Antipodal structure of the intersection of real flag manifolds in a complex flag manifold II

Osamu Ikawa (Kyoto Institute of Technology)
Hiroshi Iriyeh (Ibaraki University)
Takayuki Okuda (Hiroshima University)
Takashi Sakai (Tokyo Metropolitan University & OCAMI)
Hiroyuki Tasaki (University of Tsukuba)

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Introduction

M : homogeneous Kähler manifold

L_1, L_2 : real forms of M

i.e. $\exists \sigma_i$: anti-holomorphic involutive isometry of M $(i = 1, 2)$

s.t. $L_i = \mathrm{Fix}(\sigma_i, M)_0$

totally geodesic Lagrangian submanifold

Problems

1. Is the intersection $L_1 \cap L_2$ discrete?
2. If so, count the intersection number $\#(L_1 \cap L_2)$, and describe the geometric meaning of $\#(L_1 \cap L_2)$.
   Moreover, study the structure of the intersection $L_1 \cap L_2$.

Theorem (Tanaka-Tasaki 2012)

M : Hermitian symmetric space of compact type

$L_1, L_2 \subset M$: real forms, $L_1 \pitchfork L_2$

$\Longrightarrow$ $L_1 \cap L_2$ is an antipodal set of L_1 and L_2.

In addition, if L_1 and L_2 are congruent to each other,

$\Longrightarrow$ $L_1 \cap L_2$ is a great antipodal set of L_1 and L_2.

Theorem (Ikawa-Tanaka-Tasaki 2015)

A necessary and sufficient condition for two real forms in a compact Hermitian symmetric space to intersect transversally is given in terms of the symmetric triad $(\tilde{\Sigma}, \Sigma, W)$.

Theorem (Iriyeh-S.-Tasaki 2013)

1. *Lagrangian Floer homology of two real forms in irreducible Hermitian symmetric spacecs*
2. *Volume estimate of real forms under Hamiltonian deformations*

Antipodal sets of a compact symmetric space

M : compact Riemannian symmetric space

s_x : geodesic symmetry at $x \in M$

Definition (Chen-Nagano 1988)

1. $\mathcal{A} \subset M$: **antipodal set** $\overset{\text{def}}{\Longleftrightarrow}$ $s_x(y) = y$ for all $x, y \in \mathcal{A}$
2. $\#_2 M := \max\{\#\mathcal{A} \mid \mathcal{A} \subset M : \text{antipodal set}\}$ **2-number**
3. $\mathcal{A} \subset M$: **great** antipodal set $\overset{\text{def}}{\Longleftrightarrow}$ $\#\mathcal{A} = \#_2 M$

Theorem (Takeuchi 1989)

M : *symmetic R-space* $\Longrightarrow$ $\#_2 M = \dim H_*(M, \mathbb{Z}_2)$

Example

$\mathbb{R}P^n \subset \mathbb{C}P^n$

$$\mathcal{A} := \{\mathbb{R}e_1, \dots, \mathbb{R}e_{n+1}\} \subset \mathbb{R}P^n \quad \text{great antipodal set}$$

For $u \in U(n+1)$, $\mathbb{R}P^n \pitchfork u\mathbb{R}P^n$ in $\mathbb{C}P^n$

$$\mathbb{R}P^n \cap u\mathbb{R}P^n \cong \{\mathbb{C}e_1, \dots, \mathbb{C}e_{n+1}\} \subset \mathbb{C}P^n$$

$$\#(\mathbb{R}P^n \cap u\mathbb{R}P^n) = n+1 = \#_2\mathbb{R}P^n = \dim H_*(\mathbb{R}P^n, \mathbb{Z}_2)$$

Aim of our work

Generalizing the results on Hermitian symmetric spaces, study the intersection of two real forms in a complex flag manifold.

