Maximal antipodal sets of classical symmetric spaces

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NCTS

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1. Antipodal sets and the 2-numbers

2. Basic principle of classifying maximal antipodal sets of a compact symmetric space

3. Classification of maximal antipodal sets of compact classical symmetric spaces

4. The maximum of the cardinalities of maximal antipodal sets

1. Antipodal sets and the 2-numbers

M : a compact (Riemannian) symmetric space

s_x : the geodesic symmetry at $x \in M$

i.e., (i) s_x : an isometry, (ii) $s_x \circ s_x =$ id, (iii) x is an isolated fixed point of s_x.

$A \subset M$: a subset

A : an antipodal set $\overset{\text{def}}{\Longleftrightarrow} \forall x, y \in A,\ s_x(y) = y$

An antipodal set is finite.

The 2-number of M :

$\#_2 M := \max\{|A| \mid A \subset M : \text{an antipodal set}\}$

A : a great antipodal set $\overset{\text{def}}{\Longleftrightarrow} |A| = \#_2 M$

(Chen-Nagano 1982, 1988)

Example 1. $M = S^n \, (\subset \mathbb{R}^{n+1})$ **: the sphere**

$x \in S^n$, $\{x, -x\}$ **: a great antipodal set**

$\#_2 S^n = 2$

Example 2. $M = \mathbb{R}P^n$ **: the real proj. sp.**

$x \in \mathbb{R}P^n$, $s_x(y) = \rho_x(y) \ (y \in \mathbb{R}P^n)$

$s_x(y) = y \ \Leftrightarrow \ y = x$ **or** $y \subset x^{\perp}$

$\{e_1, \ldots, e_{n+1}\}$ **: an o.n.b. of** $\mathbb{R}^{n+1}$

$\{\langle e_1 \rangle_{\mathbb{R}}, \ldots, \langle e_{n+1} \rangle_{\mathbb{R}}\}$ **: a great antipodal set**

$\#_2 \mathbb{R}P^n = n + 1$

Example 3. G **: a compact Lie group**

$\exists g$ **: a bi-invariant Riemannian metric**

G **is a symmetric space w.r.t.** g**.**

$x \in G, \quad s_x(y) = xy^{-1}x \quad (y \in G)$

e **: the identity element,** $\quad s_e(y) = y^{-1}$

$s_e(y) = y \;\Leftrightarrow\; y^2 = e$

If $x^2 = y^2 = e$**,** $\quad s_x(y) = y \;\Leftrightarrow\; xy = yx$

$A \subset G$ **: an antipodal set,** $e \in A$

$\forall x, y \in A,\; x^2 = y^2 = e,\; xy = yx$

A **: maximal** $\;\Rightarrow\;$ **a subgroup**

$A \cong \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{r},\; |A| = 2^r$

$\#_2 G = 2^{\hat{r}}$**,** $\qquad \hat{r}$ **: the 2-rank of** G

E.g. $U(n)$ **: the unitary group**

A **: a maximal antipodal subgroup of** $U(n)$

Each eigenvalue of $\forall x \in A$ **is** ± 1 **and** A **is simultaneously diagonalizable.**

A **is conjugate to** $\Delta_n := \{\mathrm{diag}(\pm 1, \ldots, \pm 1)\}$**.**

$\#_2 U(n) = 2^n$

<u>Fact 1.</u> $N \subset M$ **: totally geodesic**

$\forall x \in N, \ s_x(N) = N \rightsquigarrow N$ **: a cpt symm. sp.**

$A \subset N$ **: antipodal** $\Rightarrow$ $A \subset M$ **: antipodal**

$\#_2 N \leq \#_2 M$

Fact 2. (Chen-Nagano 1988)

M : a connected cpt symmetric space

$\chi(M)$: the Euler characteristic of M

$$\#_2 M \geq \chi(M)$$

Fact 3. M : a conn. cpt symmetric space

M : a symmetric R-space $\Leftrightarrow$ M : an orbit of the linear isotropy representation of a symmetric space of compact type

(Takeuchi 1989)

M : a symmetric R-space

$b_k(M, \mathbb{Z}_2)$: the k-th $\mathbb{Z}_2$-Betti number

$$\#_2 M = \sum_{k \geq 0} b_k(M, \mathbb{Z}_2)$$

Fact 4. L : a symmetric R-space $\Leftrightarrow$

$\exists \tau$: an involutive anti-holomorphic isometry of a Hermit. sym. sp. M of cpt type s.t.

$L = \{x \in M \mid \tau(x) = x\}$: a real form

(T.-Tasaki 2012)

M : a Hermit. sym. sp. of cpt type

L_1, L_2 : real forms of M, $L_1 \pitchfork L_2$

$\Rightarrow L_1 \cap L_2$ is an antipodal set of L_i $(i = 1, 2)$.

