Maximal antipodal sets of G_2 and $G_2/SO(4)$ and related geometry

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Contents

1. Introduction

M : a compact Riemannian symmetric space

s_x : the geodesic symmetry at $x \in M$

$S \subset M$: a subset

S : an antipodal set $\overset{\text{def}}{\Longleftrightarrow} \forall x, y \in S,\ s_x(y) = y$

the 2-number of M

$$\#_2 M := \max\{|S| \mid S \subset M : \text{ an antipodal set}\}$$

S : a great antipodal set $\overset{\text{def}}{\Longleftrightarrow} |S| = \#_2 M$

(Chen-Nagano)

<u>Theorem A</u> (Takeuchi 1989)

If M is a symmetric R-space, $\#_2 M$ coincides with the sum of the $\mathbb{Z}_2$-Betti numbers of M.

Theorem B (Tanaka-Tasaki 2013)

M : a symmetric R-space

(1) Any antipodal set of M is included in a great antipodal set.

(2) Any two great antipodal sets are congruent, that is, if A, A' are great antipodal sets, there is $g \in I_0(M)$, the identity component of the group of isometries of M, such that $A' = gA$.

G_2, $G_2/SO(4)$ are not symmetric R-spaces.

2. Polars

M : **a compact connected Riemnnian symmetric space,** $o \in M$

A connected component of

$$F(s_o, M) := \{p \in M \mid s_o(p) = p\}$$

is called a polar of M **w.r.t.** o**.**

$F(s_o, M) = \bigcup_{i=0}^{k} M_i^+$, M_i^+ **: a polar,** $M_0^+ = \{o\}$

$\dim M_i^+ > 0 \Rightarrow$ **a totally geodesic submanifold**

A **: an antipodal set of** M

$o \in A \Rightarrow A \subset F(s_o, M)$

$A \cap M_i^+ \neq \emptyset \Rightarrow$ **an antipodal set of** M_i^+

$G := I_0(M)$

$K := \{g \in G \mid go = o\}$, $\quad M \cong G/K$

K_0 : the identity component of K

$M_i^+ = K_0 x_i \quad (x_i \in M_i^+)$

$I_0(M_i^+) = \{k|_{M_i^+} \mid k \in K_0\}$

Proposition 1 When $F(s_o, M) = \{o\} \cup M_1^+$, the assignment $A \mapsto \{o\} \cup A$ from the set of maximal antipodal sets in M_1^+ to that of in M induces a bijection between their congruence classes.

S_i^m : m-**dim sphere with radius** r_i $(i = 1, 2)$

$S_1^m \times S_2^m$: **a Riemannian symmetric space**

$s_{(x_1,x_2)}(y_1, y_2) = (s_{x_1}(y_1), s_{x_2}(y_2))$ $(x_i, y_i \in S_i^m)$

$\mathbb{Z}_2 = \{1, -1\} \curvearrowright S_1^m \times S_2^m$

-1 **acts as** $(x_1, x_2) \mapsto (-x_1, -x_2)$

$S_1^m \times S_2^m \in (x_1, x_2) \mapsto [x_1, x_2] \in (S_1^m \times S_2^m)/\mathbb{Z}_2$ **:**

the natural projection

$(S_1^m \times S_2^m)/\mathbb{Z}_2$ **: a Riem. symmetric space**

$s_{[x_1,x_2]}[y_1, y_2] = [s_{x_1}(y_1), s_{x_2}(y_2)]$ $(x_i, y_i \in S_i^m)$

By Proposition 1 we obtain :

Lemma 2 **Let** $\{e_1 \ldots, e_{m+1}\} \subset S_1^m$ **and** $\{f_1, \ldots, f_{m+1}\} \subset S_2^m$ **be orthogonal frames of** $\mathbb{R}^{m+1}$**.** **Any maximal antipodal set of** $(S_1^m \times S_2^m)/\mathbb{Z}_2$ **is congruent to** $\{[e_1, \pm f_1], \ldots, [e_{m+1}, \pm f_{m+1}]\}$ **and** $\#_2(S_1^m \times S_2^m)/\mathbb{Z}_2 = 2(m+1)$**.**