Complex flag manifolds

G : compact, connected semisimple Lie group

$x_0 (\neq 0) \in \mathfrak{g}$

$$\begin{aligned} M &:= \mathrm{Ad}(G)x_0 \subset \mathfrak{g} \quad : \textbf{complex flag manifold} \\ &\cong G/G_{x_0} \cong G^{\mathbb{C}}/P^{\mathbb{C}} \end{aligned}$$

$$G_{x_0} := \{g \in G \mid \mathrm{Ad}(g)x_0 = x_0\}$$

$$\mathfrak{g}_{x_0} = \{X \in \mathfrak{g} \mid [x_0, X] = 0\}$$

ω : Kirillov-Kostant-Souriau symplectic form on M defined by

$$\omega(X^*_x, Y^*_x) := \langle [X,Y], x \rangle \qquad (x \in M, \quad X, Y \in \mathfrak{g})$$

J : G-invariant complex structure on M compatible with ω

$(\cdot,\cdot) := \omega(\cdot, J\cdot)$: G-invariant Kähler form

Antipodal set of a complex flag manifold (1/2)

For $x \in M$ and $g \in Z(G_{x_0})$, define $s_{x,g} : M \to M$ by

$$s_{x,g}(y) := \mathrm{Ad}(g_x g g_x^{-1}) y \qquad (y \in M),$$

where $g_x \in G$ satisfying $\mathrm{Ad}(g_x)x_0 = x$.

$$\mathrm{Fix}(s_x, M) := \{y \in M \mid s_{x,g}(y) = y \ (\forall g \in Z(G_{x_0}))\}$$

Definition

$\mathcal{A} \subset M$: **antipodal set** $\overset{\text{def}}{\Longleftrightarrow}$ $y \in \mathrm{Fix}(s_x, M)$ for all $x, y \in \mathcal{A}$

Note: This definition is equivalent to the notion of an antipodal set of M defined using k-symmetric structure on M, in 2012. When M is a Hermitian symmetric space, it is also equivalent to the notion of an antipodal set introduced by Chen-Nagano.

Antipodal set of a complex flag manifold (2/2)

Proposition

For any $x \in M$,

$$\mathrm{Fix}(s_x, M) = \{y \in M \mid [x, y] = 0\}.$$

Theorem 1 (Iriyeh-S.-Tasaki)

$\mathcal{A} \subset M$: maximal antipodal set

$\Longrightarrow$ $\exists \mathfrak{t} \subset \mathfrak{g}$: maximal abelian subalgebra s.t.

$$\mathcal{A} = M \cap \mathfrak{t}.$$

Hence $\mathcal{A}$ is an orbit of the Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{t}$.
Maximal antipodal sets of M are conjugate to each other by G.

Real flag manifolds in a complex flag manifold

(G, K) : symmetric pair of compact type

θ : involution of G s.t. $\mathrm{Fix}(\theta, G)_0 \subset K \subset \mathrm{Fix}(\theta, G)$

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

$x_0 (\neq 0) \in \mathfrak{p}$

$$\begin{array}{ccccccl} L & := & \mathrm{Ad}(K)x_0 & \subset & \mathfrak{p} & & : \textbf{real flag manifold, } R\textbf{-space} \\ \cap & & \cap & & \cap & & \\ M & := & \mathrm{Ad}(G)x_0 & \subset & \mathfrak{g} & & : \textbf{complex flag manifold, } C\textbf{-space} \\ & \cong & G/G_{x_0} & \cong & G^{\mathbb{C}}/P^{\mathbb{C}} & & \end{array}$$

$\mathfrak{g}' := \mathfrak{k} + \sqrt{-1}\mathfrak{p}$ non-compact real form of $\mathfrak{g}^{\mathbb{C}}$

σ : complex conjugation of $\mathfrak{g}^{\mathbb{C}}$ w.r.t. $\mathfrak{g}'$

$\tilde{\sigma}$: anti-holomorphic involution on M.

$$L = M \cap \mathfrak{p} \cong K/K_{x_0} \cong G'/(G' \cap P^{\mathbb{C}})$$

The intersection of real flag manifolds

$(G, K_1), (G, K_2)$: symmetric pairs of compact type
θ_1, θ_2 : involutions of G

$$\mathfrak{g} = \mathfrak{k}_1 + \mathfrak{p}_1 = \mathfrak{k}_2 + \mathfrak{p}_2,$$

$x_0(\neq 0) \in \mathfrak{p}_1 \cap \mathfrak{p}_2$

$$L_1 := \mathrm{Ad}(K_1)x_0, \quad L_2 := \mathrm{Ad}(K_2)x_0 \ \subset \ M := \mathrm{Ad}(G)x_0$$

For $g \in G$, we consider the intersection of $L_1 \cap \mathrm{Ad}(g)L_2$ in M.

