Maximal antipodal sets related to G_2

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Contents

1. Introduction

M: a compact Riemannian symmetric space

s_x: the geodesic symmetry at $x \in M$

$S \subset M$: a subset

S: an antipodal set $\overset{\text{def}}{\Longleftrightarrow} \forall x, y \in S, s_x(y) = y$

the 2-number of M:

$\#_2 M := \max\{|S| \mid S \subset M : \text{an antipodal set}\}$

S: a great antipodal set $\overset{\text{def}}{\Longleftrightarrow} |S| = \#_2 M$

(Chen-Nagano)

G: a cpt. Lie group $\Rightarrow G$ is a Riem. sym. sp. w.r.t. a biinvariant Riem. metric

$s_x(y) = xy^{-1}x \ (x, y \in G)$

A: a maximal antipodal set of G, $e \in A$

$\Rightarrow A$ is a comm. subgr. isom. to $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$

$\Delta := \{\mathrm{diag}(\pm 1, \pm 1, \pm 1, \pm 1)\} \subset U(4)$ is a unique maximal antipo. subgr. up to conjugacy.

$\pi : U(4) \to U(4)/\{\pm 1_4\}$: the natural proj.

$\pi(\Delta \cup i\Delta) \subset U(4)/\{\pm 1_4\}$: a maximal antipodal subgroup, not great

$\#_2 U(4)/\{\pm 1_4\} = 2^5 > 2^4 = |\pi(\Delta \cup i\Delta)|$

Aim: to classify maximal antipodal sets of cpt. Riem. sym. sp., describe them explicitly, and determine their cardinalities.

·$U(n), O(n), Sp(n), SU(n), SO(n)$ **and their quotient groups (T.-Tasaki, 2017)**

· **Grassmann mfds.**, $Sp(n)/U(n), SO(2n)/U(n)$ **and their quotient spaces (T.-Tasaki)**

·**Oriented Grassmann mfds.**, rank ≤ 4 **(Tasaki, 2013)**

· $G_2, G_2/SO(4)$ **(T.-Tasaki-Yasukura)**

2. Antipodal sets and polars

M: **a cpt. conn. Riem. sym. sp.,** $o \in M$

A connected component of $F(s_o, M) = \{p \in M \mid s_o(p) = p\}$ **is called a polar of** M **w.r.t.** o**.**

$F(s_o, M) = \bigcup_{i=0}^{k} M_i^+$, M_i^+: **a polar,** $M_0^+ = \{o\}$

$M_i^+, \dim M_i^+ > 0$: **a totally geod. submfd.** $\Rightarrow$ **cpt. Riem. sym. sp.**

A: **a maximal antipodal set of** M

$o \in A \Rightarrow A \subset F(s_o, M)$

$\exists i$ s.t. $A \cap M_i^+ \neq \emptyset$

$\Rightarrow A \cap M_i^+$: **an antipodal set of** M_i^+

Prop. When $F(s_o, M) = \{o\} \cup M_1^+$, the assignment $A \mapsto \{o\} \cup A$ from the set of max. antipo. sets in M_1^+ to the set of max. antipo. sets containing o in M induces the bijection between their congruence classes.

G: a cpt. Lie group

A: a maximal antipodal subgroup of G

$\exists M_i^+$: a polar of G w.r.t. e s.t. $A \cap M_i^+ \neq \emptyset$

$\Rightarrow A \subset Z_G(a)$: a max. antipo. subgr. of the centralizer of $a \in A \cap M_i^+$ in G

3. Maximal antipodal sets of G_2 and $G_2/SO(4)$

$\mathbb{H}$: **the quaternions**

$1, i, j, k$: **the standard basis of $\mathbb{H}$ over $\mathbb{R}$**

$\mathbb{O} = \mathbb{H} \times \mathbb{H}$: **the octonions,** $(m, a), (n, b) \in \mathbb{O}$

$(m, a)(n, b) = (mn - \bar{b}a, a\bar{n} + bm)$

$\mathrm{Im}\mathbb{O} = \mathrm{Im}\mathbb{H} \times \mathbb{H}$

$\mathrm{Aut}(\mathbb{O}) = \{\alpha \in GL_{\mathbb{R}}(\mathbb{O}) \mid \alpha(xy) = (\alpha x)(\alpha y),\ x, y \in \mathbb{O}\}$ **is a cpt. conn. Lie group of type G_2.**

We denote $\mathrm{Aut}(\mathbb{O})$ **by G_2.**

$Sp(1) = \{p \in \mathbb{H} \mid p\bar{p} = 1\}$

$\psi : Sp(1) \times Sp(1) \to GL_{\mathbb{R}}(\mathbb{O})$: **a homo. def. by**

$\psi(p, q)(m, a) := (qm\bar{q}, pa\bar{q})$

$(p, q \in Sp(1), (m, a) \in \mathbb{O})$

$\mathrm{Im}\psi \subset G_2$, $\mathrm{Ker}\psi = \{\pm(1, 1)\}$

Fact (Yokota):

$\mathrm{Im}\psi = Z_{G_2}(\psi(1, -1))$: **the centralizer of** $\psi(1, -1)$

$Z_{G_2}(\psi(1, -1)) \cong Sp(1)^2/\{\pm(1, 1)\} \cong SO(4)$

Fact (cf. Chen-Nagano):

