Antipodal sets of compact symmetric spaces

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Quandles and Symmetric Spaces

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Contents

1. Symmetric spaces and antipodal sets

A C^∞ manifold M is called a symmetric space if for every $x \in M$ there exists a C^∞ map $s_x : M \to M$ such that

(i) $s_x \circ s_x = \mathrm{id}_M$,

(ii) x is an isolated fixed point of s_x,

(iii) $s_x \circ s_y = s_{s_x(y)} \circ s_x$ for $^\forall y \in M$,

(iv) $M \times M \ni (x, y) \mapsto s_x(y) \in M$ is a C^∞ map.

s_x is called a symmetry at x. A symmetric space is a quandle.

When a symmetric space M is a Riemannian (resp. Hermitian) manifold and every symmetry is an isometry (resp. holomorphic isometry), M is a Riemannian (resp. Hermtian) symmetric space. If M is connected, a symmetry at each point is unique.

Examples: (1) $\mathbb{R}^n$ is a symmetric space. $s_x(y) = 2x - y$.
(2) $S^n\,(\subset \mathbb{R}^{n+1})$ is a symmetric space. $\rho_x = \mathrm{id}_{\mathbb{R}x} - \mathrm{id}_{(\mathbb{R}x)^\perp}$ **induces s_x at $x \in S^n$.**

(3) A Lie group G is a symmetric space. $s_x(y) = xy^{-1}x$.

A subset S of a symmetric space M is called <u>an antipodal set</u> if $s_x(y) = y$ for every $x, y \in S$. If M is connected, $s_x(y) = y$ holds iff there exists a closed geodesic on which x, y are antipodal.

Let M be a compact symmetric space. An antipodal set of M is finite.

$\#_2 M := \max\{|S| \mid S \subset M$ antipodal set$\}$ is called the 2-number of M.

The 2-number has relation to the 2-rank of compact Lie groups.

An antipodal set S is called great if it satisfies $|S| = \#_2 M$.

Examples. (1) An antipodal set of $\mathbb{R}^n$ is a set of one point. $\#_2 \mathbb{R}^n = 1$.

(2) For each $x \in S^n$, $\{x, -x\}$ is a great antipodal set. $\#_2 S^n = 2$.

(3) Let $e_1, \ldots, e_{n+1}$ be an o.n.b. of $\mathbb{R}^{n+1}$. $\{\langle e_1 \rangle_{\mathbb{R}}, \ldots, \langle e_{n+1} \rangle_{\mathbb{R}}\}$ is a great antipodal set of $\mathbb{R}P^n$. $\#_2 \mathbb{R}P^n = n + 1$.

If N is a totally geodesic submanifold of a symmetric space M, $s_x(N) \subset N$ for $x \in N$. N is a symmetric space. If S is an antipodal set of N, S is an antipodal set of M. $\#_2 N \leq \#_2 M$.

Fact 1 (Chen-Nagano 1988) If M is a compact connected Riemmanian symmetric space,

$\#_2 M \geq \chi(M)$, **the Euler number of** M**. "="** **if** M **is a Hermitian symmetric space of compact type.**

A Riemannain symmetric space M **which has a realization as a linear isotropy orbit of a certain Riemannian symmetric space of compact type is called a symmetric** R**-space.**

Fact 2 (Takeuchi 1989) If M **is a symmetric** R**-space,** $\#_2 M = \sum_{k=0}^{\dim M} b_k(M; \mathbb{Z}_2)$**, the sum of** $\mathbb{Z}_2$**-Betti numbers.**

A symmetric R-space is a real form L of a certain Hermitian symmetric space M of compact type, i.e., there exists an involutive anti-holomorphic isometry τ of M such that $L = \{x \in M \mid \tau(x) = x\}$.

<u>Fact 3</u> (T.-Tasaki 2012) Let M be a Hermitian symmetric space of compact type. Let L_1, L_2 be real forms of M. If $L_1 \cap L_2$ is discrete, $L_1 \cap L_2$ is an antipodal set of L_1 and L_2. Moreover, if L_1, L_2 are $I_0(M)$-congruent, $L_1 \cap L_2$ is a great antipodal set.

