Fixed point sets of isometries and the intersection of real forms in a Hermitian symmetric space of compact type

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1. Introduction and known results

In S^2 any two distinct great circles intersect two antipodal points. If at least one of circles is a small circle, the intersection could be empty. S^2 is considered as $\mathbb{C}P^1$, which is a Hermitian symmetric space of compact type, and canonically embedded $\mathbb{R}P^1$ is a great circle, which is a real form in $\mathbb{C}P^1$, that is, the fixed point set of an involutive anti-holomorphic isometry. In general, we obtained the following.

Theorem 1.1 (T.-Tasaki 2012) Let M be a Hermitian symmetric space of compact type and let L_1 and L_2 be real forms in M. If $L_1 \cap L_2$ is discrete, $L_1 \cap L_2$ is an antipodal set.

Here an antipodal set is a subset on which the geodesic symmetry at each point acts trivially.

We say L_1 is congruent to L_2 if L_1 is transformed to L_2 by an element in $I_0(M)$, the identity component of the isometry group of M. If $M = G_k(\mathbb{C}^n)$, any real form is

$$G_k(\mathbb{R}^n),\ G_l(\mathbb{H}^m) \text{ if } k = 2l, n = 2m, \text{ or } U(k) \text{ if } n = 2k.$$

So in general, L_1 and L_2 are not necessarily congruent. But if they are congruent, we obtained the following.

Theorem 1.2 (T.-Tasaki 2012) If L_1 and L_2 are real forms in M which are congruent and $L_1 \cap L_2$ is discrete, then $L_1 \cap L_2$ is a great antipodal set of L_1 and L_2.

Here a great antipodal set is an antipodal set with maximal cardinality. The maximal cardinality of the antipodal sets in a compact Riemannian symmetric space M is called the 2-number of M denoted by $\#_2 M$.

For a point o in a compact Riemannian symmetric space, each connected component of the fixed point set of the geodesic symmetry s_o at o is called a polar. By making use of polars, we obtained the following.

Theorem 1.3 (T.-Tasaki 2012) Let M be an irreducible Hermitian symmetric space of compact type and let L_1 and L_2 be real forms in M. Assume that $\#_2 L_1 \leq \#_2 L_2$. If $L_1 \cap L_2$ is discrete, then $L_1 \cap L_2$ is a great antipodal set of L_1 except for the case where

$$M = G_{2m}(\mathbb{C}^{4m}) \ (m \geq 2), \ L_1 \cong G_m(\mathbb{H}^{2m}), \ L_2 \cong U(2m).$$

In this case we have

$$\#(L_1 \cap L_2) = 2^m < \binom{2m}{m} = \#_2 L_1 < 2^{2m} = \#_2 L_2.$$

On the other hand, we have another approach to investigate the intersection of real forms. Let $L_1 = F(\tau_1, M)$ and $L_2 = F(\tau_2, M)$ be real forms, then

$$L_1 \cap L_1 = F(\tau_1, M) \cap F(\tau_2, M) \subset F(\tau_2\tau_1^{-1}, M),$$

where $\tau_2\tau_1^{-1}$ is a holomorphic isometry of M. If $F(\tau_2\tau_1^{-1}, M)$ is discrete, $L_1 \cap L_2$ is discrete. Moreover, if $F(\tau_2\tau_1^{-1}, M)$ is antipodal, $L_1 \cap L_2$ is antipodal. Conversely, if $L_1 \cap L_2$ is discrete, is $F(\tau_2\tau_1^{-1}, M)$ discrete? If $L_1 \cap L_2$ is antipodal, is $F(\tau_2\tau_1^{-1}, M)$ antipodal? We will refer to these problems in Section 3.

It is known that a Hermitian symmetric space M of compact type is realized as an adjoint orbit of a compact semisimple Lie group G:

$$M = \mathrm{Ad}(G)J \subset \mathfrak{g},$$

where J satisfies $(\mathrm{ad}J)^3 = -\mathrm{ad}J$. By making use of the realization we obtained a necessary and sufficient condition for the intersection of two real forms is discrete. When the intersection is discrete, it is an orbit of a certain Weyl group, which will be mentioned in Section 4.

