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Maximal antipodal subgroups of the compact Lie group G_2 of exceptional type

(Joint work with M.S. Tanaka and H. Tasaki)

Osami Yasukura
(Faculty of Engineering, University of Fukui)

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1 Maximal antipodal sets

Let M be a connected compact Riemannian symmetric space with the identity connected component $I(M)_0$ of the isometry group. And s_x the geodesic symmetry at $x \in M$.

Definition (Chen-Nagano, *Trans. AMS*, 1988)
(1) An *antipodal set* A_2 in M is a subset of M such that $s_x y = y$ $(x, y \in A_2)$.
(2) $\sharp_2 M$ is the maximal possible cardinality $\sharp A_2$ of an antipodal set A_2 in M.

(3) A *great* antipodal set A_2 is an antipodal set in M such that $\sharp A_2 = \sharp_2 M$.
(4) A *maximal* antipodal set A is an antipodal set in M such that $A' = A$ for all antipodal subset A' in M such as $A' \supseteq A$.
(5) Two antipodal sets A, A' in M are *congruent* iff $\alpha A = A'$ for some $\alpha \in I(M)_0$.

2 Poles and polars of a set of all fixed points

Put $F(s_x, M) := \{y \in M \mid s_x y = y\}$. Then $F(s_x, M) \setminus \{x\} = \{o_i \mid 1 \leq i \leq a\} \cup (\cup_{j=1}^{b} M_j^+)$ as a disjoint union of some *poles*, *i.e.*, zero-dimensional connected components $\{o_i\}$ $(1 \leq i \leq a)$, and some *polars*, *i.e.*, positive-dimensional connected components M_j^+ $(1 \leq j \leq b)$ being compact Riemannian symmetric space with respect to the induced metric defined from the one of M.

Lemma 1. *If* $(a, b) \in \{(0,1),(1,1)\}$, *then the assignment with respect to* $x \in M$ *defined as*

$$A_1 \mapsto A_1' := \{x\} \cup \{o_i \mid 1 \leq i \leq a\} \cup A_1$$

from the set of all maximal antipodal sets in M_1^+ *to that in* M *induces a surjection between their congruent class.*

Proof. Let A be a maximal antipodal set in M containing x, $A_1 := A \setminus \{x, o_i\} \subseteq F(s_x, M) \setminus \{x, o_i\} = M_1^+$ is *a priori* a maximal antipodal set in M_1^+ such that $A_1' = A$. //

3 Maximal antipodal subgroups

Let M be a connected compact Lie group, which is a Riemannian symmetric space with a bi-invariant metric. Any two conjugate subgroups of M are congruent in M, and vice varsa if M is a simple Lie group.

Remark. (Chen-Nagano, Remarks 1.2, 1.3) *Any maximal antipodal set A in M containing the unit e is a discrete abelian subgroup of M, which is isomorphic to $(\boldsymbol{Z}_2)^t$ with $2^t < \infty$.*

4 Connected simple Lie group G_2

Theorem 1(Nagano, *Tokyo J.Math.*, 1988; p.66)
$F(s_e, G_2)\backslash\{e\} = M_1^+ \cong G_2/SO(4)$.
Moreover, $F(s_o, M_1^+)\backslash\{o\} = M_{1,1}^+ \cong S^2 \cdot S^2$ for $o \in M_1^+$, where $S^2 \cdot S^2$ is defined as $(S^2 \times S^2)/\boldsymbol{Z}_2$ by a natural action of $\boldsymbol{Z}_2 := \{\pm(1,1)\}$ on $S^2 \times S^2$ (Chen-Nagano, 3.8).

Lemma 2. *Put* $M := (S^2 \times S^2)/\boldsymbol{Z}_2$, $[\vec{x}, \vec{y}] := \{\pm(\vec{x}, \vec{y})\}$ *and* $x_{\pm i} := [\vec{e}_i, \pm\vec{e}_i]$ *for an arbitary orthonormal frame* $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ *of* $\boldsymbol{R}^3$.

Then any maximal antipodal set in M is congruent to $A := \{x_{\pm i} \mid i = 1, 2, 3\}$.

Proof. $F(s_{x_1}, M) \setminus \{x_1\} = \{x_{-1}\} \cup M_1^+$; $M_1^+ := (S^2 \cap \vec{e}_1^{\perp})^2 / \boldsymbol{Z}_2$. Any maximal antipodal set in M is congruent to $A_1' := \{x_{\pm 1}\} \cup A_1$ for some maximal antipodal set $A_1 \ni x_2$ in M_1^+ by Lemma 1 $(a = 1)$. Then $A_1 \setminus \{x_2\} \subseteq \{x_{-2}\} \cup (S^2 \cap \vec{e}_1^{\perp} \cap \vec{e}_2^{\perp})^2 / \boldsymbol{Z}_2 = \{x_{-2}, x_{\pm 3}\}$, so that $A_1' \subseteq A$ which is antipodal. Since A_1' is maximal, $A_1' = A$. //