If $M = \mathbb{C}P^1 \cong S^2$, a real form is a great circle $\cong \mathbb{R}P^1 \cong S^1$. Any two different great circles intersects at antipodal points.

A great antipodal set is a maximal antipodal set. The converse is not true in general.

Fact 4 (T.-Tasaki 2013) Let M be a symmetric R-space. (i) Any antipodal set of M is included in a great antipodal set. (ii) Any two great antipodal sets of M are $I(M)_0$-congruent. (iii) A great antipodal set of M is an orbit of the Weyl group.

Chen-Nagano determined $\#_2 M$ for most compact symmetric spaces M. One of the exceptions is the oriented real Grassmann manifolds. We have interest in not only the cardinalities but also the structures of maximal antipodal sets of M.

Our goal: Classify maximal antipodal sets of compact symmetric spaces.

2. Maximal antipodal subgroups of classical compact Lie groups

G: **a cpt. Lie gr. with biinvariant metric**

$s_x(y) = xy^{-1}x \quad (x, y \in G)$

1 : **the unit element of** G

$s_1(y) = y \Leftrightarrow y^2 = 1$

If $x^2 = y^2 = 1,\ s_x(y) = y \Leftrightarrow xy = yx$

$1 \in S$: **max. antipodal set of** $G \Rightarrow S$: **subgroup**

$S \cong \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{r} \qquad |S| = 2^r$

$r \geq \mathrm{rank}(G)$

$$\Delta_n := \left\{ \begin{bmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{bmatrix} \right\} \subset O(n)$$

$$\Delta_n^+ := \{ g \in \Delta_n \mid \det g = 1 \}$$

MAS: a maximal antipodal subgroup

MAS of $O(n)$, $U(n)$, $Sp(n)$ **is conjugate to** Δ_n. **MAS of** $SO(n)$, $SU(n)$ **is conjugate to** Δ_n^+.

$\#_2 O(n) = \#_2 U(n) = \#_2 Sp(n) = 2^n$

$\#_2 SO(n) = \#_2 SU(n) = 2^{n-1}$

$$D[4] := \left\{ \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}, \begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix} \right\} \subset O(2)$$

$n = 2^k \cdot l, \ \ l$: **odd**

$0 \leq s \leq k$

$$D(s,n) := \{d_1 \otimes \cdots \otimes d_s \otimes d_0 \mid d_1, \ldots, d_s \in D[4], d_0 \in \Delta_{n/2^s}\} = \underbrace{D[4] \otimes \cdots \otimes D[4]}_{s} \otimes \Delta_{n/2^s} \subset O(n)$$

$$Q[8] := \{\pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}\}$$

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1,$$

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \ \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \ \mathbf{ki} = -\mathbf{ik} = \mathbf{j}$$

Theorem 1 (T.-Tasaki 2017)

$\tilde{G} = U(n)$, $O(n)$, $Sp(n)$

$G = U(n)/\{\pm 1_n\}$, $O(n)/\{\pm 1_n\}$, $Sp(n)/\{\pm 1_n\}$

$\pi_n : \tilde{G} \to G$: projection

$n = 2^k \cdot l, \quad l$: odd

(1) MAS of $O(n)/\{\pm 1_n\}$ is conjugate to

$$\pi_n(D(s,n)) \quad (0 \le s \le k)$$

where $(s,n) = (k-1, 2^k)$ is excluded.

(2) MAS of $U(n)/\{\pm 1_n\}$ is conjugate to

$$\pi_n(\{1, \sqrt{-1}\}D(s,n)) \quad (0 \le s \le k)$$

where $(s,n) = (k-1, 2^k)$ is excluded.

(3) MAS of $Sp(n)/\{\pm 1_n\}$ is conjugate to

$$\pi_n(Q[8] \cdot D(s,n)) \quad (0 \leq s \leq k)$$

where $(s,n) = (k-1, 2^k)$ is excluded.

Remark. $\Delta_2 \subsetneq D[4]$.

$$D(k-1, 2^k) = \underbrace{D[4] \otimes \cdots \otimes D[4]}_{k-1} \otimes \Delta_2$$

$$\subsetneq \underbrace{D[4] \otimes \cdots \otimes D[4]}_{k-1} \otimes D[4] = D(k, 2^k)$$

Griess (1991) and Yu (2013) classified conjugate classes of elementary abelian p-subgr. of algebraic groups by algebraic methods.