2. Basic notions

Let M be a compact Riemannian symmetric space. A subset $S \subset M$ is called an **antipodal** set if

$$s_x(y) = y \quad \text{for} \quad \text{any } x, y \in S,$$

where s_x denotes the geodesic symmetry at x.

The 2-**number** $\#_2 M$ of M is defined by

$$\#_2 M = \max\{\# S \mid S \subset M \ : \ \text{antipodal set}\}.$$

An antipodal set S is called **great** if $\# S = \#_2 M$. These notions were introduced by Chen-Nagano.

In general, great antipodal sets are not necessarily congruent to each other but for symmetric R-spaces we have the following.

Theorem 2.1 (T.-Tasaki 2013) Let M be a symmetric R-space.

(1) Any antipodal set is included in a great antipodal set.

(2) Any two great antipodal sets are congruent.

Here a symmetric R-space is a compact Riemannian symmetric space which can be realized as a linear isotropy orbit of a Rieman-niam symmetric space of compact type. A Hermitian symmetric space of compact type is a symmetric R-space.

Example $\mathbb{C}P^n$

$e_1, \ldots, e_{n+1}$: a unitary basis of $\mathbb{C}^{n+1}$

$o = \langle e_1 \rangle_{\mathbb{C}} \in \mathbb{C}P^n$

s_o is induced from the reflection $\rho_o : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$

$$\rho_o = \begin{cases} \text{Id} & \text{on } \langle e_1 \rangle_{\mathbb{C}} \\ -\text{Id} & \text{on } \langle e_2, \ldots, e_{n+1} \rangle_{\mathbb{C}} \end{cases}$$

$\{\langle e_1 \rangle_{\mathbb{C}}, \ldots, \langle e_{n+1} \rangle_{\mathbb{C}}\}$ is a great antipodal set and $\#_2 \mathbb{C}P^n = n + 1$.

Let M be a Hermitian symmetric space of compact type. Let τ be an involutive anti-holomorphic isometry of M. Then the fixed point set $F(\tau, M)$ is called a **real form** in M. It is known that a real form is connected totally geodesic compact Lagrangian submanifold. A real form in a Hermitian symmetric space of compact type is a symmetric R-space, and vice versa (Takeuchi).

The classification of real forms in an irreducible Hermitian symmetric space of compact type was given by Leung and Takeuchi. As for the non-irreducible case we have the following.

Theorem 2.2 (T.-Tasaki) A real form in a Hermitian symmetric space M of compact type is a product of real forms in irreducible factors of M and diagonal real forms determined from irreducible factors of M.

Here a diagonal real form is defined as follows.

Let τ be an anti-holomorphic isometry of M. A map

$$M \times M \ni (x, y) \mapsto (\tau^{-1}(y), \tau(x)) \in M \times M$$

is an involutive anti-holomorphic isometry of $M \times M$. The real form determined by the map is

$$D_\tau(M) := \{(x, \tau(x)) \mid x \in M\},$$

which is called a **diagonal real form** determined from M.

The existence of the intersection of two real forms follows the next proposition.

Proposition (Cheng 2002) Let M be a compact Kähler manifold with positive holomorphic sectional curvature. If L_1 and L_2 are totally geodesic compact Lagrangian submanifolds in M, then $L_1 \cap L_2 \neq \emptyset$.

3. Fixed point sets of isometries of a Hermitian symmetric space of compact type

It is known that a Hermitian symmetric space M of compact type is realized as an adjoint orbit

$$M = \mathrm{Ad}(G)J \subset \mathfrak{g},$$

where G is a connected compact semisimple Lie group, $\mathfrak{g}$ is the Lie algebra of G and $J \in \mathfrak{g} - \{0\}$ satisfies $(\mathrm{ad}J)^3 = -\mathrm{ad}J$. Let K be the isotropy subgroup at J. Then the Lie algebra $\mathfrak{k}$ of K is

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid [J, X] = 0\} = \mathrm{Ker}\ \mathrm{ad}J.$$

Let

$$\mathfrak{m} = \{[J, X] \mid X \in \mathfrak{g}\} = \mathrm{Im}\ \mathrm{ad}J,$$

then we have an orthogonal direct sum decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}.$$

$\mathrm{ad}J$ is a complex structure of $\mathfrak{m}$ which can be identified with the tangent space of M at J.

