Antipodal sets of compact Riemannian symmetric spaces

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1. Introduction

M : a Riemannian symmetric space

s_x : the geodesic symmetry at x

i.e. s_x is an isometry of M, $s_x^2 =$ id and x is an isolated fixed point of s_x

$S \subset M$: a subset

S : <u>an antipodal set</u> $\Leftrightarrow {}^\forall x, y \in S,\ s_x(y) = y$

<u>the 2-number</u> of M

$\#_2 M :=$ max$\{|S| \mid S \subset M$ antipodal set$\}$

S : a <u>great</u> antipodal set $\Leftrightarrow |S| = \#_2 M$

G : a compact Lie group

p : a prime number

the p-rank of G = the maximal possible rank of the subgroup $\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ of G (Borel-Serre)

$r_2(G) :=$ the 2-rank of G, $\#_2 G = 2^{r_2(G)}$

$M \subset N$: totally geodesic $\Rightarrow \#_2 M \leq \#_2 N$ (Chen-Nagano) M : cpt. conn.

$\#_2 M \geq \chi(M)$, $\chi(M)$: the Euler number

"=" if M : Herm. sym. sp. of cpt. type

(Takeuchi) M **: a sym.** R**-sp.**

$$\Rightarrow \#_2 M = \sum_{k=0}^{\dim M} b_k(M;\mathbb{Z}_2)$$

A sym. R**-sp. is a real form** L **of some Herm. sym. sp.** M **of cpt. type, and vice versa.**

$^\exists\tau$ **: invol. anti-holo. isom. of** M

$$L = F(\tau, M) := \{x \in M \mid \tau(x) = x\}$$

(T.-Tasaki) M **: a Herm. sym. of cpt. type,** L_1, L_2 **: real forms of** M**,** $L_1 \pitchfork L_2$

$\Rightarrow L_1 \cap L_2$ **: an antipodal set of** L_i $(i = 1, 2)$

Application : calculation of Lagrangian Floer homology (Iriyeh-Sakai-Tasaki)

Fundamental problem : classification of maximal antipodal sets

2. Antipodal sets of symmetric R-spaces

Theorem 1 (T.-Tasaki 2013)

In a symmetric R-space M (i) any antipodal set is included in a great antipodal set, (ii) any two great antipodal sets are ($I_0(M)$-) congruent. (iii) a great antipodal set is an orbit of "the Weyl group".

M is a real form of $\hat{M}$

$\hat{M}$: Herm. sym. sp. of cpt. type

i.e., τ : inv. anti-holo. isometry of $\hat{M}$

$M = F(\tau, \hat{M})$

$\hat{M} = \mathrm{Ad}(G)\xi \subset \mathfrak{g}, \quad (\mathrm{ad}\xi)^3 = -\mathrm{ad}\xi$

$\cup \qquad\qquad\qquad\qquad \cup$

$M = \mathrm{Ad}(K)\xi \subset \mathfrak{p}, \quad \tau \rightsquigarrow \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$

G/K : **Riem. sym. sp. of cpt. type**

$X, Y \in M, \quad s_X(Y) = Y \Leftrightarrow [X, Y] = 0$

$A \subset M$: **MAS** $\Leftrightarrow A = M \cap \mathfrak{a}, \ \mathfrak{a} \subset \mathfrak{p}$: **max. abel.** $\rightsquigarrow$ **(i)-(iii)**

"the Weyl gr." = the Weyl group of G/K

We do not know much about antipodal sets in a cpt. Riem. sym. sp. which is not a sym. R-space.

A quotient group of a compact Lie group is not a symmetric R-space in general.

