The fixed point set of a holomorphic isometry and the intersection of two real forms in the complex Grassmann manifold

Osamu Ikawa, Makiko Sumi Tanaka and Hiroyuki Tasaki

2014 ICM Satellite Conf. & the 18th Int. Workshop on Real and Complex Submanifolds
NIMS
August 10 – 12, 2014

Contents

1. Introduction

S^2 : 2-dim sphere

$L_1,\ L_2\ \cong\ S^1$: great circles

$L_1 \neq L_2 \ \Rightarrow\ L_1 \cap L_2 = \{\text{two antipodal points}\}$

$S^2 = \mathbb{C}P^1$: Hermitian symmetric space of compact type (HSSCT)

$S^1 = \mathbb{R}P^1$: real form of $\mathbb{C}P^1$

Generally, we have the following.

[T.-Tasaki 2012] Let M be a Hermitian symmetric space of compact type. Let L_1 and L_2 be real forms of M. If $L_1 \cap L_2$ is discrete, it is an antipodal set.

A real form is the fixed point set of an involutive anti-holomorphic isometry of HSSCT M. It is known that a real form is a connected totally geodesic Lagrangian submanifold of M.

e.g. $\mathbb{R}P^n \subset \mathbb{C}P^n$

An antipodal set of compact Riemannian symmetric space M is a subset $A \subset M$ which satisfies $s_x(y) = y$ for any $x, y \in A$. Here s_x is the geodesic symmetry at x.

e.g. $v_1, \dots, v_{n+1}$: o.n.b. of $\mathbb{R}^{n+1}$

$\rightsquigarrow$ $\{\langle v_1 \rangle_{\mathbb{R}}, \dots, \langle v_{n+1} \rangle_{\mathbb{R}}\}$: antipodal set of $\mathbb{R}P^n$

An antipodal set A is great if

$$\#A = \max\{\#B \mid B \subset M : \text{antipodal set}\}.$$

$L_1,\ L_2$: real forms $\rightsquigarrow$ $L_1 \cap L_2 \neq \emptyset$

$L_1,\ L_2$ are congruent if $^{\exists} g \in \mathrm{A}_0(M)$ s.t. $gL_1 = L_2$.
$\mathrm{A}_0(M)$: the identity component of $A(M)$, the group of holomorphic isometries of M

[T.-Tasaki 2012] Let M be a Hermitian symmetric space of compact type. Let L_1 and L_2 be real forms of M which are congruent. If $L_1 \cap L_2$ is discrete, it is a great antipodal set of L_1 and L_2.

Problem When $L_1 \cap L_2$ is discrete ?

M : HSSCT

σ_1, σ_2 : involutive anti-holommorphic isometries

$F(\sigma_i, M) := \{x \in M \mid \sigma_i(x) = x\}$: real form $\quad (i = 1, 2)$

$$(*) \qquad F(\sigma_1, M) \cap F(\sigma_2, M) \subset F(\sigma_2\sigma_1^{-1}, M)$$

Note that $\sigma_2\sigma_1^{-1} = \sigma_2\sigma_1$ is holomorphic.

Problem When $F(\sigma_2\sigma_1^{-1}, M)$ is discrete ? Does the equality hold in $(*)$ when $F(\sigma_2\sigma_1^{-1}, M)$ is discrete ?

In this talk, we will show answers to these questions in the case where $M = G_k(\mathbb{C}^n)$ and L_1, L_2 are congruent to $G_k(\mathbb{R}^n)$.

2. The fixed point set of a holomorphic isometry of the complex Grassmann manifold

$\mathbb{K} = \mathbb{R}, \mathbb{C}$

$G_k(\mathbb{K}^n)$: the set of the k-dim $\mathbb{K}$-subspaces in $\mathbb{K}^n$

$G_k(\mathbb{R}^n) \subset G_k(\mathbb{C}^n)$ by $\langle v_1, \dots, v_k \rangle_{\mathbb{R}} \mapsto \langle v_1, \dots, v_k \rangle_{\mathbb{C}}$

$G_k(\mathbb{C}^n)$: HSSCT

$U(n) \curvearrowright \mathbb{C}^n \quad \rightsquigarrow \quad U(n) \curvearrowright G_k(\mathbb{C}^n)$, coincides with $A_0(G_k(\mathbb{C}^n))$

We give a necessary and sufficient condition that $F(g, G_k(\mathbb{C}^n))$ is discrete for $g \in U(n)$ in two ways.