The action of G on M coincides with the action of $I_0(M)$ on M, where $I_0(M)$ denotes the identity component of the isometry group $I(M)$ of M.

Let $A(M)$ denote the group of the holomorphic isometries and $A_0(M)$ denote the identity component of $A(M)$. Then it is known that $I_0(M) = A_0(M)$. Moreover, if M is irreducible,

$$I(M)/A(M) \cong \mathbb{Z}_2$$

and

$$A(M) = A_0(M)$$

except for the cases

$$M = Q_{2m}(\mathbb{C}) \ (m \geq 2), \quad G_m(\mathbb{C}^{2m}) \ (m \geq 2)$$

where

$$A(M)/A_0(M) \cong \mathbb{Z}_2$$

(Murakami, Takeuchi).

Theorem 3.1 (Sánchez 1997, T.-Tasaki 2013)
Let $M = \mathrm{Ad}(G)J$ be a Hermitian symmetric space of compact type. Then, a great antipodal set of M is represented as

$$M \cap \mathfrak{t}$$

for a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$.

If $g \in G$ satisfies

$$\dim\{X \in \mathfrak{g} \mid \mathrm{Ad}(g)X = X\} = \mathrm{rank}(G),$$

g is called a **regular** element.

Theorem 3.2 (T.-Tasaki) Let M be a Hermitian symmetric space of compact type and let $g \in A_0(M)$.

(1) The fixed point set $F(g, M)$ is discrete if and only if g is a regular element.

(2) If $F(g, M)$ is discrete, $F(g, M)$ is a great antipodal set of M.

If we take a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ with $J \in \mathfrak{t}$, then $\mathfrak{t}$ is also a maximal abelian subalgebra of $\mathfrak{g}$.

By using root systems we have the following lemma.

Lemma 3.3 $g \in \exp \mathfrak{t}$ is a regular element if and only if

$$F(\mathrm{Ad}(g), \mathfrak{g}) = \mathfrak{t}.$$

Hence if $g \in \exp \mathfrak{t}$,

$$F(g, M) = F(\mathrm{Ad}(g), \mathfrak{g}) \cap M = \mathfrak{t} \cap M,$$

which is a great antipodal set by Theorem 3.1.

Next, we consider the case where $g \in A(M) - A_0(M)$. The complex hyperquadric $M = Q_{2m}(\mathbb{C})$ $(m \geq 2)$ can be considered as the oriented Grassmann manifold $\tilde{G}_2(\mathbb{R}^{2m+2})$. Then

$$A(M) - A_0(M) = \{g \in O(2m+2) \mid \det g = -1\}.$$

If $g \in A(M) - A_0(M)$,

$$g \sim \begin{bmatrix} R(\theta_1) & & & & \\ & \ddots & & & \\ & & R(\theta_m) & & \\ & & & 1 & \\ & & & & -1 \end{bmatrix},$$

where $R(\theta_i) = \begin{bmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{bmatrix}$ $(1 \leq i \leq m)$.

Theorem 3.4 (T.-Tasaki) Let $M = \tilde{G}_2(\mathbb{R}^{2m+2})$ $(m \geq 2)$ and let $g \in A(M) - A_0(M)$.

(1) $F(g, \tilde{G}_2(\mathbb{R}^{2m+2}))$ is discrete if and only if $R(\theta_i) \neq R(\theta_j)$ for any i and j with $i \neq j$.

(2) When $F(g, \tilde{G}_2(\mathbb{R}^{2m+2}))$ is discrete, $F(g, \tilde{G}_2(\mathbb{R}^{2m+2}))$ is an antipodal set with

$$\#F(g, \tilde{G}_2(\mathbb{R}^{2m+2})) = 2m < 2m + 2 = \#_2\tilde{G}_2(\mathbb{R}^{2m+2}).$$

Since we do not know $A(M) - A_0(M)$ explicitly when

$$M = G_m(\mathbb{C}^{2m}) \ (m \geq 2),$$

the case of $M = G_m(\mathbb{C}^{2m})$ is unsolved.

When $M = G_k(\mathbb{C}^n)$, the complex Grassmann manifold, we obtain a refinement of Theorem 3.2.

