Antipodal sets of compact Riemannian symmetric spaces

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1. Introduction

{Riemannian manifolds}

$\bigcup$

{Riemannian homogeneous spaces}

$\bigcup \qquad M = G/K,\ G = I(M)$

{(connected) Riemannian symmetric spaces}

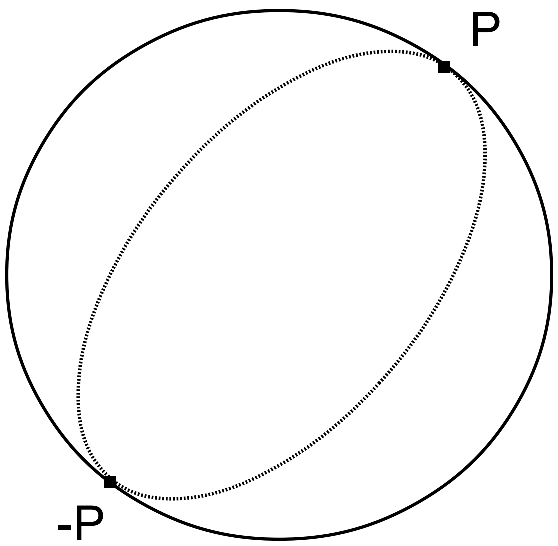
${}^\forall x \in M,\ {}^\exists s_x$: geodesic symmetry

$\langle s_x \mid x \in M \rangle \curvearrowright M$ transitively (if M connected)

e.g. $E^n,\ S^n,\ H^n,\ \mathbb{K}P^n, G_k(\mathbb{K}^n)\ (\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H})$, $U(n)$, etc.

$\gamma(t)$: geodesic, $\gamma(0) = x$

$s_x(\gamma(t)) = \gamma(-t)$



Antipodal sets in S^n

M : Riem. sym. sp., $A \subset M$

A : antipodal set $\overset{\text{def}}{\Longleftrightarrow} {}^\forall x, y \in A,\ s_x(y) = y$

e.g. $M = \mathbb{R}P^n$

$u_1, \ldots, u_{n+1}$: o.n.b. in $\mathbb{R}^{n+1}$, $\langle u_i \rangle := \mathbb{R}u_i$

$s_{\langle u_i \rangle}(v) = v \Leftrightarrow v = \langle u_i \rangle$ or $v \subset \langle u_i \rangle^\perp$ $(v \in \mathbb{R}P^n)$

$\{\langle u_1 \rangle, \ldots, \langle u_{n+1} \rangle\}$ antipodal set

A : antipodal set $\Rightarrow |A| < \infty$

M : noncompact irr. Riem. sym. sp.

$\Rightarrow {}^\forall A$: antipodal set, $|A| = 1$

A : <u>great</u> antipodal set $\overset{\text{def}}{\Longleftrightarrow}$

$|A| = \max\{|B| \mid B \subset M \text{ antipodal}\}$

$=: \#_2 M$ <u>2-number</u> (Chen-Nagano)

- M : compact Lie gp. $\Rightarrow$ $\#_2 M = 2^{r_2(M)}$

  $r_2(M)$ 2-rank (Borel-Serre)

  $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \subset M$ ($r_2(M)$-times $\mathbb{Z}_2$)

- M : symmetric R-space

$\Rightarrow$ $\#_2 M = SB(M, \mathbb{Z}_2)$ (Takeuchi)

- M : Hermitian sym. sp. of compact type

L_1, L_2 : real form, $L_1 \cap L_2$ discrete

$\Rightarrow$ $L_1 \cap L_2$ (great) antipodal set (if $L_1 \cong L_2$)

(T. -Tasaki)

- Relations to designs (Okuda-Kurihara)

An antipodal set seems to give a "nice" placement of finite points in a symmetric space.

2. Preliminaries

M : connected Riem. mfd

M : Riem. sym. sp. $\overset{\text{def}}{\Longleftrightarrow}$

$^\forall x \in M$, $^\exists s_x$: isometry s.t. (i) $s_x \circ s_x = \mathrm{id}$,

(ii) x isolated fixed point of s_x

s_x geodesic symmetry

A : antipodal set $\overset{\text{def}}{\Longleftrightarrow}$ $^\forall x, y \in A,\ s_x(y) = y$

A : great antipodal set $\overset{\text{def}}{\Longleftrightarrow}$ $|A| = \#_2 M$

$P = G/K$: Riem. sym. sp. of compact type

$G = I(P)_0$, $K = \{g \in G \mid g(o) = o\}$, $o := eK$

$\sigma : G \ni g \mapsto s_o g s_o^{-1} \in G$ involutive autom.

