複素旗多様体内の実旗多様体の交叉の構造

On the structure of the intersection of real flag manifolds in a complex flag manifold

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The intersection of real forms in a Herm. symm. space

G/K : compact symmetric space

$A \subset G/K$: **antipodal set** $\overset{\text{def}}{\Longleftrightarrow}$ $s_x(y) = y$ for all $x, y \in A$

Theorem (Tasaki (2010), Tanaka-Tasaki (2011))

G/K : Hermitian symmetric space of compact type

$L_1, L_2 \subset G/K$: real forms, L_1 and L_2 intersect transversally

$\Longrightarrow$ $L_1 \cap L_2$ is an antipodal set of L_1 and L_2.

In addition, if L_1 and L_2 are congruent to each other, then $L_1 \cap L_2$ is a great antipodal set of L_1 and L_2.

Complex flag manifolds and real flag manifolds

G : connected compact semisimple Lie group

$H \in \mathfrak{g}$

$$M_{\mathbb{C}} := \mathrm{Ad}(G)H \subset \mathfrak{g} : \textbf{complex flag manifold}$$

$$M_{\mathbb{C}} = \mathrm{Ad}(G)H \cong G/G_H \cong G^{\mathbb{C}}/P$$

G/K : Riemannian symmetric space of compact type

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

$H \in \mathfrak{p}$

$$M := \mathrm{Ad}(K)H \subset \mathfrak{m} : \textbf{real flag manifold}$$

$$M = \mathrm{Ad}(K)H \cong K/K_H$$

$$M = \mathrm{Ad}(K)H \subset \mathrm{Ad}(G)H = M_{\mathbb{C}} : \text{real form}$$

Antipodal set of a complex flag manifold (1/2)

Δ : root system of $\mathfrak{g}^{\mathbb{C}}$ w.r.t. $\mathfrak{t} \subset \mathfrak{g}$

F : fundamental system of Δ

$$\mathfrak{g}_H = \{X \in \mathfrak{g} \mid [H, X] = 0\} = \mathfrak{t} + \mathfrak{g} \cap \sum_{\substack{\alpha \in \Delta \\ \alpha(H)=0}} \mathfrak{g}_\alpha$$

$$Z := \sum_{\substack{\alpha \in F \\ \alpha(H) \neq 0}} H_\alpha \in \mathfrak{t}$$

$$g_k := \exp \frac{2\pi}{k} Z \in \exp \mathfrak{t} \subset G_H$$

$$s_H^{(k)} : M_{\mathbb{C}} \to M_{\mathbb{C}};\ x \mapsto \mathrm{Ad}(g_k)x$$

Then $s_H^{(k)}$ defines a k-symmetric structure on $M_{\mathbb{C}}$, if $k \geq k_0$.

$A_k \subset M_{\mathbb{C}}$: **antipodal set** $\overset{\text{def}}{\iff}$ $s_x^{(k)}(y) = y$ for all $x, y \in A_k$

Antipodal set of a complex flag manifold (2/2)

Theorem

For any $x \in M_{\mathbb{C}}$*, the fixed point set of* $s_x^{(k)}$ *is*

$$F(s_x^{(k)}, M_{\mathbb{C}}) = \{y \in M_{\mathbb{C}} \mid [x, y] = 0\}.$$

In particular, $F(s_x^{(k)}, M_{\mathbb{C}})$ *is independent of the choice of* $k \geq k_0$*.*

Theorem

$A \subset M_{\mathbb{C}}$: *maximal antipodal set*

$\Longrightarrow$ $\exists \mathfrak{t}' \subset \mathfrak{g}$: *maximal abelian subalgebra* *s.t.*

$$A = M_{\mathbb{C}} \cap \mathfrak{t}'.$$

Hence A *is an orbit of the Weyl group of* $\mathfrak{g}$ *with respect to* $\mathfrak{t}'$*.*

