グラスマン多様体とその商空間の極大対蹠集合

Antipodal sets of Grassmann manifolds and their quotient spaces

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1. Antipodal sets and 2-numbers

M : compact Riemann symmetric space

s_x : geodesic symmetry at x

i.e., (i) s_x is an isometry of M, (ii) ${s_x}^2 = \mathrm{id}$,

(iii) x is an isolated fixed point of s_x

$S \subset M$: subset

S : antipodal set $\overset{\text{def}}{\iff} \forall x, y \in S,\ s_x(y) = y$

2-number of M

$$\#_2 M := \max\{|S| \mid S \subset M \text{ antipodal set}\}$$

S : great antipodal set $\overset{\text{def}}{\iff} |S| = \#_2 M$

(Chen-Nagano 1988)

Examples. **(1)** $M = S^n \, (\subset \mathbb{R}^{n+1})$

$\{x, -x\}$ $(x \in S^n)$ **: great antipodal set**

$\#_2 S^n = 2$

(2) $M = \mathbb{R}P^n$

$e_1, \ldots, e_{n+1}$ **: o.n.b. of** $\mathbb{R}^{n+1}$

$\{\langle e_1 \rangle_{\mathbb{R}}, \ldots, \langle e_{n+1} \rangle_{\mathbb{R}}\}$ **: great antipodal set**

$\#_2 \mathbb{R}P^n = n + 1$

(3) $M = U(n)$ $\qquad s_x(y) = xy^{-1}x$

$s_{1_n}(y) = y \;\Leftrightarrow y^2 = 1_n$ **(**1_n **: identity matrix)**

$x^2 = y^2 = 1_n \;\Rightarrow\; s_x(y) = y$ **iff** $xy = yx$

$$\Delta_n := \left\{ \begin{bmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{bmatrix} \right\} : \textbf{ great antipodal set}$$

$$\#_2 U(n) = 2^n$$

Theorem 1. (T.-Tasaki 2013)

M **: symmetric** R**-space**

(1) Any antipodal set of M **is included in a great antipodal set.**

(2) Any two great antipodal sets are $I_0(M)$**-congruent.**

$I_0(M)$: **identity component of $I(M)$, the group of isometries of M**

(3) A great antipodal set of M is an orbit of the Weyl group.

$S^n, \mathbb{R}P^n$ **and** $U(n)$ **are symmetric** R**-spaces.**

A maximal antipodal set is not necessarily a great antipodal set.

2. Maximal antipodal sets of Grassmann manifolds (1)

$$O(n,\mathbb{K}) := \begin{cases} O(n) & (\mathbb{K}=\mathbb{R}) \\ U(n) & (\mathbb{K}=\mathbb{C}) \\ Sp(n) & (\mathbb{K}=\mathbb{H}) \end{cases}$$

$G_m(\mathbb{K}^n)$: **Grassmann manifold of m-dim $\mathbb{K}$-subspaces of $\mathbb{K}^n$**

$O(n,\mathbb{K}) \curvearrowright G_m(\mathbb{K}^n)$ **transitively**

$G_m(\mathbb{K}^n) \cong O(n,\mathbb{K})/O(m,\mathbb{K}) \times O(n-m,\mathbb{K})$

$G_m(\mathbb{K}^n)$ **is a Riemann symmetric space w.r.t. $O(n,\mathbb{K})$-invariant Riemann metric.** $G_m(\mathbb{K}^n)$ **is a symmetric R-space.**

$x \in G_m(\mathbb{K}^n)$

$\pi_x, \pi_{x^\perp}$ **: orthogonal projection onto** $x, x^\perp$

$\rho_x := \pi_x - \pi_{x^\perp} : \mathbb{K}^n \to \mathbb{K}^n$

$s_x(y) = \rho_x(y) \ \ (y \in G_m(\mathbb{K}^n))$

$e_1, \ldots, e_n$ **: o.n.b. of** $\mathbb{K}^n$

$\rho_{e_i}(e_i) = e_i, \ \ \rho_{e_i}(e_j) = -e_j \ \ (i \neq j)$

$s_{\langle e_{i_1}, \ldots, e_{i_m} \rangle_{\mathbb{K}}}(\langle e_{j_1}, \ldots, e_{j_m} \rangle_{\mathbb{K}}) = \langle e_{j_1}, \ldots, e_{j_m} \rangle_{\mathbb{K}}$