$\mathfrak{a}$: maximal abelian subspace of $\mathfrak{p}_1 \cap \mathfrak{p}_2$ containing x_0
$A := \exp \mathfrak{a} \subset G$: toral subgroup

Then $G = K_1 A K_2$, i.e. $g = g_1 a g_2 \ (g_1 \in K_1, g_2 \in K_2, a \in A)$

$$L_1 \cap \mathrm{Ad}(g)L_2 = L_1 \cap \mathrm{Ad}(g_1 a g_2)L_2 = \mathrm{Ad}(g_1)(L_1 \cap \mathrm{Ad}(a)L_2)$$

Symmetric triads

Hereafter we assume $\theta_1\theta_2 = \theta_2\theta_1$.

$$\mathfrak{g} = (\mathfrak{k}_1 \cap \mathfrak{k}_2) + (\mathfrak{p}_1 \cap \mathfrak{p}_2) + (\mathfrak{k}_1 \cap \mathfrak{p}_2) + (\mathfrak{p}_1 \cap \mathfrak{k}_2).$$

$$\mathfrak{g}_{12} := (\mathfrak{k}_1 \cap \mathfrak{k}_2) + (\mathfrak{p}_1 \cap \mathfrak{p}_2), \qquad \mathfrak{k}_{12} := (\mathfrak{k}_1 \cap \mathfrak{k}_2)$$

Then $(\mathfrak{g}_{12}, \mathfrak{k}_{12}, d\theta_1 = d\theta_2)$ is an orthogonal symmetric Lie algebra.

For $\lambda \in \mathfrak{a} \subset \mathfrak{p}_1 \cap \mathfrak{p}_2$

$$\mathfrak{p}_\lambda := \{X \in \mathfrak{p}_1 \cap \mathfrak{p}_2 \mid [H,[H,X]] = -\langle \lambda, H\rangle^2 X \ (H \in \mathfrak{a})\}$$
$$V_\lambda := \{X \in \mathfrak{p}_1 \cap \mathfrak{k}_2 \mid [H,[H,X]] = -\langle \lambda, H\rangle^2 X \ (H \in \mathfrak{a})\}$$

$$\Sigma := \{\lambda \in \mathfrak{a} \setminus \{0\} \mid \mathfrak{p}_\lambda \neq \{0\}\}$$
$$W := \{\lambda \in \mathfrak{a} \setminus \{0\} \mid V_\lambda \neq \{0\}\}$$
$$\tilde{\Sigma} := \Sigma \cup W$$

$(\tilde{\Sigma}, \Sigma, W)$: **symmetric triad** introduced by Ikawa

Hermann actions

$$\mathfrak{a}_{\mathrm{reg}} := \bigcap_{\substack{\lambda \in \Sigma \\ \alpha \in W}} \left\{ H \in \mathfrak{a} \;\middle|\; \langle \lambda, H \rangle \notin \pi\mathbb{Z}, \langle \alpha, H \rangle \notin \frac{\pi}{2} + \pi\mathbb{Z} \right\}$$

P : cell, a connected component of $\mathfrak{a}_{\mathrm{reg}}$

$$\begin{array}{ccc} & K_2 \times K_1 \curvearrowright G & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ K_2 \curvearrowright G/K_1 & & K_2\backslash G \curvearrowleft K_1 \\ \searrow & & \swarrow \\ & K_2\backslash G/K_1 \cong \overline{P} & \end{array}$$

Hermann actions

Proposition (Ikawa)

For $a = \exp H$ $(H \in \mathfrak{a})$, orbits $K_2 a K_1 \subset G$, $K_2 \pi_1(a) \subset G/K_1$, $\pi_2(a) K_1 \subset K_2\backslash G$ are regular if and only if $H \in \mathfrak{a}_{\mathrm{reg}}$.

The structure of the intersection

$$\mathfrak{a}_{\mathrm{reg}} := \bigcap_{\substack{\lambda \in \Sigma \\ \alpha \in W}} \left\{ H \in \mathfrak{a} \;\middle|\; \langle \lambda, H \rangle \notin \pi\mathbb{Z}, \langle \alpha, H \rangle \notin \frac{\pi}{2} + \pi\mathbb{Z} \right\}$$

$W(\tilde{\Sigma})$: Weyl group of the root system $\tilde{\Sigma}$ of $\mathfrak{a}$

$\mathfrak{a}_i$: maximal abelian subspace of $\mathfrak{p}_i$ containing $\mathfrak{a}$ $(i = 1, 2)$

$W(R_i)$: Weyl group of the restricted root system R_i of $(\mathfrak{g}, \mathfrak{k}_i)$ w.r.t. $\mathfrak{a}_i$