Theorem 3.5 (T.-Tasaki) Let $M = G_k(\mathbb{C}^n)$ and let $g \in U(n)$.

(1) $F(g, M)$ is discrete if and only if the multiplicity of each eigenvalue of g is 1.

(2) $F(\tau, M) \cap F(g\tau g^{-1}, M)$ is discrete if and only if $F(g\tau g^{-1}\tau^{-1}, M)$ is discrete.

(3) When $F(\tau, M) \cap F(g\tau g^{-1}, M)$ is discrete, we have

$$F(\tau, M) \cap F(g\tau g^{-1}, M) = F(g\tau g^{-1}\tau^{-1}, M)$$

and it is a great antipodal set of M.

4. The intersection of two real forms in a Hermitian symmetric space of compact type

Let $M = \mathrm{Ad}(G)J \subset \mathfrak{g}$ be the canonical embedding of a Hermitian symmetric space M of compact type. Let $L = F(\tau, M)$ be a real form in M which contains J, where τ is an involutive anti-holomorphic isometry of M.

$$I_\tau : G \to G; \;\; g \mapsto \tau g \tau^{-1}$$

is an involutive automorphism of G. Then $(G, F(I_\tau, G))$ is a compact symmetric pair.

The differential $dI_\tau : \mathfrak{g} \to \mathfrak{g}$ is an involutive automorphism of $\mathfrak{g}$. Let

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}$$

be the direct sum decomposiiton where $\mathfrak{l}$ is $(+1)$-eigenspace of dI_τ and $\mathfrak{p}$ is (-1)-engenspace of dI_τ.

Let K be the isotoropy subgroup at J. Then the Lie algebra $\mathfrak{k}$ of K is

$$\mathfrak{k} = \text{Ker ad}J$$

and set

$$\mathfrak{m} = \text{Im ad}J,$$

then

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$$

is the canonical decomposition corresponding to $M = G/K$. Then $J \in \mathfrak{k} \cap \mathfrak{p}$. We choose a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ so that $J \in \mathfrak{a}$. Let R denote the restricted root system of $(G, F(I_\tau, G))$ with respect to $\mathfrak{a}$.

Now we investigate $L \cap gL$ for $g \in G$.

Since we have a decomposition

$$G = F(I_\tau, G)(\exp \mathfrak{a})F(I_\tau, G),$$

there exit $b_1, b_2 \in F(I_\tau, G)$ and $a \in \exp \mathfrak{a}$ such that $g = b_1 a b_2$. Since $L = F(I_\tau, G)J$,

$$L \cap gL = L \cap b_1 a b_2 L = L \cap b_1 a L = b_1(L \cap aL).$$

Hence, it is enough to consider the case where $g = a = \exp H$ for $H \in \mathfrak{a}$ in order to investigate $L \cap gL$.

$H \in \mathfrak{a}$ is called a **regular** element if $\exp H$ is a regular element in G.

Theorem 4.1 (Ikawa-T.-Tasaki)

(1) $L \cap aL$ for $a = \exp H$ is discrete if and only if H is a regular element.

(2) If $L \cap aL$ is discrete,

$$L \cap aL = M \cap \mathfrak{a} = W(R)J$$

and it is a great antipodal set of L. Here $W(R)$ denotes the Weyl group of R.

Next, we consider the case where two real forms L_1 and L_2 are not congruent. Hereafter we assume that M is irreducible. Let

$$L_i = F(\tau_i, M) \quad (i = 1, 2).$$

As mentioned before, each τ_i defines an involutive automorphism I_{τ_i} of G and we obtain a compact symmetric pair $(G, F(I_{\tau_i}, G))$ and a direct sum decomposition

$$\mathfrak{g} = \mathfrak{l}_i \oplus \mathfrak{p}_i \ (i = 1, 2).$$

By the classification of real forms, it is possible to assume that

$$\tau_1 \tau_2 = \tau_2 \tau_1.$$

Then we have a direct sum decomposition

$$\mathfrak{g} = (\mathfrak{l}_1 \cap \mathfrak{l}_2) \oplus (\mathfrak{p}_1 \cap \mathfrak{p}_2) \oplus (\mathfrak{l}_1 \cap \mathfrak{p}_2) \oplus (\mathfrak{l}_2 \cap \mathfrak{p}_1).$$