Theorem 2. (Tanaka-Tasaki-Y.) *For the maximal antipodal set* A *in* $(S^2 \times S^2)/\boldsymbol{Z}_2$ *defined in Lemma 2, put* $B := \varphi(A)$ *by an isometry* $\varphi : (S^2 \times S^2)/\boldsymbol{Z}_2 \longrightarrow M^+_{1,1}$ *with respect to* $(S^2 \times S^2)/\boldsymbol{Z}_2 \cong M^+_{1,1}$ *mentioned in Theorem 1. Then:*

(1) *Any maximal antipodal set in* M^+_1 *is congruent to* $B' := \{o\} \cup B$*;*

(2) *Any maximal antipodal subgroup of* G_2 *is conjugate to* $B'' := \{e, o\} \cup B$*.*

5 Explicit description of G_2

· $\boldsymbol{H} := \boldsymbol{R}1 \oplus \boldsymbol{R}\boldsymbol{i} \oplus \boldsymbol{R}\boldsymbol{j} \oplus \boldsymbol{R}\boldsymbol{k}$: the quaternions with the Hamilton's triple $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ and the conjugation $\bar{b} := b_0 1 - b_1 \boldsymbol{i} - b_2 \boldsymbol{j} - b_3 \boldsymbol{k}$ of $b = b_0 1 + b_1 \boldsymbol{i} + b_2 \boldsymbol{j} + b_3 \boldsymbol{k} \in \boldsymbol{H}$.

· $\boldsymbol{O} := \boldsymbol{H} \times \boldsymbol{H}$: the octanions defined by Cayley-Dickson process providing the product $xy := (mn - \overline{b}a, a\overline{n} + bm)$ for $x = (m, a)$ and $y = (n, b) \in \boldsymbol{O}$ with $\bar{x} := (\bar{m}, -a) \in \boldsymbol{O}$, $(x \mid y) := (x\bar{y} + y\bar{x})/2 \in \boldsymbol{R}$ and

$G := \{\alpha \in GL_{\boldsymbol{R}}(\boldsymbol{O}) \mid \alpha(xy) = (\alpha x)(\alpha y)\}$ as the conjugation, a positive-definite $\boldsymbol{R}$-bilinear inner product and the group of all automorphisms on the $\boldsymbol{R}$-algebra $\boldsymbol{O}$. For any $\alpha \in G, x, y \in \boldsymbol{O}$, one has that $\alpha 1 = 1$, $\overline{\alpha x} = \alpha \bar{x}$ and $(\alpha x \mid \alpha y) = (x \mid y)$. Put $\mathrm{Im}\boldsymbol{O} := \{x \in \boldsymbol{O} \mid \bar{x} = -x\} \cong \boldsymbol{R}^7$, $S^6 := \{x \in \mathrm{Im}\boldsymbol{O} \mid (x \mid x) = 1\} \ni (\boldsymbol{i}, 0)$ and $H := \{\alpha \in G \mid \alpha(\boldsymbol{i}, 0) = (\boldsymbol{i}, 0)\}$.

Proposition 1. (1) G *acts transitively on* S^6

such that $H \cong SU(3)$*, so that* $G/H \cong S^6$*.*
(2) G *is a connected, simply connected, compact Lie group of dimension* 14*.*
(3) $\mathrm{rank}\ G = \mathrm{rank}\ H = 2$.

Proof. (1) e.g. Yokota, *Groups and Topology* (*Gun to isoh* in Japanese), Shōkabō, 1971, pp.250–251. (2) is a consequence of (1).

(3) By (1), there exists an isomorphism $f : SU(3) \longrightarrow H$. *Let* T^2 *be a maximal torus of* $SU(3)$*. Then* $G_2 = \cup_{\alpha \in G}\ \alpha f(T^2)\alpha^{-1}$*:*

In fact, by (2), $G \subseteq SO(\mathrm{Im}\boldsymbol{O}) \cong SO(7)$. Since any element of $SO(7)$ admits a fixed-point in S^6, any $\alpha \in G$ admits some $p \in S^6$ such that $\alpha p = p$. By (1), $\beta p = (\boldsymbol{i}, 0)$ for some $\beta \in G$. Then $(\beta\alpha\beta^{-1})(\boldsymbol{i}, 0) = (\boldsymbol{i}, 0)$. Hence, $\beta\alpha\beta^{-1} = f(A)$ for some $A \in SU(3)$. For some $B \in SU(3)$, $BAB^{-1} \in T^2$. Hence, $(f(B)\beta)\ \alpha\ (f(B)\beta)^{-1} \in f(T^2)$. //

Put $Sp(1) := \{q \in \boldsymbol{H} \mid |q| = 1\}$,

$$\psi : Sp(1) \times Sp(1) \longrightarrow GL_{\boldsymbol{R}}(\boldsymbol{O});$$

$$\psi(p,q)(m,a) := (qm\overline{q}, pa\overline{q}).$$

Moreover, put $e = \psi(1,1), \gamma := \psi(1,-1)$,

$$G^{\gamma} := \{\alpha \in G \mid \alpha\gamma = \gamma\alpha\}.$$

Proposition 2(Yokota, J.F.S.Shinshu U., 1977)

(1) $\psi(Sp(1) \times Sp(1)) = G^{\gamma}$,

(2) $\mathrm{ker}\psi = \{\pm(1,1)\}$, $G^{\gamma} \cong SO(4)$.