3. Maximal antipodal sets of G_2 and $G_2/SO(4)$

G_2 : a compact connected Lie group whose root system is of type G_2

e : the identity element

$F(s_e, G_2) = \{e\} \cup M_1^+,\ \ M_1^+ \cong G_2/SO(4)$

$o \in M_1^+$

$F(s_o, M_1^+) = \{o\} \cup M_{1,1}^+,\ \ M_{1,1}^+ \cong (S^2 \times S^2)/\mathbb{Z}_2$

G : a compact Lie group

$s_x(y) = xy^{-1}x \quad (x, y \in G)$

$A \subset G$: a maximal antipodal set

$e \in A \Rightarrow A$: a subgroup $\cong \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$

By Proposition 1 :

Theorem 3 **(1) A maximal antipodal set of** $G_2/SO(4)$ **is congruent to** $\{o\} \cup A$ **for a maximal antipodal set** A **of** $M^+_{1,1} \cong (S^2 \times S^2)/\mathbb{Z}_2$ **given in Lemma 2. Hence** $\{o\} \cup A$ **is a great antipodal set and** $\#_2 G_2/SO(4) = 7$**.**

(2) A maximal antipodal subgroup of G_2 **is conjugate to** $\{e\} \cup A'$ **for a maximal antipodal set** A' **of** $M^+_1 \cong G_2/SO(4)$ **given in (1). Hence** $\{e\} \cup A'$ **is a great antipodal subgroup and** $\#_2 G_2 = 8$**.**

4. Explicit descriptions of maximal antipodal sets

$\mathbb{O}$: **the octonions**

$\mathbb{O} = \mathbb{H} \times \mathbb{H}$, $\quad (m, a), (n, b) \in \mathbb{O}$

$(m, a)(n, b) = (mn - \bar{b}a, a\bar{n} + bm)$

$\mathbb{H} \hookrightarrow \mathbb{O}$, $\quad \mathbb{H} \ni m \mapsto (m, 0) \in \mathbb{O}$

$\mathrm{Aut}(\mathbb{O})$

$= \{\alpha \in GL_{\mathbb{R}}(\mathbb{O}) \mid \alpha(xy) = (\alpha x)(\alpha y),\ \ x, y \in \mathbb{O}\}$ **:**

a compact connected Lie group of type G_2

$G_2 \stackrel{\mathrm{id}}{=} \mathrm{Aut}(\mathbb{O})$

$Sp(1) = \{p \in \mathbb{H} \mid |p| = 1\}$

$\psi : Sp(1) \times Sp(1) \to GL_{\mathbb{R}}(\mathbb{O})$

$$\psi(p, q)(m, a) := (qm\bar{q}, pa\bar{q})$$

$(p, q \in Sp(1), m, a \in \mathbb{H})$ **(Yokota)**

ψ **is a homomorphism,** $\mathrm{Im}\psi \subset G_2$

The following is showed by Yokota :

· $\mathrm{Im}\psi = Z_{\psi(1,-1)}(G_2)$ **: the centralizer**

· $Z_{\psi(1,-1)}(G_2) \cong (Sp(1) \times Sp(1))/\{\pm(1, 1)\} \cong SO(4)$

$\mathrm{Im}\psi \overset{\mathrm{id}}{\cong} SO(4)$

$F(s_e, G_2) = \{e\} \cup M_1^+$

$\psi(1,-1)^2 = e$, **i.e.,** $s_e(\psi(1,-1)) = \psi(1,-1)$

$M_1^+ = \{g\,\psi(1,-1)\,g^{-1} \mid g \in G_2\} \cong G_2/SO(4)$

$o = \psi(1,-1) \in M_1^+$

$F(s_o, M_1^+) = \{o\} \cup M_{1,1}^+$

$M_{1,1}^+ = \{\psi(p,q) \mid p^2 = q^2 = -1\}$

$M_{1,1}^+ \cong (S^2 \times S^2)/\mathbb{Z}_2$ **where** $S^2 = \{x \in Sp(1) \mid x^2 = -1\}$