Maximal antipodal sets of $M_{\mathbb{C}}$ *are conjugate to each other by* G*.*

The intersection of real flag manifolds (1/2)

$$F_{n_1,\dots,n_r}(\mathbb{K}^n) := \left\{ (V_1,\dots,V_r) \;\middle|\; \begin{array}{l} V_1 \subset V_2 \subset \cdots \subset V_r \subset \mathbb{K}^n \\ \dim V_i = n_1 + \cdots + n_i \end{array} \right\}$$

$$\begin{aligned} F_{n_1,\dots,n_r}(\mathbb{C}^n) &\cong SU(n)/S(U(n_1)\times\cdots\times U(n_{r+1})) \\ F_{n_1,\dots,n_r}(\mathbb{R}^n) &\cong SO(n)/S(O(n_1)\times\cdots\times O(n_{r+1})) \end{aligned}$$

$F_{n_1,\dots,n_r}(\mathbb{R}^n) \subset F_{n_1,\dots,n_r}(\mathbb{C}^n)$: real form

Lemma

For any $u \in U(n)$ there exist $z_i \in U(1)$ $(1 \le i \le n)$ and positively oriented orthonormal bases $v_1,\dots,v_n$ and $w_1,\dots,w_n$ of $\mathbb{R}^n$ which satisfy

$$uw_i = z_i v_i \quad (1 \le i \le n), \qquad \det u = z_1 \cdots z_n.$$

The intersection of real flag manifolds (2/2)

Theorem

For $u \in U(n)$,

$F_{n_1,\ldots,n_r}(\mathbb{R}^n)$ *and* $F_{n_1,\ldots,n_r}(u\mathbb{R}^n)$ *intersect transversally*

$$\iff \quad z_i \neq \pm z_j \ (i \neq j).$$

$$\begin{aligned} &F_{n_1,\ldots,n_r}(\mathbb{R}^n) \cap F_{n_1,\ldots,n_r}(u\mathbb{R}^n) \\ &= \{(\langle v_{i_1},\ldots,v_{i_{n_1}}\rangle_{\mathbb{C}}, \langle v_{i_1},\ldots,v_{i_{n_1+n_2}}\rangle_{\mathbb{C}},\ldots,\langle v_{i_1},\ldots,v_{i_{n_1+\cdots+n_r}}\rangle_{\mathbb{C}}) \\ &\quad \mid 1 \leq i_1 < \cdots < i_{n_1} \leq n,\ 1 \leq i_{n_1+1} < \cdots < i_{n_1+n_2} \leq n, \ldots, \\ &\quad\ 1 \leq i_{n_1+\cdots+n_{r-1}+1} < \cdots < i_{n_1+\cdots+n_r} \leq n, \\ &\quad\ \#\{i_1,\ldots,i_{n_1+\cdots+n_r}\} = n_1+\cdots+n_r\}, \end{aligned}$$

that is a maximal antipodal set of $F_{n_1,\ldots,n_r}(\mathbb{C}^n)$.

The intersection number

Corollary

If $F_{n_1,\dots,n_r}(\mathbb{R}^n)$ *and* $F_{n_1,\dots,n_r}(u\mathbb{R}^n)$ *intersect transversally, then*

$$\begin{aligned}
&\#(F_{n_1,\dots,n_r}(\mathbb{R}^n) \cap F_{n_1,\dots,n_r}(u\mathbb{R}^n)) \\
&= \#_k(F_{n_1,\dots,n_r}(\mathbb{C}^n)) = \dim H^*(F_{n_1,\dots,n_r}(\mathbb{C}^n), \mathbb{Z}_2) \\
&= \#_I(F_{n_1,\dots,n_r}(\mathbb{R}^n)) = \dim H^*(F_{n_1,\dots,n_r}(\mathbb{R}^n), \mathbb{Z}_2) \\
&= \frac{n!}{n_1! n_2! \cdots n_{r+1}!}
\end{aligned}$$