Geometry of symmetric R-spaces

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Contents

1. Introduction

A symmetric R-space is a kind of special compact symmetric space for which several characterizations are known. One of the characterizations is that a symmetric R-space is a symmetric space which is realized as an orbit under the linear isotropy action of a certain symmetric space of compact (or noncompact) type. It is based on a result of Takeuchi-Kobayashi in 1968, to which I will mention in the former part of this talk.

In the latter part of this talk, I will mention my recent joint work with Hiroyuki Tasaki dealing with real forms in a Hermitian symmetric space of compact type, which is another characterization of symmetric R-spaces.

2. Symmetric spaces

M : a Riemannian manifold

M : a <u>(Riemannian) symmetric space</u>

$\overset{\text{def}}{\Longleftrightarrow}$ ${}^\forall x \in M$, ${}^\exists s_x : M \to M$ an isometry

s.t. (i) $s_x \circ s_x = \mathrm{id}$

(ii) x is an isolated fixed point of s_x

s_x is called the <u>symmetry</u> at x.

· A symmetric space is complete.

· The group $I(M)$ of the isometries is a Lie transformation group of M which acts transitively on M.

e.g. $\mathbb{R}^n$, S^n, T^n, $\mathbb{K}P^n$ and more generally $G_k(\mathbb{K}^N)$ are symmetric spaces, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

We assume that a symmetric space is connected.

$M = G/K$: a (connected) symmetric space

$G = I_0(M)$: the identity component of $I(M)$

$K = \{g \in G \mid g \cdot o = o\}$ for $o \in M$

$\sigma : G \to G,\ \sigma(g) := s_o g s_o^{-1}$: an involutive automorphism

$\mathfrak{g}$: the Lie algebra of G

σ induces the involutive automorphism of $\mathfrak{g}$, denoted by $d\sigma : \mathfrak{g} \to \mathfrak{g}$.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$: a direct sum decomposition

$\mathfrak{k} := \{X \in \mathfrak{g} \mid d\sigma(X) = X\}$ = the Lie algebra of K

$\mathfrak{m} := \{X \in \mathfrak{g} \mid d\sigma(X) = -X\} = T_o M$

$\cdot\, \mathrm{Ad}_G(K)\mathfrak{m} \subset \mathfrak{m}$, which is called the <u>linear isotropy action</u> of K on $\mathfrak{m}$.

3. Symmetric R-spaces

Nagano (1965) : He introduced the notion of a symmetric R-space as a compact symmetric space M which admits a transitive action of noncompact Lie group L containing $I_0(M)$ as a subgroup.
e.g. $M = S^n$ admits a transitive action of the conformal transformation group $\supset I_0(M)$.

Kobayashi-Nagano (1964, 65, 66) : They gave the structure theorem on certain filtered Lie algebras and its applications to transformation groups acting on symmetric spaces, which includes the classification of symmetric R-spaces. They also proved that if M is a noncompact irreducible symmetric space, then no Lie group acting effectively on M contains $I_0(M)$ as a proper subgroup.

Takeuchi (1965) : He used the terminology " symmetric R-space". He gave a cell decomposition of an R-space, which is a kind of generalization of a symmetric R-space.

On the other hand, Chern-Lashof introduced the total curvature of an immersed manifold in 1957.

M : a compact C^∞ manifold immersed in $\mathbb{R}^n$ with the immersion φ.

The total (absolute) curvature $\tau(\varphi)$ is defined by

$$\tau(\varphi) := \frac{1}{\mathrm{vol}(S^{n-1})} \int_B |\mathrm{det} A_\xi| \omega$$

B : the unit normal bundle of M

ω : the volume element of B

A_ξ : the shape operator of φ for $\xi \in B$

Chern-Lashof (1957, 58), Kuiper (1958) :

$$\tau(\varphi) \ \geq \ \beta(M) \ \geq \ b(M)$$

$$b(M) := \sum_{i=0}^{\mathrm{dim} M} b_i(M)$$

$b_i(M)$: the i-th Betti number for any fixed coefficient field

$$\beta(M) := \min_{f \in F} \beta(f)$$

$\beta(f)$: the total number of the critical points of f

F : the set of the Morse functions on M

$\varphi : M \to \mathbb{R}^n$ is called a <u>minimum</u> (or <u>tight</u>) immersion if $\tau(\varphi) = \beta(M)$.

