n\}$$

$$\{x \in Q[8] \cdot D(s,n) \mid x^2 = -1_n\} = ND(s,n) \cup \{\mathbf{i}, \mathbf{j}, \mathbf{k}\} PD(s,n)$$

$n = 2^k \cdot l,\ \ l$: **odd**

<u>Theorem 2</u> (T.-Tasaki) A maximal antipodal set of $CI(n)^*$ is congruent to

$$\pi_n(ND(s,n) \cup \{\mathbf{i}, \mathbf{j}, \mathbf{k}\} PD(s,n)) \quad (0 \leq s \leq k)$$

where $(s,n) = (k-1, 2^k)$ is excluded.