$d\sigma$ involutive autom. of Lie algebra $\mathfrak{g}$

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $d\sigma = \mathrm{id}$ on $\mathfrak{k}$, $d\sigma = -\mathrm{id}$ on $\mathfrak{p} \overset{\mathrm{id}}{=} T_o P$

$K \curvearrowright \mathfrak{p}$ linear isotropy action

$\xi (\neq 0) \in \mathfrak{p}$, $(\mathrm{ad}\xi)^3 = -\mathrm{ad}\xi$

$\Rightarrow K\xi \subset \mathfrak{p}$ <u>symmetric R-space</u>

M : Hermitian sym. sp. of compact type

τ : involutive anti-holo. isometry

Fix(τ) <u>real form</u>

$\rightsquigarrow$ conn. totally geodesic Lagrangian submfd

real form $\Leftrightarrow$ sym. R-space (Takeuchi)

- $\mathbb{R}P^n$ real form of $\mathbb{C}P^n$
- Real forms of $G_{2n}(\mathbb{C}^{4n})$:

  $G_{2n}(\mathbb{R}^{4n}),\ G_n(\mathbb{H}^{2n}),\ U(2n)$
- $U(n)$ sym. R-sp. but not $SU(n)$ if $n \geq 3$

3. Antipodal sets of symmetric R-spaces

Consider the following fundamental properties of antipodal sets:

(A) Any antipodal set is included in some great antipodal set.

(B) Any two great antipodal sets are congruent.

Theorem 1 (T.-Tasaki 2013) For a symmetric R-space (A) and (B) hold.

- A maximal antipodal set is not necessarily a great antipodal set.

e.g. ${}^{\exists}A \subset \mathrm{Ad}(SU(4)) \cong SU(4)/\mathbb{Z}_4$ maximal but not great.

- ${}^{\exists}$ great antipodal sets which are not congruent in the oriented Grassmann manifold (Tasaki).

$\mathfrak{g}$: compact semisimple Lie algebra

$G = \mathrm{Int}(\mathfrak{g})$ cpt semisimple Lie gp. w/o center

$G \curvearrowright \mathfrak{g}$ adjoint action

$J\ (\neq 0) \in \mathfrak{g}$, $(\mathrm{adJ})^3 = -\mathrm{adJ}$

$M = GJ$ Herm. sym. sp. of cpt type

adJ $\rightsquigarrow$ complex structure

$-B$ (B : Killing form) $\rightsquigarrow$ Hermitian metric

Conversely, every Herm. sym. sp. of cpt type is obtained in this way.

Lemma 2 $X, Y \in M$,

$s_X(Y) = Y \;\Leftrightarrow\; [X, Y] = 0$

Proposition 3 Let $A \subset M$ be a great antipodal set. Then there exists a maximal abelian subalgebra $\mathfrak{t} \subset \mathfrak{g}$ such that $A = M \cap \mathfrak{t}$. In particular, A is an orbit of the Weyl group of G. The conditions (A) and (B) hold.

τ : involutive anti-holo. isometry of M

$L = \mathrm{Fix}(\tau)$ real form

$I_\tau : G \ni g \mapsto \tau g \tau^{-1} \in G$ inv. autom. of G

dI_τ inv. autom. of $\mathfrak{g}$

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $dI_\tau = \mathrm{id}$ on $\mathfrak{k}$, $dI_\tau = -\mathrm{id}$ on $\mathfrak{p}$

K : conn. Lie subgroup of G with $\mathrm{Lie}(K) = \mathfrak{k}$

(G, K) symmetric pair $\rightsquigarrow$ restricted root system

$R \rightsquigarrow$ Weyl group $W(R)$

Lemma 4 $L = \mathfrak{p} \cap M$

Theorem 5 (T.-Tasaki 2013) Let $A \subset L$ be a great antipodal set. Then there exists a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ such that $A = M \cap \mathfrak{a}$. In particular, A is an orbit of $W(R)$. The conditions (A) and (B) hold.

Since every symmetic R-sp. is a real form of some Herm. sym. sp. of cpt type, we obtain Theorem 1.

4. Intersection of two real forms

M : H.s.s. of cpt type

Fact L_1, L_2 : real form $\Rightarrow$ $L_1 \cap L_2 \neq \emptyset$

Theorem 6 (T.-Tasaki 2012)

Let M be a H.s.s. of cpt type and L_1, L_2 be real forms of M. If $L_1 \cap L_2$ is discrete, $L_1 \cap L_2$ is an antipodal set of L_1 and L_2. Moreover, if L_1 and L_2 are congruent, $L_1 \cap L_2$ is a great antipodal set of L_1 and L_2.