3. Antipodal sets of compact Lie groups

G **: a cpt. Lie gr. with bi-invariant metric**

$x \in G,\ s_x(y) = xy^{-1}x \quad (y \in G)$

1 **: the unit element of** G

$s_1(y) = y \Leftrightarrow y^2 = 1$

If $x^2 = 1, y^2 = 1,\ s_x(y) = y \Leftrightarrow xy = yx$

$1 \in S \subset G$ **: max. antipodal set** $\Rightarrow$ **subgroup**

$S \cong \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{r} \qquad |S| = 2^r$ **(**$r = r_2(G)$**)**

$r \geq \mathrm{rank}(G)$ **(**$r > \mathrm{rank}(G)$ **can occur)**

$$\Delta_n := \left\{ \begin{bmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{bmatrix} \right\} \subset O(n)$$

$$\Delta_n^{\pm} := \{ g \in \Delta_n \mid \det g = \pm 1 \}$$

A maximal antipodal subgr. of $O(n)$, $U(n)$, $Sp(n)$ **is conjugate to** Δ_n**.**

A maximal antipodal subgr. of $SO(n)$, $SU(n)$ **is conjugate to** Δ_n^+**.**

$\#_2 O(n) = \#_2 U(n) = \#_2 Sp(n) = 2^n$

$\#_2 SO(n) = \#_2 SU(n) = 2^{n-1}$

4. Classification of maximal antipodal subgroups of quotient groups of $U(n)$, $SU(n)$, $O(n)$, $SO(n)$, $Sp(n)$

the center of $U(n) \overset{\textbf{id}}{=} \{z \in \mathbb{C} \mid |z| = 1\} = U(1)$

$\mathbb{Z}_\mu \subset U(1)$ **: cyclic gr. of order** μ

$\Rightarrow U(n)/\mathbb{Z}_\mu$ **is a cpt. Lie gr. locally isomorphic to** $U(n)$

the center of $SU(n) \overset{\textbf{id}}{=} \{z \in \mathbb{C} \mid z^n = 1\} \cong \mathbb{Z}_n$

$\mathbb{Z}_\mu \subset \mathbb{Z}_n$ **: cyclic gr. of order** μ**, where** n **is divided by** μ**.**

$\Rightarrow SU(n)/\mathbb{Z}_\mu$ **is a cpt. Lie gr. locally isomorphic to** $SU(n)$

$$D[4] := \left\{ \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}, \begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix} \right\} \subset O(2)$$

$D[4]$: **dihedral group**

$$D^{\pm}[4] := \{ g \in D[4] \mid \det g = \pm 1 \}$$

$n = 2^k \cdot l, \;\; l$: **odd**

$0 \leq s \leq k$

$$D(s,n) := \underbrace{D[4] \otimes \cdots \otimes D[4]}_{s} \otimes \Delta_{n/2^s} \subset O(n)$$

Theorem 2 (**T.-Tasaki**)

$\mathbb{Z}_\mu$: **a cyclic subgr. of the center of** $U(n)$

θ : **a primitive** 2μ**-th root of** 1

$\pi_n : U(n) \to U(n)/\mathbb{Z}_\mu$: **the natural proj.**

A max. antip. subgr. (MAS) of $U(n)/\mathbb{Z}_\mu$ **is conjugate to one of the followings.**

(1) n **or** μ **is odd**

$$\pi_n(\{1, \theta\}D(0, n)) = \pi_n(\{1, \theta\}\Delta_n)$$

(2) n **and** μ **are even**

$$\pi_n(\{1, \theta\}D(s, n)) \quad (0 \le s \le k)$$

where $(s, n) = (k - 1, 2^k)$ **is excluded.**

Remark. $\Delta_2 \subsetneq D[4]$.

$$D(k-1, 2^k) = \underbrace{D[4] \otimes \cdots \otimes D[4]}_{s} \otimes \Delta_2$$
$$\subsetneq \underbrace{D[4] \otimes \cdots \otimes D[4]}_{s} \otimes D[4] = D(k, 2^k)$$

MAS is not unique (up to conjugation)

$\Leftrightarrow$ n, μ : **even,** $n \geq 4$

A great antipodal subgroup (GAS) is unique for ${}^\forall n, {}^\forall \mu$.

