Antipodal sets of compact symmetric spaces and polars of compact Lie groups

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Submanifolds of Symmetric Spaces and Their Time Evolutions, March 5-6, 2021
Online (Zoom)

Joint work with Hiroyuki Tasaki

Contents

1. Introduction

M: a Riemannian manifold

M is called a **Riemannian symmetric space** if for $\forall x \in M$, the **point symmetry** s_x at x is given, i.e., (i) s_x is an isometry of M, (ii) $s_x \circ s_x = \mathrm{id}_M$, (iii) x is an isolated fixed point of s_x.

- The differential $(ds_x)_x$ is $-\mathrm{id}_{T_xM}$.
- When M is connected, s_x is uniquely determined by (i)-(iii) and s_x is the geodesic symmetry.

$F(s_x, M) := \{y \in M | s_x(y) = y\}$

A connected component of $F(s_x, M)$ is called a polar w.r.t. x.

By (iii), $\{x\}$ **is a polar w.r.t. x, called the trivial polar.**

- **A polar M^+ of positive dimension is a totally geodesic submanifold and hence M^+ is a Riemannian symmetric space. The point symmetry at $y \in M^+$ is given by $s_y|_{M^+}$.**

- $\mathbb{R}^n$**: Euclidean space,** $F(s_x, \mathbb{R}^n) = \{x\}$
- S^n**: a sphere,** $F(s_x, S^n) = \{x, -x\}$

• P^n: **the projective space,** $F(s_x, P^n) = \{x\} \cup P^{n-1}$

(∵) **Set** $\mathbb{K} = \mathbb{R}, \mathbb{C}$, **or** $\mathbb{H}$ **and denote** P^n **by** $\mathbb{K}P^n$. **Since** s_x **is induced by the reflection along** x **in** $\mathbb{K}^{n+1}$, $F(s_x, \mathbb{K}P^n) =$
$\{x\} \cup \{1-$dim. subspaces in $x^{\perp}\} (= \mathbb{K}P^{n-1})$.

• **If** M **is of noncompact type,** $F(s_x, M) = \{x\}$.

Hereafter we consider the case where M **is compact.**

• **A compact connected Riem. sym. sp. M is** (i) **of compact type (**$I(M)$ **is compact and semisimple),** (ii) **a torus, or a product of** (i) **and** (ii) **locally.**

A**: a subset of** M

A **is called an antipodal set if for** $\forall x, y \in A$**,** $s_x(y) = y$ **holds.**

For $\forall x \in A$**,** $A \subset F(s_x, M)$**.** x **is an isolated point in** $F(s_x, M)$ **as well as in** A**. Thus** A **is discrete. Hence an antipodal set is finite.**

The 2-number of M is $\#_2 M := \max\{|A| \mid A \subset M$: an antipodal set$\}$.
If A satisfies $|A| = \#_2 M$, A is called great.
If $A \subset A'$ implies $A = A'$, we say A is maximal.

- **A great antipodal set is maximal but the converse is not true.**
- **$\#_2 S^n = 2$ and $\{x, -x\}$ is a great antipodal set.**

Bang-Yen Chen and Tadashi Nagano gave detailed studies of the 2-numbers (Chen-Nagano, 1988).

In the past ten years there was progress in the research of antipodal sets. Our interest is in maximal antipodal sets themselves rather than their cardinalities. We are working on the classification of maximal antipodal sets.

- In (T.-Tasaki, 2017) we classified max. antip. subgr. of some classical cpt. Lie groups G.
- In (T.-Tasaki, 2020) we classified max. antip. sets of some classical cpt. Riem. sym. sp. M.

The basic principle is to make use of an embedding of M into G as a polar w.r.t. the identity element and apply the classification of max. antip. subgr. of G.

- In order to continue the classification of max. antip. sets for some other classical cpt. Riem. sym. sp. M, we need a realization of M as a polar of a disconnected cpt. Lie gr.
- Chen-Nagano and Nagano gave detailed studies of polars of connected cpt. Riem. symmetric spaces.
- We studied polars of disconnected cpt. Lie groups (T.-Tasaki, submitted).

2. Relations between antipodal sets and polars

G: a compact Lie group

e: the identity element of G

G_0: the identity component of G

$\exists$ a biinvariant Riemannian metric on G

G is a compact Riem. symmetric space.

$\forall x \in G$, $s_x(y) = xy^{-1}x \ (y \in G)$

- $s_e(y) = y^{-1}, \ s_x(y) = L_x \circ s_e \circ L_{x^{-1}}(y)$

$F(s_e, G) = \{x \in G \mid x^2 = e\}$

$F(s_e, G) = \bigcup_{j=0}^{r} G_j^+$, G_j^+: a polar, $G_0^+ = \{e\}$

In general, when a polar consists of a single point x, we call x a pole.