$A := \{\langle e_{i_1}, \ldots, e_{i_m} \rangle_{\mathbb{K}} \mid 1 \leq i_1 < \cdots < i_m \leq n\}$ **is an antipodal set of** $G_m(\mathbb{K}^n)$**.**

$|A| = \binom{n}{m} = \#_2 G_m(\mathbb{K}^n)$ **(Chen-Nagano)**

A **is a unique great antipodal set up to** $I_0(G_m(\mathbb{K}^n))$**-congruence by Theorem 1.**

3. Maximal antipodal sets of Grassmann manifolds (2)

$O(n,\mathbb{K})$ **is a Riemann symmetric space w.r.t. a bi-invariant metric.**

$x \in O(n,\mathbb{K})$, $s_x(y) = xy^{-1}x \ (y \in O(n,\mathbb{K}))$

1_n **: identity matrix,** $s_{1_n}(y) = y \Leftrightarrow y^2 = 1_n$

$$F(s_{1_n}, O(n,\mathbb{K})) = \bigcup_{0\leq k\leq n} \bigcup_{g\in O(n,\mathbb{K})} g \begin{bmatrix} 1_k & \\ & -1_{n-k} \end{bmatrix} g^{-1}$$

$\iota : G_m(\mathbb{K}^n) \ni x \mapsto \rho_x \in O(n,\mathbb{K})$ **: embedding**

$$F(s_{1_n}, O(n,\mathbb{K})) = \bigcup_{0\leq k\leq n} \iota(G_k(\mathbb{K}^n))$$

G **: compact Lie group,** e **: identity element**

G_0 **: identity component of** G

M **: connected component of** $F(s_e, G)$

M **is a polar of** G **w.r.t.** e**.**

$x \in M,\ \ M = \{I_g(x) \mid g \in G_0\},\ \ I_g(x) = gxg^{-1}$

$I_0(M) = \{I_g|_M \mid g \in G_0\}$

$\iota(G_m(\mathbb{K}^n)) \underset{\text{id}}{=} G_m(\mathbb{K}^n)$ **is a polar of** $O(n, \mathbb{K})$ **w.r.t.** 1_n**.**

A **: maximal antipodal set of** $G_m(\mathbb{K}^n)$

$\{1_n\} \cup A$ **: antipodal set of** $O(n, \mathbb{K})$

$\exists \tilde{A}$: **maximal antipodal subgroup of** $O(n, \mathbb{K})$

s.t. $\{1_n\} \cup A \subset \tilde{A}$

$\exists g \in O(n, \mathbb{K})$ **s.t.** $\tilde{A} = g\, \Delta_n\, g^{-1}$

When $\mathbb{K} = \mathbb{R}$**, we can take** $g \in SO(n)$**.**

$A = \tilde{A} \cap G_m(\mathbb{K}^n)$

$A = g\, \Delta_n\, g^{-1} \cap G_m(\mathbb{K}^n) = g(\Delta_n \cap G_m(\mathbb{K}^n))g^{-1}$

$$\Delta_n \cap G_m(\mathbb{K}^n)$$
$$= \Delta_n^m := \left\{ \begin{bmatrix} \varepsilon_1 & & \\ & \ddots & \\ & & \varepsilon_n \end{bmatrix} \in \Delta_n \middle| |\{i \mid \varepsilon_i = 1\}| = m \right\}$$

$A = g\, \Delta_n^m\, g^{-1}$

Δ_n^m is a unique maximal antipodal set (hence a unique great antipodal set) of $G_m(\mathbb{K}^n)$ up to $I_0(G_m(\mathbb{K}^n))$-congruence.