Theorem (Ikawa-Iriyeh-Okuda-S.-Tasaki)

For $a = \exp H$ $(H \in \mathfrak{a})$, the intersection $L_1 \cap \mathrm{Ad}(a)L_2$ is discrete if and only if $H \in \mathfrak{a}_{\mathrm{reg}}$. Moreover, if $L_1 \cap \mathrm{Ad}(a)L_2$ is discrete, then

$$L_1 \cap \mathrm{Ad}(a)L_2 = W(\tilde{\Sigma})x_0 = W(R_1)x_0 \cap \mathfrak{a} = W(R_2)x_0 \cap \mathfrak{a},$$

in particular, $L_1 \cap \mathrm{Ad}(a)L_2$ is an antipodal set of M.

Example

$$(G, K_1, K_2) = (SU(2n), SO(2n), Sp(n))$$

$$\theta_1(g) = \bar{g}, \quad \theta_2(g) = J_n \bar{g} J_n^{-1} \quad (g \in G) \quad \text{where} \quad J_n := \begin{bmatrix} O & I_n \\ -I_n & O \end{bmatrix}$$

$$\mathfrak{p}_1 \cap \mathfrak{p}_2 = \left\{ \begin{bmatrix} \sqrt{-1}X & \sqrt{-1}Y \\ -\sqrt{-1}Y & \sqrt{-1}X \end{bmatrix} \;\middle|\; \begin{array}{l} X, Y \in M_n(\mathbb{R}), \ \mathrm{trace} X = 0 \\ {}^tX = X, \ {}^tY = -Y \end{array} \right\}$$

Fix a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}_1 \cap \mathfrak{p}_2$ as

$$\mathfrak{a} = \left\{ H = \begin{bmatrix} \sqrt{-1}X & O \\ O & \sqrt{-1}X \end{bmatrix} \;\middle|\; \begin{array}{l} X = \mathrm{diag}(t_1, \ldots, t_n), \\ t_1, \ldots, t_n \in \mathbb{R}, \ t_1 + \cdots + t_n = 0 \end{array} \right\}$$

Then

$$\widetilde{\Sigma} = \Sigma = W = \{\pm(e_i - e_j) \mid 1 \leq i < j \leq n\}$$

where $e_i - e_j \in \mathfrak{a}$ $(i \neq j)$ is defined by $\langle e_i - e_j, H \rangle = t_i - t_j$.

$$x_0 = \begin{bmatrix} \sqrt{-1}X & O \\ O & \sqrt{-1}X \end{bmatrix} \in \mathfrak{a}$$

where $X = \mathrm{diag}(x_1 I_{n_1}, \ldots, x_{r+1} I_{n_{r+1}})$ and x_i are distinct real numbers satisfying $n_1 x_1 + \cdots + n_{r+1} x_{r+1} = 0$.

$$L_1 = \mathrm{Ad}(K_1)x_0 \cong F^{\mathbb{R}}_{2n_1,\ldots,2n_r}(\mathbb{R}^{2n})$$
$$L_2 = \mathrm{Ad}(K_2)x_0 \cong F^{\mathbb{H}}_{n_1,\ldots,n_r}(\mathbb{H}^{n})$$
$$M = \mathrm{Ad}(G)x_0 \cong F^{\mathbb{C}}_{2n_1,\ldots,2n_r}(\mathbb{C}^{2n})$$

$\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$

$n, n_1, \ldots, n_r$ satisfying $n_{r+1} := n - (n_1 + \cdots + n_r) > 0$

$$F^{\mathbb{K}}_{n_1,\ldots,n_r}(\mathbb{K}^n) = \left\{ (V_1, \ldots, V_r) \;\middle|\; \begin{array}{l} V_j \text{ is a } \mathbb{K}\text{-subspace of } \mathbb{K}^n, \\ \dim_{\mathbb{K}} V_j = n_1 + \cdots + n_j, \\ V_1 \subset V_2 \subset \cdots \subset V_r \subset \mathbb{K}^n \end{array} \right\}$$

$$a = \exp H, \quad H = \begin{bmatrix} \sqrt{-1}Y & O \\ O & \sqrt{-1}Y \end{bmatrix} \in \mathfrak{a}$$

where $Y = \mathrm{diag}(t_1, \dots, t_n)$ and $t_1, \dots, t_n \in \mathbb{R}$ which satisfy $t_1 + \cdots + t_n = 0$. By our theorem,