Sketch of Proof :

$A \subset U(n)/\mathbb{Z}_\mu$: **MAS**

$B := \pi_n^{-1}(A) \subset U(n)$

Case 1. B **is abelian.**

A **is conjugate to** $\pi_n(\Delta_n \cup \theta\Delta_n)$**.**

Case 2. B **is not abelian.**

$^\exists a, b \in B$ **s.t.** $ab \neq ba$

$n = 2n'$: **even**

$\langle a, b \rangle \cong D[4] \otimes 1_{n'}$

A **is conjugate to** $\pi_n(D[4] \otimes B')$,

$B' = \pi_n^{-1}($**MAS in** $U(n')/\mathbb{Z}_\mu)$

$\rightsquigarrow$ **induction**

$SU(n)/\mathbb{Z}_\mu \subset U(n)/\mathbb{Z}_\mu$ **(**n **is divided by** μ**)**

Theorem 2 $\rightsquigarrow$ **classification of MAS of** $SU(n)/\mathbb{Z}_\mu$

When $n = 8$**,** $\exists$ **two GAS's:**

$$\pi_8(\{1, \theta\}\Delta_8^+), \qquad \pi_8(\{1, \theta\}D(3, 8))$$

$$Q[8] := \{\pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}\}$$

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1,$$

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k},\ \mathbf{jk} = -\mathbf{kj} = \mathbf{i},\ \mathbf{ki} = -\mathbf{ik} = \mathbf{j}$$

Theorem 3 (**T.-Tasaki**)

$\tilde{G} = O(n),\ SO(n),\ Sp(n),\ G = O(n)/\{\pm 1_n\}$, $SO(n)/\{\pm 1_n\}\,(n :$ **even**$)$, $Sp(n)/\{\pm 1_n\}$

$\pi_n : \tilde{G} \to G$ **: the natural proj.**

$n = 2^k \cdot l$, l **: odd**

(I) MAS of $G = O(n)/\{\pm 1_n\}$ **is conjugate to**

$$\pi_n(D(s,n))\ (0 \leq s \leq k),$$

where $(s,n) = (k-1, 2^k)$ **is excluded.**

(II) MAS of $G = SO(n)/\{\pm 1_n\}$ is conjugate to

(II-1) $k = 1$

$$\pi_n(\Delta_n^+), \quad \pi_n(D^+[4] \otimes \Delta_l),$$

where $\pi_2(\Delta_2^+)$ **is excluded when** $n = 2$**.**

(II-2) $k \geq 2$

$$\pi_n(\Delta_n^+), \quad \pi_n(D(s,n)) \quad (1 \leq s \leq k),$$

where $(s,n) = (k-1, 2^k)$ **is excluded &** $\pi_4(\Delta_4^+)$ **is excluded when** $n = 4$**.**

(III) MAS of $G = Sp(n)/\{\pm 1_n\}$

is conjugate to

$$\pi_n(Q[8] \cdot D(s,n)) \quad (0 \leq s \leq k),$$

where $(s,n) = (k-1, 2^k)$ **is excluded.**

Corollary 4

(I) $G = O(n)/\{\pm 1_n\}$

$n = 2$, $\pi_2(D[4])$ **is a unique GAS.**

$n = 4$, $\pi_4(D(2,4))$ **is a unique GAS.**

$n \neq 2, 4$, $\pi_n(\Delta_n)$ **is a unique GAS.**

(II) $G = SO(n)/\{\pm 1_n\}$

$n = 2,\ \ \pi_2(D^+[4])$ **is a unique GAS.**

$n = 4,\ \ \pi_4(D(2,4))$ **is a unique GAS.**

$n = 8, \pi_8(\Delta_8^+)$ **and** $\pi_8(D(3,8))$ **are the GAS's.**

$n \neq 2,4,8,\ \ \pi_n(\Delta_n^+)$ **is a unique GAS.**

(III) $G = Sp(n)/\{\pm 1_n\}$

$n = 2,\ \ \pi_2(Q[8] \cdot D[4])$ **is a unique GAS.**

$n = 4,\ \ \pi_4(Q[8] \cdot D(2,4))$ **is a unique GAS.**

$n \neq 2,4,\ \ \pi_n(Q[8] \cdot \Delta_n)$ **is a unique GAS.**

5. Classification of MAS's of the automorphism groups of Lie algebras

G : conn. cpt. semisimple Lie gr.