Proposition 1

$Z_G(G_0)$: the centralizer of G_0 in G

$\tilde{Z}_2(G) := Z_G(G_0) \cap F(s_e, G)$

- The set of poles coincides with $\tilde{Z}_2(G)$.
- For a point x in G_j^+, $G_j^+ = \{I_g(x) \mid g \in G_0\}$, where $I_g(x) = gxg^{-1}$.

Hence each polar is a G_0-conjugacy class of involutive elements.

A: an antipodal set of G

We can assume $e \in A$ by left (or right) translations. Then,

- $x^2 = e \ (x \in A), \ xy = yx \ (x, y \in A)$.
- If A is maximal, A is a subgroup $\cong \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$.

We call such A a maximal antipodal subgroup.

Example. $G = O(n)$: **the orthogonal group**

$G_0 = SO(n)$

1_n: **the identity matrix**

$I_j = \mathrm{daig}(\underbrace{-1, \ldots, -1}_{j}, 1, \ldots, 1) \in O(n)$

$G_j^+ = \{gI_jg^{-1} \mid g \in SO(n)\}$

$\cong SO(n)/S(O(j) \times O(n-j))$

$= G_j(\mathbb{R}^n)$: **the real Grassmann mfd.**

$A_0 = \{\mathrm{diag}(\epsilon_1, \ldots, \epsilon_n) \mid \epsilon_i = \pm 1\}$ **is a maximal antipodal subgroup of** $O(n)$**.**

$\tilde{Z}_2(O(n)) = \{\pm 1_n\}$

• A_0 **is a unique max. antip. subgr. of** $O(n)$ **up to conjugation, while a max. antip. subgr. of** $O(n)/\{\pm 1_n\}$ **is not unique up to conjugation when** n **is even and** $n \geq 4$**.**

$M = G_j^+$**: a polar of positive dim.**
M **is a connected cpt. Riem. sym. sp.**
$x_0 \in M$, $M = \{I_g(x_0) \mid g \in G_0\}$
• $I_0(M) = \{I_g|_M \mid g \in G_0\}$
• **If** A **is an antip. set of** M**, then** $A \cup \{e\}$ **is an antip. set of** G**.**

- $\exists \tilde{A}$: **a max. antip. subgr.** $A \cup \{e\} \subset \tilde{A}$
- **If** A **is maximal in** M**, then** $A = M \cap \tilde{A}$**.**

$C_1, \ldots, C_k$: G_0**-conjugacy classes of maxi. antip. subgr. of** G

B_s: **a representative of** C_s $(1 \leq s \leq k)$

(We gave their explicit descriptions for some classical G**.)**

$\exists g \in G_0,\ 1 \leq \exists s \leq k, \quad \tilde{A} = I_g(B_s)$

$A = M \cap \tilde{A} = M \cap I_g(B_s) = I_g(M \cap B_s)$

Hence A **is** $I_0(M)$**-congruent to** $M \cap B_s$**.**

Therefore, a representative of an $I_0(M)$-congruence class of maximal antipodal sets of M is one of $M \cap B_1, \ldots, M \cap B_k$.

- **Using this principle, for some classical cpt. Riem. sym. sp. M, we determined $I_0(M)$-cong. classes of max. antip. sets of M and gave explicit descriptions of their representatives.**
- **$\exists M$, realized as a polar not of a connected G but of a disconnected G.**

e.g., $U(n)/O(n), U(2n)/Sp(n)$

3. Polars of disconnected compact Lie groups

G: a compact Lie group

G_0: the identity component of G

$G = G_0 \cup \bigcup_{\lambda \in \Lambda} G_\lambda$, G_λ: a conn. component

$$F(s_e, G) = (F(s_e, G) \cap G_0) \cup \bigcup_{\lambda \in \Lambda} (F(s_e, G) \cap G_\lambda)$$

We know $F(s_e, G) \cap G_0$ by Chen-Nagano.

We study $F(s_e, G) \cap G_\lambda$.

If $F(s_e, G) \cap G_\lambda \neq \emptyset$, for $\forall x_\lambda \in G_\lambda \cap F(s_e, G)$ we have $G_\lambda = G_0 x_\lambda = x_\lambda G_0$.

I_{x_λ} ($I_{x_\lambda}(y) = x_\lambda y x_\lambda^{-1}$) is an involutive automorphism of G_0.