Kobayashi (1967) : He proved that every compact homogeneous Kähler manifold can be embedded into a Euclidean space with a minimum embedding.

Kobayashi (1968) : He gave an explicit construction of minimum embeddings of some symmetric R-spaces.

Takeuchi-Kobayashi (1968) :

$M = G/U$: an R-space

G : a connected real semisimple Lie group without center

U : a parabolic subgroup of G

$\mathfrak{g}$ (resp. $\mathfrak{u}$) : the Lie algebra of G (resp. U)

$^{\exists}Z \in \mathfrak{g}$ s.t. eigenvalues of ad$Z$ are all real and $\mathfrak{u}$ is the direct sum of all eigenspaces corresponding to the non-negative eigenvalues of ad$Z$

K : a maximal compact subgroup of G

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: a Cartan decomposition with $Z \in \mathfrak{p}$

$K_0 := \{k \in K \mid \mathrm{Ad}(k)Z = Z\}$

Then, we have $M = K/K_0$. (Takeuchi)

Takeuchi-Kobayashi defined the map

$$\varphi : M = K/K_0 \to \mathfrak{p} \quad \text{by} \quad \varphi(kK_0) := \mathrm{Ad}(k)(Z),$$

which is a K-equivariant embedding of M into $\mathfrak{p}$. They proved that φ is a minimum embedding, which is nowadays called the <u>standard embedding</u> of an R-space M. They also proved that if M is a symmetric R-space with simple G particularly, then $\varphi(M)$ is a minimal submanifold of a hypersphere in $\mathfrak{p}$.

For the standard embedding $\varphi : M \to \mathfrak{p}$ of an R-space, we have $\tau(\varphi) = \beta(M) = b(M, \mathbb{Z}_2)$, where $b(M, \mathbb{Z}_2)$ denotes the sum of the $\mathbb{Z}_2$-Betti numbers.

4. Antipodal sets

M : a compact symmetric space

$S \subset M$ is called an antipodal set if $s_x(y) = y$ for any $x, y \in S$.

The 2-number $\#_2 M$ of M is defined by

$$\#_2 M := \sup\{\#S \mid S : \text{an antipodal set in } M\} \ (< \infty)$$

An antipodal set S which satisfies $\#S = \#_2 M$ is called a great antipodal set.

e.g. $S = \{x, -x\} \subset S^n$ is a great antipodal set and $\#_2 S^n = 2$.

$e_1, \ldots, e_{n+1}$: o.n.b. of $\mathbb{R}^{n+1}$

$S = \{\langle e_1 \rangle, \ldots, \langle e_{n+1} \rangle\} \subset \mathbb{R}P^n$ is a great antipodal set and $\#_2 \mathbb{R}P^n = n + 1$.

Takeuchi (1989) : He proved that if M is a symmetric R-space, then $\#_2 M = b(M, \mathbb{Z}_2)$, the sum of the $\mathbb{Z}_2$-Betti numbers.

Tanaka-Tasaki (2013) :

If M is a symmetric R-space,

(i) any antipodal set is included in a great antipodal set,

(ii) any two great antipodal sets are congruent.

$S_1,\ S_2 \subset M$ are <u>congruent</u> if $^\exists g \in I_0(M)$ such that $g \cdot S_1 = S_2$.

We proved it by making use of the standard embedding $\varphi : M \to \mathfrak{p}$ of a symmetric R-space M.

An essential point is :

$$s_x(y) = y \iff [x, y] = 0 \quad (x, y \in M)$$

under the identification of M with $\varphi(M)$.