Z : the center of G (discrete subgr. of G)

$G/Z \cong \mathrm{Inn}(\mathfrak{g}) \cong \mathrm{Ad}(G)$

How many involutive inner automorphisms of $\mathfrak{g}$ which commute to each other can we take ? $\rightsquigarrow$ MAS of $\mathrm{Inn}(\mathfrak{g})$

Section 4 $\rightsquigarrow$ classification of MAS of $\mathrm{Inn}(\mathfrak{g})$,

$\mathfrak{g} = \mathfrak{su}(n)$, $\mathfrak{so}(n)$, $\mathfrak{sp}(n)$

$\mathrm{Aut}(\mathfrak{g})$: **the group of automorphisms of** $\mathfrak{g}$

$\mathrm{Aut}(\mathfrak{g})_0 = \mathrm{Inn}(\mathfrak{g})$

$\mathrm{Ad} : G \to G/Z$: **the natural proj.**

<u>Theorem 5</u> (**T.-Tasaki**)

$n = 2^k \cdot l, \quad l$: **odd**

(**I**) $\tau : \mathfrak{su}(n) \to \mathfrak{su}(n)$; $X \mapsto \bar{X}$

MAS of $\mathrm{Aut}(\mathfrak{su}(n))$ **is conjugate to**

$$\{e, \tau\}\mathrm{Ad}(D(s, n)) \quad (0 \leq s \leq k),$$

where $(s, n) = (k - 1, 2^k)$ **is excluded.**

(**II**) **MAS of** $\mathrm{Aut}(\mathfrak{so}(n))$: **is conjugate to**

$$\mathrm{Ad}(D(s,n)) \quad (0 \le s \le k),$$

where $(s,n) = (k-1, 2^k)$ **is excluded.**

(**III**) **MAS of** $\mathrm{Aut}(\mathfrak{sp}(n))$: **is conjugate to**

$$\mathrm{Ad}(Q[8] \cdot D(s,n)) \quad (0 \le s \le k),$$

where $(s,n) = (k-1, 2^k)$ **is excluded.**

6. Classification of maximal antipodal subsets of G_2 and $G_2/SO(4)$

M : **a cpt. Rieman. sym. sp.,** $o \in M$

Assume $F(s_o, M) = \{o\} \cup M_1^+$

$o \in A \subset M$

A : **MAS of** M $\Leftrightarrow$ $A \cap M_1^+$: **MAS of** M_1^+

e : **the unit element**

$F(s_e, G_2) = \{e\} \cup M_1^+, \quad M_1^+ \cong G_2/SO(4)$

$o \in M_1^+$

$F(s_o, M_1^+) = \{o\} \cup M_{1,1}^+, \quad M_{1,1}^+ \cong (S^2 \times S^2)/\mathbb{Z}_2$

$S^2 \times S^2 \ni (p, q) \mapsto [p, q] \in (S^2 \times S^2)/\mathbb{Z}_2$

$(u_i, v_i) \in S^2 \times S^2 \quad (i = 1, 2, 3)$

$u_i \perp u_j, \quad v_i \perp v_j \quad (i \neq j)$

$A := \{[u_1, \pm v_1], [u_2, \pm v_2], [u_3, \pm v_3]\}$ **is a unique MAS of** $(S^2 \times S^2)/\mathbb{Z}_2$ **up to congruence.**

$A \ \leftrightarrow \ A_{1,1} \subset M^+_{1,1}$

Theorem 6 (T.-Tasaki)

MAS of $G_2/SO(4)$ **is congruent to** $\{o\} \cup A_{1,1}$**.**

MAS of G_2 **is conjugate to** $\{e, o\} \cup A_{1,1}$.