$L_1 \cap \mathrm{Ad}(a)L_2$ is discrete

$$\iff \quad H \in \mathfrak{a}_{\mathrm{reg}} = \left\{ H \in \mathfrak{a} \;\middle|\; \langle e_i - e_j, H \rangle \notin \frac{\pi}{2}\mathbb{Z} \ (1 \le i < j \le n) \right\}$$

$$L_1 \cap \mathrm{Ad}(a)L_2 = W(\tilde{\Sigma})x_0 = W(R_1)x_0 \cap \mathfrak{a} = W(R_2)x_0 \cap \mathfrak{a}$$

In this case, a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}_1 \cap \mathfrak{p}_2$ is also a maximal abelian subspace in $\mathfrak{p}_2$, i.e. $\mathfrak{a} = \mathfrak{a}_2$ and $\widetilde{\Sigma} = R_2$.

We shall express the intersection in the flag model $F^{\mathbb{C}}_{2n_1,\dots,2n_r}(\mathbb{C}^{2n})$.

$v_1, \dots, v_{2n}$: standard basis of $\mathbb{C}^{2n}$

$W_i := \langle v_i, v_{n+i} \rangle_{\mathbb{C}} = \langle v_i \rangle_{\mathbb{H}}$ $(1 \leq i \leq n)$

Proposition

For $a = \exp H$ $(H \in \mathfrak{a}_{\mathrm{reg}})$,

$$
\begin{aligned}
&F^{\mathbb{R}}_{2n_1,\dots,2n_r}(\mathbb{R}^{2n}) \cap a F^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n) \\
&= \{(W_{i_1} \oplus \cdots \oplus W_{i_{n_1}}, W_{i_1} \oplus \cdots \oplus W_{i_{n_1+n_2}}, \cdots \\
&\qquad \cdots, W_{i_1} \oplus \cdots \oplus W_{i_{n_1+\cdots+n_r}}) \\
&\quad \mid 1 \leq i_1 < \cdots < i_{n_1} \leq n, 1 \leq i_{n_1+1} < \cdots < i_{n_1+n_2} \leq n, \dots, \\
&\quad 1 \leq i_{n_1+\cdots+n_{r-1}+1} < \cdots < i_{n_1+\cdots+n_r} \leq n, \\
&\quad \#\{i_1, \dots, i_{n_1+\cdots+n_r}\} = n_1 + \cdots + n_r\},
\end{aligned}
$$

which is an antipodal set of $F^{\mathbb{C}}_{2n_1,\dots,2n_r}(\mathbb{C}^{2n})$.

Corollary

For $g \in SU(2n)$, *if* $F^{\mathbb{R}}_{2n_1,\dots,2n_r}(\mathbb{R}^{2n})$ *and* $gF^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n)$ *intersect transversally in* $F^{\mathbb{C}}_{2n_1,\dots,2n_r}(\mathbb{C}^{2n})$, *then*

$$
\begin{aligned}
&\#\big(F^{\mathbb{R}}_{2n_1,\dots,2n_r}(\mathbb{R}^{2n}) \cap gF^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n)\big) \\
&= \#_I(F^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n)) = \dim H_*(F^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n), \mathbb{Z}_2) \\
&= \frac{n!}{n_1!n_2!\cdots n_{r+1}!} \\
&< \#_I(F^{\mathbb{R}}_{2n_1,\dots,2n_r}(\mathbb{R}^{2n})) = \dim H_*(F^{\mathbb{R}}_{2n_1,\dots,2n_r}(\mathbb{R}^{2n}), \mathbb{Z}_2) \\
&= \#_k(F^{\mathbb{C}}_{2n_1,\dots,2n_r}(\mathbb{C}^{2n})) = \dim H_*(F^{\mathbb{C}}_{2n_1,\dots,2n_r}(\mathbb{C}^{2n}), \mathbb{Z}_2) \\
&= \frac{(2n)!}{(2n_1)!(2n_2)!\cdots(2n_{r+1})!}.
\end{aligned}
$$

Further problems

1. Study the intersection of two real flag manifolds in the case where $\theta_1\theta_2 \neq \theta_2\theta_1$.
2. Calculate Lagrangian Floer homologies of pairs of real flag manifolds in complex flag manifolds.
3. Determine Hamiltonian volume minimizing properties of all real forms in irreducible Hermitian symmetric spaces, more generally, in complex flag manifolds.

Thank you very much for your attention

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