$|\Delta_n^m| = \binom{n}{m} = \#_2 G_m(\mathbb{K}^n)$

$\iota^{-1}(\Delta_n^k)$

$= \{\langle e_{i_1}, \ldots, e_{i_m}\rangle_{\mathbb{K}} \mid 1 \leq i_1 < \cdots < i_m \leq n\}$

4. The quotient space $G_m(\mathbb{K}^{2m})^*$ $(\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H})$

$\gamma : G_m(\mathbb{K}^{2m}) \ni x \mapsto x^\perp \in G_m(\mathbb{K}^{2m})$: **isometry**

$G_m(\mathbb{K}^{2m})^* := G_m(\mathbb{K}^{2m})/\{\mathrm{id}, \gamma\}$

$s_{[x]}([y]) = [s_x(y)] \quad ([x], [y] \in G_m(\mathbb{K}^{2m})^*)$

$O(2m, \mathbb{K})^* := O(2m, \mathbb{K})/\{\pm 1_{2m}\}$

$\pi_{2m} : O(2m, \mathbb{K}) \to O(2m, \mathbb{K})^*$: **projection**

$\iota \circ \gamma(x) = \iota(x^\perp) = \rho_{x^\perp} = -\rho_x \quad (x \in G_m(\mathbb{K}^{2m}))$

$G_m(\mathbb{K}^{2m})^* \underset{\mathrm{id}}{=} \iota(G_m(\mathbb{K}^n))/\{\pm 1_{2m}\} \subset O(2m, \mathbb{K})^*$

$G_m(\mathbb{K}^{2m})^*$ **is a polar of** $O(2m, \mathbb{K})^*$ **w.r.t.** $\pi_{2m}(1_{2m})$.

5. Maximal antipodal sets of $G_m(\mathbb{K}^{2m})^*$

$$D[4] := \left\{ \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}, \begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix} \right\}$$

$n = 2^k \cdot l,\ \ l$ **: odd**

$0 \le s \le k$

$$D(s,n) := \underbrace{D[4] \otimes \cdots \otimes D[4]}_{s} \otimes \Delta_{n/2^s} \subset O(n)$$

$$= \{ d_1 \otimes \cdots \otimes d_s \otimes d_0 \mid d_1, \ldots, d_s \in D[4], d_0 \in \Delta_{n/2^s} \}$$

$\Delta_2 \subsetneq D[4]$

$$D(k-1, 2^k) = \underbrace{D[4] \otimes \cdots \otimes D[4]}_{k-1} \otimes \Delta_2$$

$$\subsetneq \underbrace{D[4] \otimes \cdots \otimes D[4]}_{k-1} \otimes D[4] = D(k, 2^k)$$

Theorem 2. (T.-Tasaki 2017)

A maximal antipodal subgroup (MAS) of $O(n, \mathbb{K})^*$ is given as follows.

(1) $\mathbb{K} = \mathbb{R}$. MAS of $O(n)^*$ is $O(n)^*$-conjugate to one of the following.

$$\pi_n(D(s, n)) \quad (0 \leq s \leq k),$$

where the case $(s, n) = (k-1, 2^k)$ is excluded.

(2) $\mathbb{K} = \mathbb{C}$. MAS of $U(n)^*$ is $U(n)^*$-conjugate to one of the following.

$$\pi_n(\{1, \sqrt{-1}\}D(s, n)) \quad (0 \leq s \leq k),$$

where the case $(s, n) = (k-1, 2^k)$ is excluded.

(3) $\mathbb{K} = \mathbb{H}$. **MAS of** $Sp(n)^*$ **is** $Sp(n)^*$**-conjugate to one of the following.**

$$\pi_n(\{1, \mathrm{i}, \mathrm{j}, \mathrm{k}\}D(s, n)) \quad (0 \leq s \leq k),$$

where the case $(s, n) = (k-1, 2^k)$ **is excluded.**

In (1) we can replace "$O(n)^*$**-conjugate" by "**$SO(n)^*$**-conjugate".**

A **: maximal antipodal set of** $G_m(\mathbb{K}^{2m})^*$

$\{\pi_{2m}(1_{2m})\} \cup A$ **: antipodal set of** $O(2m, \mathbb{K})^*$

$\exists \tilde{A}$ **: maximal antipodal subgroup of** $O(2m, \mathbb{K})^*$

s.t. $\{\pi_{2m}(1_{2m})\} \cup A \subset \tilde{A}$

$$A = \tilde{A} \cap G_m(\mathbb{K}^{2m})^*$$

$$2m = 2^k \cdot l, \ \ l \ \textbf{: odd}$$

$$\Gamma_{\mathbb{K}} := \begin{cases} \{1\} & (\mathbb{K} = \mathbb{R}) \\ \{1, \sqrt{-1}\} & (\mathbb{K} = \mathbb{C}) \\ \{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\} & (\mathbb{K} = \mathbb{H}) \end{cases}$$