The action defined by $g.h = ghI_{x_\lambda}(g)^{-1}$ **(**$g, h \in G_0$**) is called the twisted conjugate action by** I_{x_λ}**. (It is a Hermann action.)**

T_λ**: a maximal torus of the identity comp. of** $F(I_{x_\lambda}, G_0)$**.**

By a property of Hermann actions we have:

Proposition 2 $G_\lambda = \bigcup_{g \in G_0} g(x_\lambda T_\lambda)g^{-1}$

(It is well-known $G_0 = \bigcup_{g \in G_0} gTg^{-1}$ **for a maximal torus** T **of** G_0**.)**

$$F(s_e, G) \cap G_\lambda = \bigcup_{g \in G_0} g\{x \in x_\lambda T_\lambda \,|\, x^2 = e\} g^{-1}$$

In order to determine $F(s_e, G) \cap G_\lambda$, it is enough to determine $\{x \in x_\lambda T_\lambda \,|\, x^2 = e\}$ and G_0-conjugacy classes of each element of the set.
We can carry out them for each G on a case-by-case argument.

On the other hand, we have the following:
Proposition 3 **Assume $G_\lambda \cap F(s_e, G) \neq \emptyset$.**
(1) $G_0 \cup G_\lambda$ **is a subgroup.**

(2) **For** $x_\lambda \in G_\lambda \cap F(s_e, G)$**,** $G_0 \cup G_\lambda$ **is isomorphic to** $G_0 \rtimes \langle I_{x_\lambda} \rangle$**, where** $\langle I_{x_\lambda} \rangle$ **is the subgroup of** $\mathrm{Aut}(G_0)$ **generated by** I_{x_λ}**.**

Hence, the determination of polars of G **is reduced to the determination of polars of** $G_0 \rtimes \langle I_{x_\lambda} \rangle$**.**

$G_0 \rtimes \langle I_{x_\lambda} \rangle$ **consists of two connected components:**

$$G_0 \rtimes \langle I_{x_\lambda} \rangle = \{(g, \mathrm{id}) \mid g \in G_0\} \cup \{(g, I_{x_\lambda}) \mid g \in G_0\}$$

The group operation of $G_0 \rtimes \langle I_{x_\lambda} \rangle$:

For $g, h \in G_0,\ e' := \mathrm{id},\ \tau := I_{x_\lambda}$,

$(g, e')(h, e') = (gh, e'),\ (g, e')(h, \tau) = (gh, \tau)$,

$(g, \tau)(h, e') = (g\tau(h), \tau),\ (g, \tau)(h, \tau) = (g\tau(h), e')$.

Proof of Prop. 3: (1) **is easily seen by the group operation.** (2) $\varphi : G_0 \rtimes \langle I_{x_\lambda} \rangle \to G_0 \cup G_\lambda$ **defined by** $\varphi(g, \mathrm{id}) = g, \varphi(g, I_{x_\lambda}) = gx_\lambda$ **gives a Lie group isomorphism.**

G**: a connected cpt. Lie group**

σ**: an involutive atumorphism of** G

$\widehat{e} = (e, \mathrm{id})$**: the identity element of** $G \rtimes \langle \sigma \rangle$

Theorem 4

$F(s_{\hat{e}}, G \rtimes \langle\sigma\rangle) = (F(s_e, G), \mathrm{id}) \cup (F(s_e \circ \sigma, G), \sigma)$

In particular, each connected component of $(F(s_e \circ \sigma, G), \sigma)$ **is a polar of** $G \rtimes \langle\sigma\rangle$**. Moreover, the conn. comp. of** $(F(s_e \circ \sigma, G), \sigma)$ **containing** (e, σ) **coincides with** $(\rho_\sigma(G) \cdot e, \sigma)$**, where** ρ_σ **is the twisted conjugate action by** σ**, and** $\rho_\sigma(G) \cdot e \cong G/F(\sigma, G)$**.**

Proof of Thm. 4 :

$F(s_{\hat{e}}, G \rtimes \langle\sigma\rangle) = F(s_{\hat{e}}, (G, \mathrm{id})) \cup F(s_{\hat{e}}, (G, \sigma))$

$$F(s_{\hat{e}}, (G, \mathrm{id})) = (F(s_e, G), \mathrm{id})$$

$$F(s_{\hat{e}}, (G, \sigma)) = (F(s_e \circ \sigma, G), \sigma)$$

$(\because)\ \forall g \in G,$

$$s_{\hat{e}}(g, \sigma) = (g, \sigma)$$

$$\Leftrightarrow\ (g, \sigma) = (g, \sigma)^{-1} = (\sigma(g^{-1}), \sigma)$$

$$\Leftrightarrow\ g = \sigma(g^{-1})$$

$$\Leftrightarrow s_e \circ \sigma(g) = g$$

As stated before, if we obtain the classification of max.antip. sugr. of $G \rtimes \langle \sigma \rangle$, we can determine max. antip. sets of $G/F(\sigma, G)$.