5. Intersection of real forms

M : a Hermitian symmetric space of compact type

A Hermitian manifold M is a Hermitian symmetric space if for each $x \in M$ there exists the symmetry s_x which is an holomorphic isometry.
A symmetric space $M = G/K$ is of compact type iff G is compact and semisimple.

τ : an involutive anti-holomorphic isometry of M

Then, $F(\tau, M) := \{x \in M \mid \tau(x) = x\}$ is connected and a totally geodesic Lagrangian submanifold, which is called a real form of M.

e.g. $\mathbb{R}P^n$ is a real form of $\mathbb{C}P^n$.

M : a Hermitian symmetric space of compact type

τ : an anti-holomorphic isometry of M

$M \times M \ni (x, y) \mapsto (\tau^{-1}(y), \tau(x)) \in M \times M$ is an involutive anti-holomorphic isometry of $M \times M$

A real form $\quad D_\tau(M) := \{(x, \tau(x)) \mid x \in M\} \quad$ of $M \times M$ is called a <u>diagonal real form</u>.

A real form of a Hermitian symmetric space M of compact type is a product of real forms in irreducible factors of M and diagonal real forms determined from irreducible factors of M.

Takeuchi (1984) : Every real form of a Hermitian symmetric space of compact type is a symmetric R-space. Conversely, every symmetric R-space is realized as a real form of a Hermitian symmetric space of compact type. The correspondence is one-to-one.

$M = \mathbb{C}P^1 = S^2$: a Herm. sym. space of compact type
$L = \mathbb{R}P^1 = S^1$: a real form of M

Every real form of S^2 is a great circle.

Any two distinct great circles in S^2 intersect at two points which are antipodal to each other. That is, the intersection is an antipodal set.

The following is a generalization of the fact.

Theorem 1 (Tanaka-Tasaki 2012)
Let L_1, L_2 be real forms of a Hermitian symmetric space of compact type whose intersection is discrete. Then $L_1 \cap L_2$ is an antipodal set of L_1 and L_2.

Furthermore, if L_1 and L_2 are congruent, then $L_1 \cap L_2$ is a great antipodal set, that is, $\#(L_1 \cap L_2) = \#_2 L_1 = \#_2 L_2$.

Theorem 2 (Tanaka-Tasaki 2012)

Let M be an irreducible Hermitian symmetric space of compact type and let L_1, L_2 be real forms of M with $\#_2 L_1 \leq \#_2 L_2$ and we assume that $L_1 \cap L_2$ is discrete.

(1) If $M = G_{2m}(\mathbb{C}^{4m})$ $(m \geq 2)$, L_1 is congruent to $G_m(\mathbb{H}^{2m})$ and L_2 is congruent to $U(2m)$,

$$\#(L_1 \cap L_2) = 2^m < \binom{2m}{m} = \#_2 L_1 < 2^{2m} = \#_2 L_2.$$

(2) Otherwise,

$$\#(L_1 \cap L_2) = \#_2 L_1 \ (\leq \#_2 L_2).$$

M : a compact symmetric space

$o \in M$

$F(s_o, M) = \bigcup_{j=0}^{r} M_j^+$, M_j^+ : a connected component

M_j^+ is called a polar of o. Since $\{o\}$ is a connected component of $F(s_o, M)$, we set $M_0^+ = \{o\}$.

e.g. For a point o in $M = S^n$, $F(s_o, M) = \{o, -o\}$ and $M_0^+ = \{o\}$ and $M_1^+ = \{-o\}$.

For a point $o = \langle e_1 \rangle$ in $M = \mathbb{R}P^n$,

$$F(s_o, M) = \{o\} \cup \{1 - \text{dim subspaces } \subset \langle e_2, \ldots, e_{n+1} \rangle\}$$

and $M_0^+ = \{o\}$ and $M_1^+ \cong \mathbb{R}P^{n-1}$.