$F(s_e, G_2) = \{e\} \cup M_1^+$,

$M_1^+ = \{g\,\psi(1, -1)\,g^{-1} \mid g \in G_2\} \cong G_2/SO(4)$

$\Psi := \{\psi(p, \pm p) \mid p = 1, i, j, k\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ **is a unique maximal antipodal subgroup of** $Z_{G_2}(\psi(1, -1))$ **up to conjugacy.**

Since M_1^+ **is the unique polar(**$\neq \{e\}$**) of** G_2**,**

Ψ **is a unique maximal antipodal subgroup of** G_2 **up to conjugacy.**

$\Psi_1 := \Psi \setminus \{\psi(1, 1)\}$ **is a unique maximal antipodal set of** M_1^+ **up to congruence.**

$\#_2 G_2 = |\Psi| = 8$

$\#_2 G_2/SO(4) = \#_2 M_1^+ = |\Psi_1| = 7$

4. $G_2/SO(4)$ as the associative Grassmannian

$\tilde{G}_{\text{ass}} := \{V \subset \text{Im}\mathbb{O} \mid V : \text{an associative 3-plane}\}$

V: **associative** $\overset{\text{def}}{\Longleftrightarrow}$ $\forall x, y, z \in V, (xy)z = x(yz)$

$\Leftrightarrow \mathbb{R} \oplus V \subset \mathbb{O}$: **a quaternionic subspace**

V **has a natural orientation.**

Fact (Harvey-Lawson):

$\tilde{G}_{\text{ass}} \cong G_2/SO(4)$

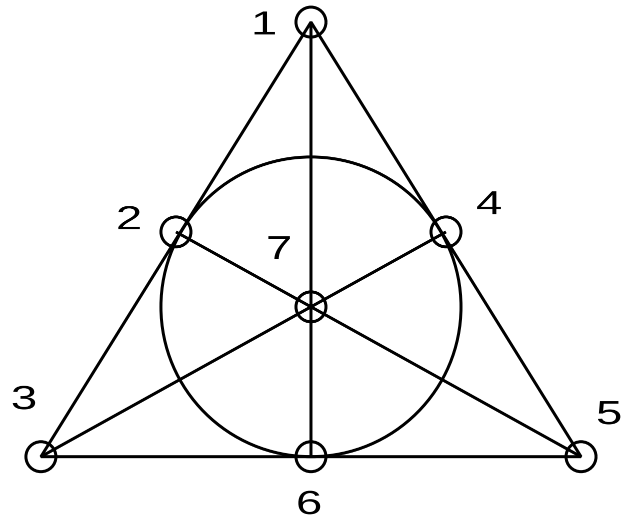
$M_1^+ \cong \tilde{G}_{\text{ass}}$ **by** $\xi \mapsto V(\xi) = \{x \in \text{Im}\mathbb{O} \mid \xi x = x\}$

$M_1^+ \supset \Psi_1 \leftrightarrow \tilde{\Psi}_1 := V(\Psi_1) \subset \tilde{G}_{\text{ass}}$

$\tilde{G}_{\text{ass}} \subset \tilde{G}_3(\mathbb{R}^7)$: **a totally geodesic submfd. of the oriented 3-palne Grassmannian**

$\tilde{\Psi}_1$ **is an antipodal set of** $\tilde{G}_3(\mathbb{R}^7)$**.**

Fano plane: the finite proj. plane with 7 lines and 7 points



Fact (Tasaki):

$\Phi := \{\{1,2,3\},\{1,4,5\},\{2,4,6\},\{3,5,6\},\{1,6,7\},$
$\{2,5,7\},\{3,4,7\}\}$: **the set of lines**

$e_1,\ldots,e_7$: **an o.n.b. of** $\mathbb{R}^7$

$\mathcal{A}(\Phi) := \{\pm\langle e_{\alpha_1}, e_{\alpha_2}, e_{\alpha_3}\rangle_{\mathbb{R}} \mid \{\alpha_1,\alpha_2,\alpha_3\} \in \Phi\}$ **is a unique max. antipo. set of** $\tilde{G}_3(\mathbb{R}^7)$ **up to congruence.**

$\#_2\tilde{G}_3(\mathbb{R}^7) = |\mathcal{A}(\Phi)| = 14$

$$M_1^+ \leftrightarrow \tilde{G}_{\mathrm{ass}} \quad \subset \quad \tilde{G}_3(\mathbb{R}^7)$$

$$\cup \qquad\qquad \cup \qquad\qquad\qquad \cup$$

$$\Psi_1 \ \leftrightarrow \ \tilde{\Psi}_1 \to \Phi \leftrightarrow \mathcal{A}(\Phi)$$

$$|\Psi_1| = |\tilde{\Psi}_1| = |\Phi| = 7$$

Under the correspondence $1 \leftrightarrow (i,0), 2 \leftrightarrow (j,0), 3 \leftrightarrow (k,0), 4 \leftrightarrow (0,k), 5 \leftrightarrow (0,j), 6 \leftrightarrow (0,i), 7 \leftrightarrow (0,1)$**,** $\tilde{\Psi}_1$ **(hence** Ψ_1**) determines a multiplication table of the unit octonions through the Fano plane.**

Thank you for your kind attention.