$$\exists g \in O(2m, \mathbb{K}) \ (\exists g \in SO(2m) \ \textbf{when} \ \mathbb{K} = \mathbb{R}),$$

$$\exists s \in \{0, \ldots, k\} \ \textbf{s.t.}$$

$$\tilde{A} = \pi_{2m}(g)\, \pi_{2m}(\Gamma_{\mathbb{K}} D(s, 2m))\, \pi_{2m}(g)^{-1}$$

$$A = \pi_{2m}(g)\, \pi_{2m}(\Gamma_{\mathbb{K}} D(s, 2m)) \pi_{2m}(g)^{-1} \cap G_m(\mathbb{K}^{2m})^*$$

$$= \pi_{2m}(g) \pi_{2m}(\Gamma_{\mathbb{K}} D(s, 2m) \cap G_m(\mathbb{K}^{2m})) \pi_{2m}(g)^{-1}$$

$$PD(s, 2m) := \{d \in D(s, 2m) \mid d^2 = 1_{2m}\}$$

$$ND(s,2m) := \{d \in D(s,2m) \mid d^2 = -1_{2m}\}$$

$$D(s,2m) \cap G_m(\mathbb{R}^{2m})$$

$$= \{d \in D(s,2m) \mid d^2 = 1_{2m}, \mathrm{Tr} d = 0\}$$

$$= \{d_1 \otimes \cdots \otimes d_s \otimes d_0 \in PD(s,2m) \mid$$

$$\exists d_i (0 \leq i \leq s) \ \ \mathbf{Tr} d_i = 0\}$$

$$AG(s,2m) := \pi_{2m}(D(s,2m) \cap G_m(\mathbb{R}^{2m}))$$

MAS:=maximal antipodal set

Theorem 3. (T.-Tasaki)

(1) MAS of $G_m(\mathbb{R}^{2m})^*$ is $SO(2m)^*$-congruent to $AG(s,2m)$ $(0 \leq s \leq k)$ with exceptions $(*)$.

(2) MAS of $G_m(\mathbb{C}^{2m})^*$ **is** $U(2m)^*$**-congruent to** $AG(s,2m) \cup \pi_{2m}(\sqrt{-1}ND(s,2m))$ $(0 \le s \le k)$ **with exceptions** $(*)$**.**

(3) MAS of $G_m(\mathbb{H}^{2m})^*$ **is** $Sp(2m)^*$**-congruent to** $AG(s,2m) \cup \pi_{2m}(\{\mathbf{i},\mathbf{j},\mathbf{k}\}ND(s,2m))$ $(0 \le s \le k)$ **with exceptions** $(*)$**.**

$(*)$ **:** $AG(k-1,2^k)$ **when** $2m = 2^k$ **and** $AG(0,4)$ **when** $2m = 4$**.**

$AG(0,4) = \pi_4(\{\pm I_1 \otimes 1_2, \pm 1_2 \otimes I_1, \pm I_1 \otimes I_1\})$

$\subsetneq AG(2,4)$**.**

6. The 2-number and great antipodal sets of $G_m(\mathbb{K}^{2m})^*$

$AG_{\mathbb{C}}(s, 2m) := AG(s, 2m) \cup \pi_{2m}(\sqrt{-1}ND(s, 2m))$

$AG_{\mathbb{H}}(s, 2m) := AG(s, 2m) \cup \pi_{2m}(\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}ND(s, 2m))$