4. Examples

$U(n)$**: the unitary group**

$F(s_{1_n}, U(n)) =$

$\{x \in U(n) \,|\, x^2 = 1_n\} = \bigcup_{j=0}^{n} \{g\, I_j\, g^{-1} \,|\, g \in U(n)\}$

$I_j = \mathrm{diag}(\underbrace{-1, \ldots, -1}_{j}, 1, \ldots, 1) \in U(n)$

The polars of $U(n)$ **w.r.t.** 1_n **is:**

$\{1_n\}$, $\{-1_n\}$,

$U(n)/(U(j) \times U(n-j)) = G_j(\mathbb{C}^n) \quad (1 \leq j \leq n-1)$ **the complex Grassmann mfd.**

$\tau(g) := \bar{g} \ (g \in U(n))$

τ is an involutive autom. of $U(n)$

$G = U(n) \rtimes \langle\tau\rangle, \ \langle\tau\rangle = \{e', \tau\}$

$G = \{(g, e') \mid g \in U(n)\} \cup \{(g, \tau) \mid g \in U(n)\} \cdots (*)$

We write (g, e') by g, and (g, τ) by $g\tau$.

$(*) \rightsquigarrow G = U(n) \cup U(n)\tau$

$F(s_{\hat{e}}, G) = (F(s_{\hat{e}}, G) \cap U(n)) \cup (F(s_{\hat{e}}, G) \cap U(n)\tau)$

$F(s_{\hat{e}}, G) \cap U(n) = F(s_{1_n}, U(n)) = \bigcup_{j=0}^{n} G_j(\mathbb{C}^n)$

We study $F(s_{\hat{e}}, G) \cap U(n)\tau$ by using Thm. 4.

T**: a maximal torus of $F(\tau, U(n)) = O(n)$**

$U(n)\tau = \bigcup_{g \in U(n)} g(\tau T)g^{-1}$ **(by Prop. 2)**

$$F(s_{\hat{e}}, G) \cap U(n)\tau = \bigcup_{g \in U(n)} g\{x \in \tau T \mid x^2 = 1_n\} g^{-1}$$

So we study $\{x \in \tau T \mid x^2 = 1_n\}$**. We can take** $T \subset O(n)$ **as**

$$T = \left\{ \begin{bmatrix} R(\theta_1) & & & \\ & \ddots & & \\ & & R(\theta_k) & \\ & & & (1) \end{bmatrix} \middle| \theta_1, \dots, \theta_k \in \mathbb{R} \right\},$$

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \qquad k = \lfloor \tfrac{n}{2} \rfloor$$

$$\forall t \in T,\ \ \tau t = (1_n, \tau)(t, e') = (\tau(t), \tau) = t\tau,$$

$$(\tau t)^2 = \tau^2 t^2 = t^2$$

Hence, $\{x \in \tau T \mid x^2 = 1_n\} = \tau\{t \in T \mid t^2 = 1_n\}$

$$= \tau \left\{ \begin{bmatrix} \epsilon_1 1_2 & & & \\ & \ddots & & \\ & & \epsilon_k 1_2 & \\ & & & (1) \end{bmatrix} \,\middle|\, \epsilon_1, \ldots, \epsilon_k = \pm 1 \right\}.$$

$$F(s_{\widehat{e}}, G) \cap U(n)\tau = \bigcup_{g \in U(n)} g\tau\{t \in T \mid t^2 = 1_n\}g^{-1}$$

- $\forall t \in T, \forall g \in U(n),\ \ g(\tau t)g^{-1} = g\, t\, {}^t g\, \tau$
- **Since** $(i1_2)(-1_2)(i1_2) = 1_2$,

$\forall t \in T, t^2 = 1_n,\ \ \exists h \in U(n)$ s.t. $h\, t\, {}^t h = 1_n$.

Hence, if $t \in T, t^2 = 1_n,\ \ \{g(\tau t)g^{-1} \mid g \in U(n)\} =$
$\{g\, t\, {}^t g \mid g \in U(n)\}\tau = \{g\, 1_n\, {}^t g \mid g \in U(n)\}\tau.$

So $F(s_{\hat{e}}, G) \cap U(n)\tau = \{g\,1_n\,{}^t g \mid g \in U(n)\}\tau$

Here $g1_n\,{}^t g = g1_n\bar{g}^{-1} = g1_n\tau(g)^{-1} = \rho_\tau(g)(1_n)$.

ρ_τ**: the twisted conjugate action by** τ**.**

Hence $\{g\,1_n\,{}^t g \mid g \in U(n)\}$ **is an orbit of** $\rho_\tau(G)$ **through** 1_n**.**

$g\,1_n\,{}^t g = 1_n \Leftrightarrow {}^t g = g^{-1} = {}^t\bar{g} \Leftrightarrow g \in O(n)$

$F(s_{\hat{e}}, G) \cap U(n)\tau \cong U(n)/O(n)$ **(connected)**

$U(n)/O(n)$ **is realized as a polar of** $U(n) \rtimes \langle\tau\rangle$**.**

($U(n)/O(n)$ **is not realized as a polar of a connected compact Lie group.)**

Thank you for your kind attention.