A maximal antipodal set of $M^+_{1,1}$ is congruent to $\{\psi(\mathrm{i}, \pm\mathrm{i}), \psi(\mathrm{j}, \pm\mathrm{j}), \psi(\mathrm{k}, \pm\mathrm{k})\}$ where $\mathrm{i}, \mathrm{j}, \mathrm{k}$ are elements of the standard basis of $\mathbb{H}$.
A maximal antipodal set of M^+_1 is congruent to $\{\psi(1, -1), \psi(\mathrm{i}, \pm\mathrm{i}), \psi(\mathrm{j}, \pm\mathrm{j}), \psi(\mathrm{k}, \pm\mathrm{k})\}$.
A maximal antipodal subgroup of G_2 is conjugate to $\{\psi(1, \pm 1), \psi(\mathrm{i}, \pm\mathrm{i}), \psi(\mathrm{j}, \pm\mathrm{j}), \psi(\mathrm{k}, \pm\mathrm{k})\} (\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \subset SO(4)$.

5. $G_2/SO(4)$ as an oriented real Grassmann manifold

H : a quaternion subalgebra of $\mathbb{O}$

$\{1\}^\perp$ in H is a canonically oriented 3-plane in $\mathrm{Im}\mathbb{O} = \mathrm{Im}\mathbb{H} \times \mathbb{H} \cong \mathbb{R}^7$, called an associative 3-plane.

$\tilde{G}_{\mathrm{ass}}$: the set of associative 3-planes in $\mathrm{Im}\mathbb{O}$

$\tilde{G}_{\mathrm{ass}} \subset \tilde{G}_3(\mathbb{R}^7)$: totally geodesic

$\tilde{G}_{\mathrm{ass}} \cong G_2/SO(4)$ (Harvey-Lawson)

$M_1^+ \cong \tilde{G}_{\mathrm{ass}}$ by $\xi \mapsto V(\xi) = \{x \in \mathrm{Im}\mathbb{O} \mid \xi x = x\}$

The classification of maximal antipodal sets of $\tilde{G}_3(\mathbb{R}^7)$ is given by Tasaki.

$[7] = \{1, 2, 3, 4, 5, 6, 7\}$

$\binom{[7]}{3} = \{\alpha \subset [7] \mid |\alpha| = 3\}$

$A \subset [7]$ **: an antipodal set** $\overset{\text{def}}{\Leftrightarrow} \forall \alpha, \beta \in \binom{[7]}{3},\ |\alpha \backslash \beta|$ **: even**

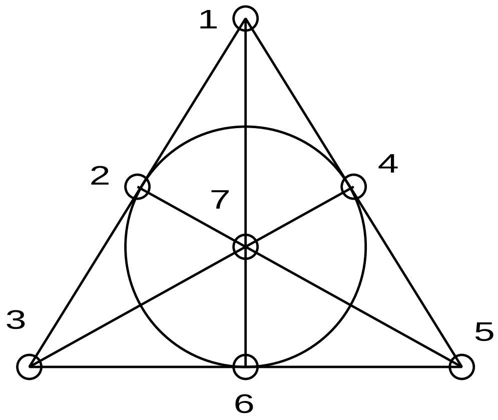
$A \subset \binom{[7]}{3}$ **: a maximal antipodal set**

$e_1, \ldots, e_7$ **: an orthonormal basis of** $\mathbb{R}^7$

$\Rightarrow \{\pm\langle e_{\alpha_1}, e_{\alpha_2}, e_{\alpha_3}\rangle_{\mathbb{R}} \mid \{\alpha_1, \alpha_2, \alpha_3\} \in A\}$ **is a maximal antipodal set of** $\tilde{G}_3(\mathbb{R}^7)$**, and vice versa.**

The set of maximal antipodal sets of $\binom{[7]}{3}$

coincides with the set of lines of the Fano plane under the actions of permutations.



A maximal antipodal set of $M_1^+ = G_2/SO(4)$

$\leftrightarrow$ **a maximal antipodal set** A **of** $\tilde{G}_{\text{ass}}$

$\rightarrow$ **a maximal antipodal set** $\{\pm\alpha \mid \alpha \in A\}$ **of** $\tilde{G}_3(\mathbb{R}^7)$

$\leftrightarrow$ **the set of lines of the Fano plane**

Thank you for your kind attention.