Maximal antipodal sets of the bottom space of $Sp(n)/U(n)$

Makiko Sumi Tanaka

Tokyo Univ. of Science

The 13th OCAMI-RIRCM Joint Differential Geometry Workshop on Submanifolds and Lie Theory

March 27-30, 2017

Joint with Hiroyuki Tasaki

1. **Introduction**

2. **Maximal antipodal subgroups of compact classical Lie groups**

3. **The bottom space** $CI(n)^*$ **of** $CI(n) = Sp(n)/U(n)$

4. **Maximal antipodal sets of** $CI(n)^*$

5. **Other results**

1. Introduction

M: a compact Riemannian symmetric space

s_x: the geodesic symmetry at x

i.e., (i) s_x is an isometry of M, (ii) ${s_x}^2 = \mathrm{id}$,

(iii) x is an isolated fixed point of s_x

$S \subset M$: a subset

S: an antipodal set $\overset{\text{def}}{\Longleftrightarrow} \forall x, y \in S,\ s_x(y) = y$

The 2-number $\#_2 M$ of M

$\#_2 M := \max\{|S| \mid S \subset M \text{ antipodal set}\}$

S: great $\overset{\text{def}}{\Longleftrightarrow} |S| = \#_2 M$

(Chen-Nagano 1988)

Examples. (1) $M = S^n \, (\subset \mathbb{R}^{n+1})$

$\{x, -x\}$: **a great antipodal set for** $\forall x \in S^n$

(2) $M = \mathbb{R}P^n$

$e_1, \ldots, e_{n+1}$: **an o.n.b. of** $\mathbb{R}^{n+1}$

$\{\langle e_1 \rangle_{\mathbb{R}}, \ldots, \langle e_{n+1} \rangle_{\mathbb{R}}\}$: **a great antipodal set**

(3) $M = U(n) \qquad s_x(y) = xy^{-1}x$

$s_{1_n}(x) = x \;\Leftrightarrow x^2 = 1_n$ **(**1_n**: the unit matrix)**

$x^2 = y^2 = 1_n \;\Rightarrow\; s_x(y) = y$ **iff** $xy = yx$

$$\left\{ \begin{bmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{bmatrix} \right\}$$: **a great antipodal set**

$M \subset N$: **totally geodesic**

$S \subset M$: **an antip. set** $\Rightarrow$ $S \subset N$: **an antip. set**

$\rightsquigarrow \#_2 M \leq \#_2 N$

(Chen-Nagano) M: **cpt. conn.**

$\#_2 M \geq \chi(M)$, $\chi(M)$: **the Euler number**

"=" **if** M: **a Herm. sym. sp. of cpt. type**

(Takeuchi) M: **a symmetric** R**-space**

$$\Rightarrow \#_2 M = \sum_{k=0}^{\dim M} b_k(M; \mathbb{Z}_2)$$

b_k: **the k-th Betti number**

Remark. S^n, $\mathbb{R}P^n$, $U(n)$: **sym.** R**-sp.**

Remark. **Generally** $\exists$**maximal antp. set, not great**

(T.-Tasaki 2013) In a symmetric R**-space** M **(i) any antipodal set is included in a great antipodal set, (ii) any two great antipodal sets are (**$I(M)_0$**-)congruent, (iii) a great antipodal set is an orbit of "the Weyl group".**

A sym. R-sp. is a real form L of some Herm. sym. sp. M of cpt. type,

i.e., $\exists\tau$: an invol. anti-holo. isom. of M;

$L = F(\tau, M) := \{x \in M \mid \tau(x) = x\}$ **(connected)**

(T.-Tasaki 2012)

M: a Herm. sym. of cpt. type

L_1, L_2: real forms of M, $L_1 \pitchfork L_2$

$\Rightarrow$ $L_1 \cap L_2$: an antipodal set of L_i $(i = 1, 2)$

Moreover, if L_1, L_2: congruent, then $L_1 \cap L_2$: great.

Chen-Nagano determined $\#_2 M$ for most cpt. Riem. sym. sp. M but we don't know much about antipodal sets themselves, especially when M is not a sym. R-space.

Problem: Classification of maximal antipodal sets of M which is not a sym. R-sp.