$M = GJ \subset \mathfrak{g}$: H.s.s. of cpt type

L : real form of M, $J \in L$

τ : inv. anti-holo. isom. with $L = \mathrm{Fix}(\tau)$

$I_\tau \rightsquigarrow \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\mathrm{Lie}(K) = \mathfrak{k}$

$\mathfrak{a} \subset \mathfrak{p}$: maximal abelian subspace

$A = \exp \mathfrak{a} \subset G$ torus

$Ao \subset G/K$, $o = eK$ maximal torus

Lemma 7 $G = KAK$

$g \in G, \quad g = k_1 a k_2 \ (k_1, k_2 \in K, \ a \in A)$

$L \cap gL = L \cap k_1 a k_2 L = k_1(k_1^{-1} L \cap a k_2 L)$

$= k_1(L \cap aL)$

Theorem 8 (Ikawa-T.-Tasaki to appear)
Let L be a real form of M and $a = \exp H$ ($H \in \mathfrak{a}$). Then $L \cap aL$ is discrete iff H is a regular element. In this case $L \cap aL = M \cap \mathfrak{a} = W(R)J$.

$H \in \mathfrak{a}$ regular $\Leftrightarrow$ $\mathfrak{p} \cap a\mathfrak{p} = \mathfrak{a}$

L_1, L_2 : real form of irr. M, not congruent

$J \in L_1 \cap L_2$

τ_1, τ_2 : inv. anti-holo. isom.

$L_i = \mathrm{Fix}(\tau_i) \quad (i = 1, 2)$

Fact $\tau_1 \tau_2 = \tau_2 \tau_1$

$I_{\tau_i} \rightsquigarrow \mathfrak{g} = \mathfrak{k}_i \oplus \mathfrak{p}_i$, $\mathrm{Lie}(K_i) = \mathfrak{k}_i$ $(i = 1, 2)$

$\mathfrak{g} = \mathfrak{k}_1 \cap \mathfrak{k}_2 \oplus \mathfrak{k}_1 \cap \mathfrak{p}_2 \oplus \mathfrak{p}_1 \cap \mathfrak{k}_2 \oplus \mathfrak{p}_1 \cap \mathfrak{p}_2$

$\mathfrak{a} \subset \mathfrak{p}_1 \cap \mathfrak{p}_2$: maximal abelian subspace

$A = \exp \mathfrak{a} \subset G$ torus

Lemma 9 (Heintze-Palais-Terng -Thorbergsson)

$G = K_1 A K_2$

$g \in G,\ \ g = k_1 a k_2\ (k_1 \in K_1,\ a \in A, k_2 \in K_2)$

$L_1 \cap gL_2 = L_1 \cap k_1 a k_2 L_2 = k_1(k_1^{-1} L_1 \cap a k_2 L_2)$

$= k_1(L_1 \cap aL_2)$

Theorem 10 (Ikawa-T.-Tasaki to appear)

Let L_1, L_2 be real forms of an irreducible H.s.s. M of cpt type and $a = \exp H$ $(H \in \mathfrak{a})$. Then

$L_1 \cap aL_2$ is discrete iff H is a regular element. In this case $L_1 \cap aL_2 = W(R_1)J = W(R_2)J$.

$H \in \mathfrak{a}$ regular $\Leftrightarrow$ $\mathfrak{p}_1 \cap a\mathfrak{p}_2 = \mathfrak{a}$

To prove Thm 10 we use "symmetric triads" introduced by Ikawa in 2011. Since a symmetric triad is a notion which generalizes an irreducible restricted root system, we need the assumption that M is irreducible in Thm 10.

Theorem 11 (Iriyeh-Sakai-Tasaki 2013) Let L_1, L_2 be real forms of a H.s.s. M of cpt type which intersect discretely. Assume that M is monotone as a symplectic manifold and the minimal Maslov numbers of L_1 and L_2 are greater than or equal to 3. Then the Lagrangian Floer homology $HF(L_1, L_2 : \mathbb{Z}_2)$ is:

$$HF(L_1, L_2 : \mathbb{Z}_2) \cong \bigoplus_{p \in L_1 \cap L_2} \mathbb{Z}_2[p].$$

M irreducible $\Rightarrow$ the assumptions for M, L_1, L_2 are satisfied

$p \in L_1 \cap L_2 \Rightarrow$

- s_p holo. isom.
- $s_p(L_i) = L_i \ (i = 1, 2)$
- $s_p(q) = q \quad ({}^\forall q \in L_1 \cap L_2)$

u : J-holo. strip connecting p to q

$\Rightarrow s_p \circ u$ J-holo. strip connecting p to q

$\rightsquigarrow \partial = 0$