We take a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}_1 \cap \mathfrak{p}_2$. Under this situation we obtain a "symmetric triad" $(\tilde{\Sigma}, \Sigma, W)$, which is introduced by Ikawa. Σ is the restricted root system of $(\mathfrak{l}_1 \cap \mathfrak{l}_2) \oplus (\mathfrak{p}_1 \cap \mathfrak{p}_2)$ with respect to $\mathfrak{a}$. W is a certain subset in $\mathfrak{a}$ invariant under $-\mathrm{Id}$. $\tilde{\Sigma} = \Sigma \cup W$ which is an irreducible root system of $\mathfrak{a}$.

Theorem 4.2 (Ikawa-T.-Tasaki)

(1) $L_1 \cap aL_2$ for $a = \exp H$ is discrete if and only if H is a regular element.

(2) If $L_1 \cap aL_2$ is discrete,

$$L_1 \cap aL_2 = M \cap \mathfrak{a} = W(\tilde{\Sigma})J = W(R_1)J \cap \mathfrak{a} = W(R_2)J \cap \mathfrak{a}.$$

By the result, we obtain Theorem 1.3 again.

Moreover, by using the classification of irreducible root systems, we can show that an orbit of the Weyl group through J is two-point homogeneous. Consequently, a great antipodal set of an irreducible Hermitian symmetric space of compact type and the intersection of two real forms in an irreducible Hermitian symmetric space of compact type are two-point homogeneous.

Example $M = \mathbb{C}P^1$, $L = \mathbb{R}P^1$

$$\mathfrak{g} = \mathfrak{su}(2) = \left\{ \begin{bmatrix} ix & y+iz \\ -y+iz & -ix \end{bmatrix} \Big| \; x, y, z \in \mathbb{R} \right\}$$

$$\cong \{(x, y, z) \mid x, y, z \in \mathbb{R}\} = \mathbb{R}^3$$

$$J = \frac{1}{2} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \; (\mathrm{ad}J)^3 = -\mathrm{ad}J$$

$$M = \mathrm{Ad}(SU(2))J = SU(2)/S(U(1) \times U(1)) = \mathbb{C}P^1 \cong S^2 \subset \mathbb{R}^3$$

$$\tau : \mathfrak{su}(2) \to \mathfrak{su}(2); \; X \mapsto -\bar{X}$$

$$\tau(J) = J, \quad \tau(M) = M$$

$$\begin{aligned} F(\tau, M) = F(\tau, \mathfrak{su}(2)) \cap M &\cong \{(x, 0, z) \mid x, z \in \mathbb{R}\} \cap S^2 \subset \mathbb{R}^3 \\ &= \{(\cos\theta, 0, \sin\theta) \mid \theta \in \mathbb{R}\} = S^1 \end{aligned}$$

$$I_\tau : SU(2) \to SU(2);\ \ g \mapsto \tau g \tau^{-1} = \bar{g}$$

$$F(I_\tau, SU(2)) = SO(2)$$

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}$$

$$\mathfrak{l} = \left\{ \begin{bmatrix} 0 & y \\ -y & 0 \end{bmatrix} \Bigm| y \in \mathbb{R} \right\} = \mathfrak{so}(2)$$

$$\mathfrak{p} = \left\{ \begin{bmatrix} ix & iz \\ iz & -ix \end{bmatrix} \Bigm| x, z \in \mathbb{R} \right\}$$

$$\mathfrak{a} = \left\{ \begin{bmatrix} ix & 0 \\ 0 & -ix \end{bmatrix} \Bigm| x \in \mathbb{R} \right\} = \mathbb{R}J$$

$\alpha = 4J$, $R = \{\pm\alpha\} = A_1$, $W(R) = \{\pm 1\}$

For $H \in \mathfrak{a}$, if $\langle \alpha, H \rangle \in \pi\mathbb{Z}$, $S^1 = \mathrm{Ad}(\exp H) S^1$

if $\langle \alpha, H \rangle \notin \pi\mathbb{Z}$, $S^1 \cap \mathrm{Ad}(\exp H) S^1 = \{\pm J\} = W(R)J$