Moreover, if L_1, L_2 are $I_0(M)$-congruent, $L_1 \cap L_2$ is a great antipodal set.

Application : Determination of Lagrangian Floer homology (Iriyeh-Sakai-Tasaki 2013)

Fact 5. **(T.-Tasaki 2013)**

M : a symmetric R-space

(i) Any antipodal set of M is contained in a great antipodal set.

(ii) Any two great antipodal sets of M are $I_0(M)$-congruent.

A : an antipodal set, A : maximal $\Leftrightarrow$
A' : an antipodal set, $A \subset A' \Rightarrow A = A'$
On a symmetric R-space, a maximal antipodal set is a great antipodal set.
In general, a maximal antipodal set is not necessarily great.

Aim : To understand maximal antipodal sets of compact symmetric spaces (classifications, cardinalities, properties, etc.).

2. Basic principle of classifying maximal antipodal sets of a compact symmetric space

G : a compact Lie group

e : the identity element of G

G_0 : the identity component of G

Each connected component of $F(s_e, G) := \{g \in G \mid s_e(g) = g\}$ is called a polar of G.

$\{e\}$: a trivial polar

A polar : a totally geodesic submanifold

$\Rightarrow$ a compact symmetric space

Example. $G = U(n)$

$$F(s_{1_n}, U(n)) = \bigcup_{j=0}^{n} \{g\, I_j\, g^{-1} \,|\, g \in U(n)\}$$

$$I_j = \mathrm{diag}(\underbrace{-1, \ldots, -1}_{j}, \underbrace{1, \ldots, 1}_{n-j})$$

$g \in G$, $\tau_g(h) := ghg^{-1} \ (h \in G)$

Lemma. M **: a polar of** G, $x \in M$

$$\Rightarrow \quad M = \{\tau_g(x) \,|\, g \in G_0\}$$

$$I_0(M) = \{\tau_g|_M \,|\, g \in G_0\}$$

A **: a maximal antipodal set of a polar** M

$A \cup \{e\}$ **is an antipo. set of** G **by** $A \subset F(s_e, G)$.

$\exists \tilde{A}$: a max. antipo. subgr. (MAS) of G s.t.

$A \cup \{e\} \subset \tilde{A}$

$A = M \cap \tilde{A}$ by the maximality of A

$B_1, \ldots, B_k$: the representatives of each G_0-conj. class of MAS of G

$1 \leq \exists s \leq k,\ \exists g \in G_0$ s.t. $\tilde{A} = \tau_g(B_s)$

$A = M \cap \tilde{A} = M \cap \tau_g(B_s) = \tau_g(M \cap B_s)$

A representative of an $I_0(M)$-congruence class of maximal antipodal sets of M is one of $M \cap B_1, \ldots, M \cap B_k$.

3. Classification of maximal antipodal sets of classical compact symmetric spaces

$\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$

$O(n, \mathbb{K}) := O(n), U(n), Sp(n)$

$G_m(\mathbb{K}^n)$ **: the Grassman manifold of the** m**-dim.** $\mathbb{K}$**-subpsaces in** $\mathbb{K}^n$

$G_m(\mathbb{K}^n) \cong O(n, \mathbb{K})/O(m, \mathbb{K}) \times O(n-m, \mathbb{K})$

$\iota : G_m(\mathbb{K}^n) \ni x \mapsto \rho_x \in O(n, \mathbb{K})$ **: an embedding, the image is a polar.**

Any maximal antipodal subgroup of $O(n, \mathbb{K})$ **is conjugate to** Δ_n**.**

$$\iota(G_m(\mathbb{K}^n)) \cap \Delta_n$$

$$= \Delta_n^m := \{\mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_n) \in \Delta_n \mid$$

$$|\{i|\varepsilon_i = 1\}| = m,\ |\{i|\varepsilon_i = -1\}| = n - m\}$$

By taking the inverse image under ι :

Theorem 1. **Any maximal antipodal set of $G_m(\mathbb{K}^n)$ is $O(n, \mathbb{K})$-congruent to**

$$\{\langle e_{i_1}, \ldots, e_{i_m}\rangle_{\mathbb{K}} \mid 1 \leq i_1 < \cdots < i_m \leq n\}.$$

$\{e_1, \ldots, e_n\}$ **: the standard o.n.b. of** $\mathbb{K}^n$

$$\#_2 G_m(\mathbb{K}^n) = \binom{n}{m}$$

$\gamma : G_m(\mathbb{K}^{2m}) \ni x \mapsto x^{\perp} \in G_m(\mathbb{K}^{2m})$ **: an isometry**

$G_m(\mathbb{K}^{2m})^* := G_m(\mathbb{K}^{2m})/\{\mathrm{id}, \gamma\}$ **: cpt. sym. sp.**