(1) By linear algebra

$\mathbb{K} = \mathbb{R}, \mathbb{C}$

$W_1, \dots, W_s$: $\mathbb{K}$-subspaces of $\mathbb{K}^n$

$\mathbb{K}^n = W_1 \oplus \cdots \oplus W_s$

$k_1, \ldots, k_s$: positive integers, $k_1 + \cdots + k_s = k$

$G_{k_1}(W_1) \times \cdots \times G_{k_s}(W_s)$

$= \{U_1 \oplus \cdots \oplus U_s \in G_k(\mathbb{C}^n) \mid U_a \in G_{k_a}(W_a)\ (1 \leq a \leq s)\}$

Lemma 2.1

$g \in U(n)$

$\alpha_1, \ldots, \alpha_s$: distinct eigenvalues of g

V_a : the eigenspace of g with eigenvalue α_a $(1 \leq a \leq s)$

$$F(g, G_k(\mathbb{C}^n)) = \bigcup_{\substack{k_1 + \cdots + k_s = k \\ 0 \leq k_a \leq \dim V_a\ (1 \leq a \leq s)}} G_{k_1}(V_1) \times \cdots \times G_{k_s}(V_s)$$

Theorem 2.2

$g \in U(n)$

$F(g, G_k(\mathbb{C}^n))$ is discrete

$\Longleftrightarrow$

(the multiplicity of α_a) $= 1$ $\;(1 \leq {}^\forall a \leq s)$

In this case

$$F(g, G_k(\mathbb{C}^n)) = \{\langle v_{i_1}, \ldots, v_{i_k} \rangle_{\mathbb{C}} \mid 1 \leq i_1 \leq \cdots \leq i_k \leq n\}$$

is a great antipodal set, where v_i $(1 \leq i \leq n)$ is a unit vector of each eigenspace of g.

(2) By the use of realization as adjoint orbit

$$G = SU(n) = \{X \in M(n, \mathbb{C}) \mid X^t \bar{X} = I, \ \mathrm{det} X = 1\}$$

$$\mathfrak{g} = \mathfrak{su}(n) = \{X \in M(n, \mathbb{C}) \mid X = -{}^t\bar{X},\ \mathrm{tr}(X) = 0\}$$

$\langle X, Y \rangle = -\mathrm{tr}(XY)$: Ad(G)-inv inner product on $\mathfrak{su}(n)$

$$J := \sqrt{-1} \begin{bmatrix} (1 - \frac{k}{n})1_k & 0 \\ 0 & -\frac{k}{n}1_{n-k} \end{bmatrix} \in \mathfrak{su}(n)$$

$$\Rightarrow (\mathrm{ad}J)^3 = -\mathrm{ad}J$$

$$M := \mathrm{Ad}(SU(n))J \subset \mathfrak{su}(n)$$

$$\Rightarrow M = SU(n)/S(U(k) \times U(n-k)) = G_k(\mathbb{C}^n)$$

$$\mathfrak{k} := \mathrm{Ker}(\mathrm{ad}J)$$

$$= \left\{ \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \middle| X = -{}^t\bar{X}, Y = -{}^t\bar{Y}, \mathrm{tr}(X) + \mathrm{tr}(Y) = 0 \right\}$$

$$\mathfrak{m} := \mathrm{Im}(\mathrm{ad}J)$$

$$= \left\{ \begin{bmatrix} 0 & Z \\ -{}^t\bar{Z} & 0 \end{bmatrix} \middle| \, Z \in M(k, n-k, \mathbb{C}) \right\}$$

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$: canonical decomposition w.r.t. $e^{\pi \mathrm{ad}J}$

$$\mathfrak{t} := \left\{ \sqrt{-1}\mathrm{diag}(t_1, \ldots, t_n) \middle| \, t_i \in \mathbb{R}, \sum_{i=1}^{n} t_i = 0 \right\} \subset \mathfrak{k}$$

: maximal abelian

$$T := \exp \mathfrak{t} = \left\{ \mathrm{diag}(e^{\sqrt{-1}t_1}, \ldots, e^{\sqrt{-1}t_n}) \middle| \, t_i \in \mathbb{R}, \sum_{i=1}^{n} t_i = 0 \right\}$$

$\subset SU(n)$: maximal torus

Theorem 2.3

$G_k(\mathbb{C}^n) = \mathrm{Ad}(SU(n))J \subset \mathfrak{su}(n)$

$g \in SU(n)$

$F(g, G_k(\mathbb{C}^n))$ is discrete

$\Longleftrightarrow$

${}^{\exists}g_1 \in SU(n),\ {}^{\exists}a \in T$ s.t. $g = g_1 a g_1^{-1}$ and

$a = \exp\sqrt{-1}\mathrm{diag}(t_1, \ldots, t_n),\ t_i - t_j \notin 2\pi\mathbb{Z}\ (1 \leq i < j \leq n)$

In this case $F(g, G_k(\mathbb{C}^n)) = \mathfrak{t} \cap G_k(\mathbb{C}^n)$ is a great antipodal set.