3. Maximal antipodal sets of classical compact symmetric spaces

We use an appropriate totally geodesic embedding of a classical compact symmetric space into a classical compact Lie group.

$CI(n) := \{x \in Sp(n) \mid x^2 = -1_n\} \cong Sp(n)/U(n)$
$CI(n)$ is a Hermitian symmetric space of compact type. $\mathrm{i}\Delta_n$ is a unique maximal antipodal set of $CI(n)$ up to congruence.
$\#_2 CI(n) = 2^n$.

$Sp(n)^* := Sp(n)/\{\pm 1_n\}$

$\pi_n : Sp(n) \to Sp(n)^*$ **projection,** $1_n^* := \pi_n(1_n)$

$CI(n)^* := \pi_n(CI(n)) = CI(n)/\{\pm 1_n\}$

$CI(n)^* \subset \{x \in Sp(n)^* \mid x^2 = 1_n^*\} = \mathsf{Fix}(s_{1_n^*})$

Let $S \subset CI(n)^*$**be a maximal antipodal set.** $\{1_n^*\} \cup S$ **is an antipodal set of** $Sp(n)^*$**. There exists a maximal antipodal subgroup** $\tilde{S}$ **of** $Sp(n)^*$ **such that** $\{1_n^*\} \cup S \subset \tilde{S}$**. By Theorem 1,** $^\exists g \in Sp(n)$**,** $^\exists s \in \{0, \ldots, k\}$ **such that**

$$\tilde{S} = \pi_n(g)\, \pi_n(Q[8] \cdot D(s,n))\, \pi_n(g)^{-1}.$$

$$\{1_n^*\} \cup \pi_n(g)^{-1}\, S\, \pi_n(g) \subset \pi_n(Q[8] \cdot D(s,n)).$$

By the maximality of S

$$\pi_n(g)^{-1}\, S\, \pi_n(g) = \pi_n(Q[8] \cdot D(s,n)) \cap CI(n)^*.$$

RHS$= \pi_n(\{x \in Q[8] \cdot D(s,n) \mid x^2 = -1_n\})$.

$PD(s,n) := \{d \in D(s,n) \mid d^2 = 1_n\}$

$ND(s,n) := \{d \in D(s,n) \mid d^2 = -1_n\}$

$$\{x \in Q[8] \cdot D(s,n) \mid x^2 = -1_n\}$$
$$= ND(s,n) \cup \{\mathbf{i}, \mathbf{j}, \mathbf{k}\} PD(s,n)$$

Theorem 2 (T.-Tasaki) Let $n = 2^k \cdot l$ where l is odd. A maximal antipodal set of $CI(n)^*$ is congruent to

$$\pi_n(ND(s,n) \cup \{\mathbf{i}, \mathbf{j}, \mathbf{k}\} PD(s,n)) \quad (0 \leq s \leq k)$$

where $(s,n) = (k-1, 2^k)$ is excluded.

Other cases: $DIII(n) = SO(2n)/U(n)$ is regarded as one of two connected components of $\{x \in SO(2n) \mid x^2 = -1_{2n}\}$. When $n = 2m$, $DIII(n)^* \subset \{x \in SO(2n)^* \mid x^2 = 1_{2n}^*\}$.

$G_k(\mathbb{K}^n)$, $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, **is regarded as one of** $(n+1)$ **connected components of** $\{x \in G \mid x^2 = 1_n\}, G = O(n), U(n), Sp(n)$ **respectively. When** $n = 2k$**,** $G_k(\mathbb{K}^n)^* \subset \{x \in G^* \mid x^2 = 1_n^*\}$**. When** $M = G/K$ **is one of** $UI(n) = U(n)/O(n)$, $AI(n) = SU(n)/SO(n)$, $UII(n) = U(2n)/Sp(n)$, $AII(n) = SU(2n)/Sp(n)$**, we use the fact that** $\{x \in G \rtimes \langle\sigma\rangle \mid x^2 = 1\}$ **contains** $M\sigma$ **as a connected component, where** σ **is an involutive automorphism for a symmetric pair** (G, K)**.**