$s_{[x]}([y]) = [s_x(y)] \quad ([x],[y] \in G_m(\mathbb{K}^{2m})^*)$

$O(2m,\mathbb{K})^* := O(2m,\mathbb{K})/\{\pm 1_{2m}\}$

$\pi_{2m} : O(2m,\mathbb{K}) \to O(2m,\mathbb{K})^*$ **: the projection**

$\iota \circ \gamma(x) = \iota(x^{\perp}) = -\rho_x \ (x \in G_m(\mathbb{K}^{2m}))$

$G_m(\mathbb{K}^{2m})^* \underset{\textbf{id}}{=} \iota(G_m(\mathbb{K}^{2m}))/\{\pm 1_{2m}\} \subset O(2m,\mathbb{K})^*$

: a polar

A **: maximal antipodal set of** $G_m(\mathbb{K}^{2m})^*$

$\{e\} \cup A$ **: antipodal set of** $O(2m,\mathbb{K})^*$

$\exists \tilde{A}$ **: maximal antipodal subgroup of** $O(2m,\mathbb{K})^*$

s.t. $\{e\} \cup A \subset \tilde{A}$

$A = \tilde{A} \cap G_m(\mathbb{K}^{2m})^*$

We classified maximal antipodal subgroups of $O(2m, \mathbb{K})^*$ (J. Lie Theory 2017).

$$D[4] := \left\{ \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}, \begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix} \right\}$$

$n := 2m = 2^k \cdot l$, l **: odd**

$0 \leq s \leq k$

$$D(s,n) := \underbrace{D[4] \otimes \cdots \otimes D[4]}_{s} \otimes \Delta_{n/2^s} \subset O(n)$$

$$= \{d_1 \otimes \cdots \otimes d_s \otimes d_0$$

$$\mid d_1, \ldots, d_s \in D[4], d_0 \in \Delta_{n/2^s}\}$$

$$\Delta_2 \subsetneq D[4]$$

$$D(k-1, 2^k) = \underbrace{D[4] \otimes \cdots \otimes D[4]}_{k-1} \otimes \Delta_2$$
$$\subsetneq \underbrace{D[4] \otimes \cdots \otimes D[4]}_{k-1} \otimes D[4] = D(k, 2^k)$$

$$\Gamma_{\mathbb{K}} := \begin{cases} \{1\} & (\mathbb{K} = \mathbb{R}) \\ \{1, \sqrt{-1}\} & (\mathbb{K} = \mathbb{C}) \\ \{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\} & (\mathbb{K} = \mathbb{H}) \end{cases}$$

$1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ **: the standard basis of** $\mathbb{H}$

MAS of $O(2m, \mathbb{K})^*$ **is conjugate to**

$$\pi_{2m}(\Gamma_{\mathbb{K}} D(s, 2m)) \quad (0 \leq s \leq k)$$

with some exceptions.

$$\exists g \in O(2m, \mathbb{K}) \ (\exists g \in SO(2m) \ \textbf{when} \ \mathbb{K} = \mathbb{R}),$$

$$\exists s \in \{0, \ldots, k\} \ \textbf{s.t.}$$

$$\tilde{A} = \pi_{2m}(g)\, \pi_{2m}(\Gamma_{\mathbb{K}} D(s, 2m))\, \pi_{2m}(g)^{-1}$$

$$A = \pi_{2m}(g)\pi_{2m}(\Gamma_{\mathbb{K}} D(s, 2m) \cap G_m(\mathbb{K}^{2m}))\pi_{2m}(g)^{-1}$$

$$PD(s, 2m) := \{d \in D(s, 2m) \mid d^2 = 1_{2m}\}$$

$$ND(s, 2m) := \{d \in D(s, 2m) \mid d^2 = -1_{2m}\}$$

$$D(s, 2m) \cap G_m(\mathbb{R}^{2m})$$

$$= \{d \in D(s, 2m) \mid d^2 = 1_{2m}, \mathrm{Tr} d = 0\}$$

$$= \{d_1 \otimes \cdots \otimes d_s \otimes d_0 \in PD(s, 2m) \mid$$

$$\exists d_i (0 \leq i \leq s) \ \textbf{Tr} d_i = 0\}$$

$AG(s,2m) := \pi_{2m}(D(s,2m) \cap G_m(\mathbb{R}^{2m}))$

MAS:= a maximal antipodal set

<u>**Theorem 2.**</u> **(T.-Tasaki)**

(1) MAS of $G_m(\mathbb{R}^{2m})^*$ **is** $SO(2m)^*$**-congruent to** $AG(s,2m)$ $(0 \leq s \leq k)$ **with the exceptions** $(*)$**.**

(2) MAS of $G_m(\mathbb{C}^{2m})^*$ **is** $U(2m)^*$**-congruent to** $AG(s,2m) \cup \pi_{2m}(\sqrt{-1}ND(s,2m))$ $(0 \leq s \leq k)$ **with the exceptions** $(*)$**.**

(3) MAS of $G_m(\mathbb{H}^{2m})^*$ **is** $Sp(2m)^*$**-congruent to** $AG(s,2m) \cup \pi_{2m}(\{\mathrm{i},\mathrm{j},\mathrm{k}\}ND(s,2m))$ $(0 \leq s \leq$

k) **with the exceptions** $(*)$.