Remark $A(M) = A_0(M)$ when $M \neq G_m(\mathbb{C}^{2m})$ and $A(M)/A_0(M) \cong \mathbb{Z}_2$ when $M = G_m(\mathbb{C}^{2m})$ $(m \geq 2)$.

3. The intersection of two real Grassmann manifolds in the complex Grassmann manifold

$\tau : \mathbb{C}^n \to \mathbb{C}^n,\ \tau(z) = \bar{z}$

$\rightsquigarrow\ \ \tau : G_k(\mathbb{C}^n) \to G_k(\mathbb{C}^n)$: involutive anti-holomorphic isometry

$F(\tau) := F(\tau, G_k(\mathbb{C}^n)) = G_k(\mathbb{R}^n)$

$u \in U(n)$

$uF(\tau) = uG_k(\mathbb{R}^n) = G_k(u\mathbb{R}^n) = F(u\tau u^{-1})$

Lemma 3.1

(1) ${}^{\forall}u \in U(n),\ \ {}^{\exists}z_i \in U(1)\ (1 \leq i \leq n)$,

$^\exists$ positively oriented o.n.b. $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ of $\mathbb{R}^n$ s.t. $uw_i = z_i v_i$ $(1 \leq i \leq n)$, $\det u = z_1 \cdots z_n$

(2) Under (1),

$i, j \in \{1, \ldots, n\}$, $\quad i \sim j \overset{\text{def}}{\Longleftrightarrow} z_i = \pm z_j$

$\{1, \ldots, n\} = N_1 \cup \cdots \cup N_s$

: decomposition to the equivalent classes

$v, w \in \mathbb{R}^n$, $\|v\| = \|w\| = 1$, $z \in \mathbb{C}$, $uw = zv$

$\Longrightarrow$

$1 \leq {}^\exists a \leq s$ s.t.

$$v \in \bigoplus_{i \in N_a} \langle v_i \rangle_{\mathbb{R}}, \quad w \in \bigoplus_{i \in N_a} \langle w_i \rangle_{\mathbb{R}}, \quad z = \pm z_i \ (i \in N_a)$$

Theorem 3.2 [Iriyeh-Sakai-Tasaki]

In $G_k(\mathbb{C}^n)$ $(0 \leq k \leq n)$, for $u \in U(n)$

$$G_k(\mathbb{R}^n) \cap G_k(u\mathbb{R}^n)$$

$$= \bigcup_{\substack{k_1 + \cdots + k_s = k \\ 0 \leq k_a \leq \# N_a \ (1 \leq a \leq s)}} G_{k_1}\left(\bigoplus_{i_1 \in N_1} \langle v_{i_1} \rangle_{\mathbb{R}}\right) \times \cdots \times G_{k_s}\left(\bigoplus_{i_s \in N_s} \langle v_{i_s} \rangle_{\mathbb{R}}\right)$$

$G_k(\mathbb{R}^n) \cap G_k(u\mathbb{R}^n)$ is discrete

$\Longleftrightarrow$

$\# N_a = 1$ for $1 \leq {}^{\forall} a \leq s$

In this case

$G_k(\mathbb{R}^n) \cap G_k(u\mathbb{R}^n) = \{\langle v_{i_1}, \ldots, v_{i_k} \rangle_{\mathbb{C}} \mid 1 \leq i_1 < \cdots < i_k \leq n\}$

is a great antipodal set of $G_k(\mathbb{C}^n)$.

4. The intersection and the fixed point set

$$F(\tau) \cap uF(\tau) = F(\tau) \cap F(u\tau u^{-1}) \subset F(u\tau u^{-1}\tau^{-1})$$

Theorem 4.1

$u \in U(n)$

$F(\tau) \cap uF(\tau)$ is discrete

$\Longleftrightarrow$

$F(u\tau u^{-1}\tau^{-1})$ is discrete.

In this case $F(\tau) \cap uF(\tau) = F(u\tau u^{-1}\tau^{-1})$ and they are great antipodal sets of $G_k(\mathbb{C}^n)$.