Proof. e.g. Yokota, *Tsukuba J.Math.* **14**-1 (1990), 185–223; 1.3.3, 1.3.4. //

Corollary. *G is a connected, simply connected, compact, simple Lie group of type G_2 with $z(G) = \{e\}$.*

Proof. (1) (Yokota, arXiv:0902.0431v1, Theorem 1.11.1) $z(G) = \{e\}$: In fact, $z(G) \subset z(G^\gamma) = z(\psi(Sp(1) \times Sp(1))) = \{\psi(1, \pm 1)\} = \{e, \gamma\}$ and $\gamma \notin z(G)$ by $\mathrm{dim} G^\gamma = 6 < 14 = \mathrm{dim} G$. (2) By (1), G is semisimple, so that the type is $A_1 \oplus A_1$, A_2, G_2. By Proposition 1 (2), $G = G_2$. //

6 Explicit description of polars in G_2

Theorem 3. (Tanaka-Tasaki-Y.)

(1) $F(s_e, G)\backslash\{e\} = M_1^+ = \{g\gamma g^{-1} \mid g \in G\}$
$\cong G_2/SO(4)$.

(2) $o := \gamma \in M_1^+, F(s_o, M_1^+)\backslash\{o\} = M_{1,1}^+$
$= \{\psi(p,q) \mid p^2 = q^2 = -1\} \cong (S^2 \times S^2)/\boldsymbol{Z}_2$.

(3) *Any maximal antipodal set in* $M_{1,1}^+$ *is congruent to* $B := \{\psi(p, \pm p) \mid p = \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$.

(4) *Any maximal antipodal set in* M_1^+ *is congruent to* $B' := \{\psi(1,-1)\} \cup B$.

(5) *Any maximal antipodal subgroup of* G_2 *is conjugate to* $B'' := \{\psi(1, \pm 1)\} \cup B$.

Proof. (1) Take a maximal torus of $SU(3)$ as $T^2 := \{A = \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3) \mid A \in SU(3)\}$; $F(s_e, T^2) = \{\mathrm{diag}(\pm 1, \pm 1, \pm 1) \mid \det = 1\} = \{e\} \cup \{A_i \mathrm{diag}(1, -1, -1) A_i^{-1} \mid i = 1, 2, 3\}$ with some $A_i \in SU(3)$. By Proposition 1 (3), $\gamma \in F(s_e, G) \backslash \{e\} = \cup_{g \in G}\, g f(F(s_e, T^2)) g^{-1} \backslash \{e\} = \cup_{g \in G} \{g \gamma g^{-1} \mid g \in G\} \cong G/G^{\gamma}$, which is connected since G is connected.

Hence, $G_2/SO(4) \cong F(s_e, G)\backslash\{e\} = M_1^+$.
(2) $F(s_\gamma, M_1^+)\backslash\{\gamma\} = M_1^+ \cap G^\gamma\backslash\{\gamma\} = \{\psi(p,q) \mid (p^2, q^2) = \pm(1,1)\}\backslash\{e,\gamma\} = \{\psi(p,q) \mid (p^2, q^2) = -(1,1)\} \cong (S^2 \times S^2)/\boldsymbol{Z}_2$, because of $e = \psi(1,1)$, $\gamma = \psi(1,-1)$ and $\{p \in Sp(1) \mid p^2 = -1\}$
$= \{p \in Sp(1) \mid p = -\bar{p}\}$
$= \{p = p_1\boldsymbol{i} + p_2\boldsymbol{j} + p_3\boldsymbol{k} \mid \sum_{i=1}^3 p_i^2 = 1\}$.
(3) follows from Lemma 2 because of (2). (4) (resp. (5)) follows from Lemma 1 with $a = 0$ because of (3) (resp. (4)) and (1). //

7 Conclusion.

Corollary. (Chen-Nagano, 3.13)
$\sharp_2(S^2 \cdot S^2) = 6$, $\sharp_2 G_2/SO(4) = 7$, $\sharp_2 G_2 = 8$.

Proof. In $S^2 \cdot S^2$ (resp. $G_2/SO(4)$, G_2), B (resp. B', B'') is a great antipodal set as unique maximal one up to congruence. //

Remark. *Posteriorly*, Theorem 3 (5) is verified by heavy use of weights of B'' on $\boldsymbol{O} = \boldsymbol{R}^8$. By the use of Lemma 1, Theorem 3 provides *apriori* classification for G_2.

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