$(*)$ **:** $AG(k-1, 2^k)$ **when** $2m = 2^k$ **and** $AG(0,4)$ **when** $2m = 4$.

$AG(0,4) = \pi_4(\{\pm I_1 \otimes 1_2, \pm 1_2 \otimes I_1, \pm I_1 \otimes I_1\})$
$\subsetneq AG(2,4)$

$CI(n) := \{x \in Sp(n) \mid x^2 = -1_n\} \cong Sp(n)/U(n)$

<u>Theorem 3.</u> Any maximal antipodal set of $CI(n)$ **is** $Sp(n)$**-congruent to** $\mathrm{i}\Delta_n$.

$\#_2 CI(n) = 2^n$

$Sp(n)^* := Sp(n)/\{\pm 1_n\}$

$\pi_n : Sp(n) \to Sp(n)^*$ **: the projection**

$CI(n)^* := \pi_n(CI(n)) = CI(n)/\{\pm 1_n\}$

$CI(n)^* \subset \{x \in Sp(n)^* \mid x^2 = \pi_n(1_n)\}$ **: a polar**

By the classification of MAS of $Sp(n)^*$ **we can determine the maximal antipodal sets of** $CI(n)^*$**.**

$DIII(n) := \{x \in SO(2n) \mid x^2 = -1_{2n}, \mathrm{Pf}(x) = 1\} \cong SO(2n)/U(n)$

$\mathrm{Pf}(x)$ **: the Pfaffian of** x

Theorem 4. **Any maximal antipodal set of $DIII(n)$ is $SO(2n)$-congruent to**

$$\left\{ \begin{bmatrix} \epsilon_1 J & & \\ & \ddots & \\ & & \epsilon_n J \end{bmatrix} \in SO(2n) \middle| \epsilon_i = \pm 1, \epsilon_1 \cdots \epsilon_n = 1 \right\}.$$

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Assume n is even.

$SO(2n)^* := SO(2n)/\{\pm 1_{2n}\}$

$\pi_{2n} : SO(2n) \to SO(2n)^*$ **: the projection**

$DIII(n)^* := \pi_{2n}(DIII(n)) = DIII(n)/\{\pm 1_{2n}\}$

$DIII(n)^* \subset SO(2n)^*$ **: a polar**

4. The maximum of the cardinalities of maximal antipodal sets

$M = G_m(\mathbb{R}^{2m})^*$

MAS of M is $SO(2m)^*$-congruent to $AG(s, 2m)$ $(0 \leq s \leq k)$ with some exceptions.

$|AG(0, 2m)| = |\pi_{2m}(\Delta_{2m}^m)| = \binom{2m}{m} \Big/ 2$

$|AG(s, 2m)|$

$= 2^{\frac{m}{2^{s-1}}-1}(2^{2s-1} + 2^{s-1} - 1) + \binom{m/2^{s-1}}{m/2^s} \Big/ 2$

$(1 \leq s \leq k)$

Set $\binom{m/2^{s-1}}{m/2^s} = 0$ **when** $m/2^s \notin \mathbb{Z}$.

We can show: $m \geq 5 \Rightarrow$

$|AG(0, 2m)| > |AG_{\mathbb{H}}(s, 2m)| > |AG_{\mathbb{C}}(s, 2m)| > |AG(s, 2m)| \ (1 \leq s \leq k)$

$m \leq 4 \Rightarrow$ **case-by-case**

Theorem 5. **(T.-Tasaki)**

Great antipodal sets of $G_m(\mathbb{R}^{2m})^*$ **(up to congruence) and** $\#_2 G_m(\mathbb{R}^{2m})^*$ **are as follows.**

(1) $m = 1 : AG(1,2), \#_2 G_1(\mathbb{R}^2)^* = 2 (= \#_2 S^1)$

(2) $m = 2 : AG(2,4),$

$$\#_2 G_2(\mathbb{R}^4)^* = 9 (= \#_2 \mathbb{R}P^2 \times \mathbb{R}P^2)$$

(3) $m = 4 : AG(0,8), AG(3,8),$

$$\#_2 G_4(\mathbb{R}^8)^* = 35$$

(4) $m \neq 1,2,4 : AG(0,2m),$

$$\#_2 G_m(\mathbb{R}^{2m})^* = \binom{2m}{m}/2$$

Thank you for your kind attention.