E.g. $\tilde{G}_k(\mathbb{R}^n)$: oriented real Grassmann mfd. ($k = 3, 4$ Tasaki), G/Γ: quotient groups of cpt. Lie gr. (G: classical T.-Tasaki), G_2 (T.-Tasaki-Yasukura), quotient spaces of Herm. sym. sp. of cpt. type

2. Maximal antip. subgr. of cpt. classical Lie groups

G: **a cpt. Lie gr. with bi-invariant metric**

$x \in G,\ s_x(y) = xy^{-1}x \quad (y \in G)$

1 : **the unit element of** G

$s_1(y) = y \Leftrightarrow y^2 = 1$

If $x^2 = y^2 = 1,\ s_x(y) = y \Leftrightarrow xy = yx$

$1 \in S \subset G$: **max. antipodal set** $\Rightarrow$ **subgroup**

$$S \cong \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{r} \qquad |S| = 2^r$$

$r \geq \mathrm{rank}(G)$ **(**$r > \mathrm{rank}(G)$ **can occur)**

$$\Delta_n := \left\{ \begin{bmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{bmatrix} \right\} \subset O(n)$$

$$\Delta_n^+ := \{ g \in \Delta_n \mid \det g = 1 \}$$

A maximal antipodal subgr.(MAS) of $O(n)$, $U(n)$, $Sp(n)$ **is conjugate to** Δ_n**.**

A MAS of $SO(n)$, $SU(n)$ **is conjugate to** Δ_n^+**.**

$\#_2 O(n) = \#_2 U(n) = \#_2 Sp(n) = 2^n$

$\#_2 SO(n) = \#_2 SU(n) = 2^{n-1}$

$$D[4] := \left\{ \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}, \begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix} \right\} \subset O(2)$$

the dihedral group

$n = 2^k \cdot l, \;\; l$: **odd**

$0 \leq s \leq k$

$$D(s,n) := \underbrace{D[4] \otimes \cdots \otimes D[4]}_{s} \otimes \Delta_{n/2^s} \subset O(n)$$

$$Q[8] := \{\pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}\}$$

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1,$$

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \; \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \; \mathbf{ki} = -\mathbf{ik} = \mathbf{j}$$

Theorem 1 (**T.-Tasaki**)

$\tilde{G} = U(n)$, $O(n)$, $Sp(n)$

$G = U(n)/\{\pm 1_n\}$, $O(n)/\{\pm 1_n\}$, $Sp(n)/\{\pm 1_n\}$

$\pi_n : \tilde{G} \to G$: **the projection**

$n = 2^k \cdot l, \quad l$: **odd**

(I) MAS of $G = O(n)/\{\pm 1_n\}$ **is conjugate to**

$$\pi_n(D(s,n)) \quad (0 \leq s \leq k)$$

where $(s,n) = (k-1, 2^k)$ **is excluded.**

(II) MAS of $G = U(n)/\{\pm 1_n\}$ **is conjugate to**

$$\pi_n(\{1, \sqrt{-1}\}D(s,n)) \quad (0 \leq s \leq k)$$

where $(s,n) = (k-1, 2^k)$ **is excluded.**

(III) MAS of $G = Sp(n)/\{\pm 1_n\}$ **is conjugate to**

$$\pi_n(Q[8] \cdot D(s,n)) \quad (0 \leq s \leq k)$$

where $(s,n) = (k-1, 2^k)$ **is excluded.**

Remark. $\Delta_2 \subsetneq D[4]$.

$$D(k-1, 2^k) = \underbrace{D[4] \otimes \cdots \otimes D[4]}_{s} \otimes \Delta_2$$

$$\subsetneq \underbrace{D[4] \otimes \cdots \otimes D[4]}_{s} \otimes D[4] = D(k, 2^k)$$