If M is a symmetric R-space, $\#_2 M = \sum_{j=0}^{r} \#_2 M_j^+$ (Takeuchi).

Lemma Let M be a Hermitian symmetric space of compact type and $o \in M$.

$$F(s_o, M) = \bigcup_{j=0}^{r} M_j^+, \quad M_j^+ : \text{a polar}$$

(1) If $\dim M_j^+ > 0$, M_j^+ is a Hermitian symmetric space of compact type.

(2) Let L be a real form of M containing o. Then,

$$F(s_o, L) = \bigcup_{j=0}^{r} L \cap M_j^+$$

and if $L \cap M_j^+ \neq \emptyset$, $L \cap M_j^+$ is a real form of M_j^+.

$$\#_2 L = \sum_{j=0}^{r} \#_2(L \cap M_j^+).$$

(3) Let L_1, L_2 be real forms of M with $o \in L_1 \cap L_2$ and we assume that $L_1 \cap L_2$ is discrete. Then,

$$L_1 \cap L_2 = \bigcup_{j=0}^{r} \Big\{(L_1 \cap M_j^+) \cap (L_2 \cap M_j^+)\Big\},$$

$$\#(L_1 \cap L_2) = \sum_{j=0}^{r} \# \Big\{(L_1 \cap M_j^+) \cap (L_2 \cap M_j^+)\Big\}.$$

Based on this Lemma, in the proof of Theorem 1 and Theorem 2 we use an induction on polars.

The classification of real forms in an irreducible Hermitian symmetric space of compact type was given by Leung and Takeuchi. We classified all the possible pairs of real forms in a Hermitian symmetric space M of compact type in the case where M is not irreducible. They are essentially four cases. To investigate the intersection of two real forms is reduced to an irreducible case for three of those four cases. The remaining case is reduced to the case where both real forms are diagonal real forms.

M : a Hermitian symmetric space of compact type (or noncompact type)

$A(M)$: the group of the holomorphic isometries of M

$A_0(M)$: the idenitity component of $A(M)$

It is known that $A_0(M) = I_0(M)$.

Theorem 3 (Tanaka-Tasaki 2013)

Let M be an irreducible Hermitian symmetric space of compact type. Let τ_1, τ_2 be anti-holomorphic isometries of M, which determine diagonal real forms $D_{\tau_1}(M)$, $D_{\tau_2^{-1}}(M)$ in $M \times M$ respectively. We assume that $D_{\tau_1}(M)$ $\cap$ $D_{\tau_2^{-1}}(M)$ is discrete. Then,

(1) if $M = Q_{2m}(\mathbb{C})$ $(m \geq 2)$ and $\tau_2\tau_1 \notin A_0(M)$,

$$\#(D_{\tau_1}(M) \cap D_{\tau_2^{-1}}(M)) = 2m < 2m+2 = \#_2 M,$$

(2) if $M = G_m(\mathbb{C}^{2m})$ $(m \geq 2)$ and $\tau_2\tau_1 \notin A_0(M)$,

$$\#(D_{\tau_1}(M) \cap D_{\tau_2^{-1}}(M)) = 2^m < \binom{2m}{m} = \#_2 M,$$

(3) otherwise,

$$\#(D_{\tau_1}(M) \cap D_{\tau_2^{-1}}(M)) = \#_2 M.$$

It is known by Murakami and Takeuchi that if M is an irreducible Hermitian symmetric space of compact type except for $Q_{2m}(\mathbb{C})$ and $G_m(\mathbb{C}^{2m})$ with $m \geq 2$, then $I(M)/I_0(M) \cong \mathbb{Z}_2$ and $A(M) = A_0(M)$. If M is $Q_{2m}(\mathbb{C})$ or $G_m(\mathbb{C}^{2m})$ with $m \geq 2$, then $I(M)/I_0(M) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $A(M)/A_0(M) \cong \mathbb{Z}_2$.