Since $ND(0, 2m) = \emptyset$**,** $AG(0, 2m) = AG_{\mathbb{C}}(0, 2m)$ $= AG_{\mathbb{H}}(0, 2m)$**.**

$$AG(s, 2m) \subsetneq AG_{\mathbb{C}}(s, 2m) \subsetneq AG_{\mathbb{H}}(s, 2m) \quad (1 \leq s \leq k)$$

$$|AG(0, 2m)| = |\pi_{2m}(\Delta_{2m}^{m})| = \binom{2m}{m} \Big/ 2$$

$|AG(s,2m)|$
$= 2^{\frac{m}{2^{s-1}}-1}(2^{2s-1}+2^{s-1}-1) + \binom{m/2^{s-1}}{m/2^s}\Big/ 2$

$|AG_{\mathbb{C}}(s,2m)|$
$= 2^{\frac{m}{2^{s-1}}-1}(2^{2s}-1) + \binom{m/2^{s-1}}{m/2^s}\Big/ 2$

$|AG_{\mathbb{H}}(s,2m)|$
$= 2^{\frac{m}{2^{s-1}}-1}(2^{2s+1}-2^s-1) + \binom{m/2^{s-1}}{m/2^s}\Big/ 2$

$(1 \le s \le k)$

We set $\binom{m/2^{s-1}}{m/2^s} = 0$ **when** $m/2^s \notin \mathbb{Z}$**.**

We can show: when $m \geq 5$**,**

$|AG(0,2m)| > |AG_{\mathbb{H}}(s,2m)| > |AG_{\mathbb{C}}(s,2m)| >$ $|AG(s,2m)|$ $(1 \le s \le k)$.

We need a case-by-case argument when $m \leq 4$.

GAS:=great antipodal set

Theorem 4. (T.-Tasaki)

GAS of $G_m(\mathbb{K}^{2m})^*$ (up to congruence) and $\#_2 G_m(\mathbb{K}^{2m})^*$ are as follows.

I. $G_m(\mathbb{R}^{2m})^*$

(1) $m = 1$ $\quad AG(1,2),\ \#_2 G_1(\mathbb{R}^2)^* = 2$

(2) $m = 2$ $\quad AG(2,4),\ \#_2 G_2(\mathbb{R}^4)^* = 9$

(3) $m = 4$ $\quad AG(0,8), AG(3,8),\ \#_2 G_4(\mathbb{R}^8)^* =$

(4) $m \neq 1,2,4$ $\quad AG(0,2m)$, $\#_2 G_m(\mathbb{R}^{2m})^* = \binom{2m}{m}/2$

II. $G_m(\mathbb{C}^{2m})^*$

(1) $m = 1$ $\quad AG_{\mathbb{C}}(1,2)$, $\#_2 G_1(\mathbb{C}^2)^* = 3$

(2) $m = 2$ $\quad AG_{\mathbb{C}}(2,4)$, $\#_2 G_2(\mathbb{C}^4)^* = 15$

(3) $m = 3$ $\quad AG_{\mathbb{C}}(1,6)$, $\#_2 G_3(\mathbb{C}^6)^* = 12$

(4) $m = 4$ $\quad AG_{\mathbb{C}}(3,8)$, $\#_2 G_4(\mathbb{C}^8)^* = 63$

(5) $m \neq 1,2,3,4$ $\quad AG_{\mathbb{C}}(0,2m)$, $\#_2 G_m(\mathbb{C}^{2m})^* = \binom{2m}{m}/2$

III. $G_m(\mathbb{H}^{2m})^*$

(1) $m = 1$ $AG_{\mathbb{H}}(1, 2)$, $\#_2 G_1(\mathbb{H}^2)^* = 5$

(2) $m = 2$ $AG_{\mathbb{H}}(2, 4)$, $\#_2 G_2(\mathbb{H}^4)^* = 27$

(3) $m = 3$ $AG_{\mathbb{H}}(1, 6)$, $\#_2 G_3(\mathbb{H}^6)^* = 20$

(4) $m = 4$ $AG_{\mathbb{H}}(3, 8)$, $\#_2 G_4(\mathbb{H}^8)^* = 119$

(5) $m \neq 1, 2, 3, 4$ $AG_{\mathbb{H}}(0, 2m)$, $\#_2 G_m(\mathbb{H}^{2m})^* = \binom{2m}{m}/2$

Thank you for your kind attention.