$\#_2 G$ **and great antipodal subgroups (GAS) of** G **are given as follows.**

<u>**Theorem 2**</u> **(T.-Tasaki)**

(I) $G = O(n)/\{\pm 1_n\}$

If $n = 2$, $\#_2 G = 2^2$, **GAS:** $\pi_2(D[4])$

If $n = 4$, $\#_2 G = 2^4$, **GAS:** $\pi_4(D(2,4))$

If $n \neq 2, 4$, $\#_2 G = 2^{n-1}$, **GAS:** $\pi_n(\Delta_n)$

(II) $G = U(n)/\{\pm 1_n\}$

If $n = 2$, $\#_2 G = 2^3$, **GAS:** $\pi_2(\{1, \sqrt{-1}\} D[4])$

If $n = 4$, $\#_2 G = 2^5$, **GAS:** $\pi_4(\{1, \sqrt{-1}\} D(2,4))$

If $n \neq 2, 4$, $\#_2 G = 2^n$, **GAS:** $\pi_n(\{1, \sqrt{-1}\} \Delta_n)$

(III) $G = Sp(n)/\{\pm 1_n\}$

If $n = 2$, $\#_2 G = 2^4$, **GAS:** $\pi_2(Q[8] \cdot D[4])$

If $n = 4$, $\#_2 G = 2^6$, **GAS:** $\pi_4(Q[8] \cdot D(2,4))$

If $n \neq 2, 4$, $\#_2 G = 2^{n+1}$, **GAS:** $\pi_n(Q[8] \cdot \Delta_n)$

Griess (1991) , Yu (2013)

They classified conjugate classes of elementary abelian p-subgroups of algebraic groups by algebraic methods.

Antipodal subgroups = Elementary abelian 2-subgroups

3. The bottom space $CI(n)^*$ of $CI(n)$

$CI(n) := \{x \in Sp(n) \mid x^2 = -1_n\} \cong Sp(n)/U(n)$

an irr. Herm. sym. sp. of cpt. type

$\mathrm{i}\Delta_n \subset CI(n)$: **unique max. antip. set**

$\rightsquigarrow$ **a great antip. set** $\qquad \#_2 CI(n) = 2^n$

$Sp(n)^* := Sp(n)/\{\pm 1_n\}, \ 1_n^* := \pi_n(1_n)$

$\pi_n : Sp(n) \to Sp(n)^*$ **the projection**

$CI(n)^* := \pi_n(CI(n)) = CI(n)/\{\pm 1_n\}$

a Riem. sym. sp. but not a Hermit. sym. sp.

$CI(n)^* \subset F(s_{1_n^*}, Sp(n)^*)$

4. Maximal antipodal sets of $CI(n)^*$

$S \subset CI(n)^*$: a max. antipodal set

$\{1_n^*\} \cup S$: an antipodal set of $Sp(n)^*$

$\exists \tilde{S}$: a max. antipodal subgroup of $Sp(n)^*$;

$$\{1_n^*\} \cup S \subset \tilde{S}$$

$\exists g \in Sp(n)$; $\quad \tilde{S} = \pi_n(g(Q[8] \cdot D(s,n))g^{-1})$

$\{1_n^*\} \cup \pi_n(g)^{-1} S \pi_n(g) \subset \pi_n(Q[8] \cdot D(s,n))$

$\pi_n(g)^{-1} S \pi_n(g) \subset \pi_n(Q[8] \cdot D(s,n)) \cap CI(n)^*$

The r.h.s. is an antip. subset of $CI(n)^*$.

By the maximality of S we obtain "$=$".

$$\pi_n(g)^{-1} S \pi_n(g) = \pi_n(Q[8] \cdot D(s,n)) \cap CI(n)^*$$
$$= \pi_n(\{x \in Q[8] \cdot D(s,n) \mid x^2 = -1_n\})$$

$$PD(s,n) := \{d \in D(s,n) \mid d^2 = 1_n\}$$
$$ND(s,n) := \{d \in D(s,n) \mid d^2 = -1_n\}$$

$$\{x \in Q[8] \cdot D(s,n) \mid x^2 = -1_n\}$$
$$= ND(s,n) \cup \{\mathbf{i}, \mathbf{j}, \mathbf{k}\} PD(s,n)$$

$n = 2^k \cdot l, \ l$ **: odd**

Theorem 3 (T.-Tasaki) A maximal antipodal subset of $CI(n)^*$ is congruent to

$$\pi_n(ND(s,n) \cup \{\mathbf{i}, \mathbf{j}, \mathbf{k}\} PD(s,n)) \quad (0 \leq s \leq k)$$

where $(s,n) = (k-1, 2^k)$ is excluded.

Theorem 3 $\rightsquigarrow \#_2 CI(n)^*$ **& great antipodal sets (GAS) of** $CI(n)^*$

$$|\pi_n(ND(s,n) \cup \{\mathbf{i},\mathbf{j},\mathbf{k}\}PD(s,n))| = (2^{s+1}+1)2^{s-1+2^{k-s}\cdot l}$$

$$J_1 := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad K_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Theorem 4 (T.-Tasaki)

(1) $\#_2 CI(2)^* = 10$

GAS: $\pi_2(\{\pm J_1\} \cup \{\mathbf{i},\mathbf{j},\mathbf{k}\}(\Delta_2 \cup \{\pm K_1\}))$

(2) $\#_2 CI(4)^* = 36$

GAS: $\pi_4(ND(2,4) \cup \{\mathbf{i}, \mathbf{j}, \mathbf{k}\} PD(2,4))$

(3) If $n \neq 2, 4$**,** $\#_2 CI(n)^* = 3 \cdot 2^{n-1}$

GAS: $\pi_n(\{\mathbf{i}, \mathbf{j}, \mathbf{k}\} \Delta_n)$

In particular, a GAS of $CI(n)^*$ **is unique up to congruence.**

Remark. $CI(2)^* \cong G_2(\mathbb{R}^5)$

5. Other results

$\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$

$G_m(\mathbb{K}^{2m})$: **Grassmann mfd. of m-dim. subspaces of $\mathbb{K}^{2m}$**

$\gamma : G_m(\mathbb{K}^{2m}) \to G_m(\mathbb{K}^{2m}), \ \ \gamma(x) = x^{\perp}$

$G_m(\mathbb{K}^{2m})^* := G_m(\mathbb{K}^{2m})/\{\mathbf{id}, \gamma\}$

The corresponding sym. pair of $G_m(\mathbb{K}^{2m})$:

$(G, K) = (O(2m), O(m) \times O(m))$ **if** $\mathbb{K} = \mathbb{R}$

$(G, K) = (U(2m), U(m) \times U(m))$ **if** $\mathbb{K} = \mathbb{C}$

$(G, K) = (Sp(2m), Sp(m) \times Sp(m))$ **if** $\mathbb{K} = \mathbb{H}$

Consider $G_m(\mathbb{K}^{2m}) \subset G$ **by the correspondence** $x \mapsto \mathbf{id}_x - \mathbf{id}_{\gamma(x)}$.

$G_m(\mathbb{K}^{2m})^* \subset G^* := G/\{\pm 1_{2m}\}$

$G_m(\mathbb{K}^{2m}) \subset F(s_{1_{2m}}, G)$

$G_m(\mathbb{K}^{2m})^* \subset F(s_e, G^*)$, $e := \pi_{2m}(1_{2m})$

$2m = 2^k \cdot l,\ \ l$: **odd**

<u>Theorem 5</u> (T.-Tasaki) (I) A MAS of $G_m(\mathbb{R}^{2m})^*$ **is cong. to**

$$\pi_{2m}(\{d_1 \otimes \cdots \otimes d_s \otimes d_0 \in PD(s, 2m) \mid \exists d_i (0 \le i \le s)\ \mathrm{Tr} d_i = 0\}) \quad (0 \le s \le k)$$

where $(s, 2m) = (k-1, 2^k)$ **is excluded.**

(II) A MAS of $G_m(\mathbb{C}^{2m})^*$ **is cong. to**

$$\pi_{2m}(\{d_1 \otimes \cdots \otimes d_s \otimes d_0 \in PD(s, 2m) \mid \exists d_i (0 \leq i \leq s) \ \mathbf{Tr} d_i = 0\} \cup \sqrt{-1} ND(s, 2m)) \quad (0 \leq s \leq k)$$

where $(s, 2m) = (k-1, 2^k)$ **is excluded.**

(III) A MAS of $G_m(\mathbb{H}^{2m})^*$ **is cong. to**

$$\pi_{2m}(\{d_1 \otimes \cdots \otimes d_s \otimes d_0 \in PD(s, 2m) \mid \exists d_i (0 \leq i \leq s) \ \mathbf{Tr} d_i = 0\} \cup \{\mathbf{i}, \mathbf{j}, \mathbf{k}\} ND(s, 2m)) \quad (0 \leq s \leq k)$$

where $(s, 2m) = (k-1, 2^k)$ **is excluded.**