Maximal antipodal subgroups of compact Lie groups II

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MSJ Autumn Meeting 2015: Classification of maximal antipodal subgroups of $U(n)/\mathbb{Z}_\mu$, $SU(n)/\mathbb{Z}_\mu$

This time: Classification of maximal antipodal subgroups of $O(n)/\{\pm 1_n\}$, $SO(n)/\{\pm 1_n\}$, $Sp(n)/\{\pm 1_n\}$, G_2

M : **compact Riemannian symmetric space**

s_x : **the geodesic symmetry at** $x \in M$

$S \subset M$: **antipodal set** $\Leftrightarrow$ $^\forall x, y \in S,\ s_x(y) = y$

S : **great antipodal set** $\Leftrightarrow$

$|S| =$ **max**$\{|A| \mid A \subset M$ **antipodal set**$\} =: \#_2 M$

e.g. $M = S^n$, $\{x, -x\}$: **great antipodal set**

$M = \mathbb{R}P^n$, $\{\mathbb{R}e_1, \ldots, \mathbb{R}e_{n+1}\}$: **great antip. set**

Theorem 1 **(T.-Tasaki 2013)**

In a symmetric R-space (i) any antipodal set is included in a great antipodal set, (ii) any two great antipodal sets are congruent, and (iii) a great antipodal set is an orbit of the Weyl group.

A great antipodal set is a maximal antipodal set. The converse is not true in general.

We do not know much about antipodal sets in a compact Riemmanian symmetric space which is not a symmetric R-space.
A quotient group of a compact Lie group is not a symmetric R-space generally.

G : compact Lie group
$x \in G, \quad s_x(y) = xy^{-1}x \quad (y \in G)$
$s_e(y) = y \iff y^2 = e \qquad (e$: unit element$)$
if $x^2 = y^2 = e, \quad s_x(y) = y \iff xy = yx$
A maximal antipodal set $S \subset G, \ e \in S$ is a finite abelian subgroup of G.

$$\Delta_n := \left\{ \begin{bmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{bmatrix} \right\} \subset O(n)$$

$$\Delta_n^{\pm} := \{ g \in \Delta_n \mid \det g = \pm 1 \}$$

Δ_n is a unique great antipodal subgroup of $O(n), U(n), Sp(n)$ up to conjugation.

Δ_n^{+} is a unique great antipodal subgroup of $SO(n), SU(n)$ up to conjugation.

$$D[4] := \left\{ \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}, \begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix} \right\} \subset O(2)$$

$$D^{\pm}[4] := \{g \in D[4] \mid \det g = \pm 1\}$$

$D[4]$: **dihedral group**

$n = 2^k \cdot l,\ l$: **odd** $\qquad 0 \le s \le k$

$$C(s,n) := \underbrace{D[4] \otimes \cdots \otimes D[4]}_{s} \otimes \Delta_{n/2^s} \subset O(n)$$

$$Q[8] := \{\pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}\}$$

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1,$$

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k},\ \mathbf{jk} = -\mathbf{kj} = \mathbf{i},\ \mathbf{ki} = -\mathbf{ik} = \mathbf{j}$$

<u>**Theorem 2**</u> $\tilde{G} = O(n),\ SO(n),\ Sp(n),\ G =$ $O(n)/\{\pm 1_n\}, SO(n)/\{\pm 1_n\}\,(n$: **even**$),\ Sp(n)/\{\pm 1_n\}$

$\pi_n : \tilde{G} \to G$ **: natural projection**

$n = 2^k \cdot l, \quad l$ **: odd**

(I) $G = O(n)/\{\pm 1_n\}$

A **is a maximal antipodal subgroup (MAS) of** G **iff** A **is conjugate to**

$$\pi_n(C(s,n)) \ (0 \leq s \leq k),$$

where $(s,n) = (k-1, 2^k)$ **is excluded.**

(II) $G = SO(n)/\{\pm 1_n\}$

A **is a MAS of** G **iff** A **is conjugate to**

(II-1) $k = 1$

$$\pi_n(\Delta_n^+), \quad \pi_n(D^+[4] \otimes \Delta_l),$$

where $\pi_2(\Delta_2^+)$ **is excluded when** $n = 2$**.**

(II-2) $k \geq 2$

$$\pi_n(\Delta_n^+), \quad \pi_n(C(s,n)) \quad (1 \leq s \leq k),$$

where $(s,n) = (k-1, 2^k)$ **is excluded &**

$\pi_4(\Delta_4^+)$ **is excluded when** $n = 4$**.**

(III) $G = Sp(n)/\{\pm 1_n\}$

A **is a MAS of** G **iff** A **is conjugate to**

$$\pi_n(Q[8] \cdot C(s,n)) \quad (0 \leq s \leq k),$$

where $(s,n) = (k-1, 2^k)$ **is excluded.**

Corollary 3

(I) $G = O(n)/\{\pm 1_n\}$

(I-1) $n = 2$

$\pi_2(D[4])$ **is a unique great antipodal subgroup (GAS) up to conjugation.** $\#_2 G = 2^2 = 2^n$

(I-2) $n = 4$

$\pi_4(C(2,4))$ **is a unique GAS.** $\#_2 G = 2^4 = 2^n$

(I-3) the others

$\pi_n(\Delta_n)$ **is a unique GAS.** $\#_2 G = 2^n$

(II) $G = SO(n)/\{\pm 1_n\}$

(II-1) $n = 2$

$\pi_2(D^+[4])$ **is a unique GAS.** $\#_2 G = 2^1 = 2^{n-1}$

(II-2) $n = 4$

$\pi_4(C(2,4))$ **is a unique GAS.** $\#_2 G = 2^4 = 2^n$

(II-3) $n = 8$

$\pi_8(\Delta_8^+)$ **and** $\pi_8(C(3,8))$ **are the GAS's.** $\#_2 G = 2^6 = 2^{n-2}$

(II-4) the others

$\pi_n(\Delta_n^+)$ **is a unique GAS.** $\#_2 G = 2^{n-2}$

(III) $G = Sp(n)/\{\pm 1_n\}$

(III-1) $n = 2$

$\pi_2(Q[8] \cdot D[4])$ **is a unique GAS.** $\#_2 G = 2^4 = 2^{n+2}$

(III-2) $n = 4$

$\pi_4(Q[8] \cdot C(2,4))$ **is a unique GAS.** $\#_2 G = 2^6 = 2^{n+2}$

(III-3) the others

$\pi_n(Q[8] \cdot \Delta_n)$ **is a unique GAS.** $\#_2 G = 2^{n+1}$

Maximal antipodal subgroups of the compact Lie group G_2 of exceptional type

e : **unit element**

$F(s_e, G_2) = \{x \in G_2 \mid x^2 = e\} = \{e\} \cup M_1^+$

$M_1^+ \cong G_2/SO(4)$

$o \in M_1^+$

$F(s_o, M_1^+) = \{x \in M_1^+ \mid xo = ox\} = \{o\} \cup M_{1,1}^+$

$M_{1,1}^+ \cong (S^2 \times S^2)/\mathbb{Z}_2$

$S^2 \times S^2 \ni (p, q) \mapsto [p, q] \in (S^2 \times S^2)/\mathbb{Z}_2$

$(u_i, v_i) \in S^2 \times S^2 \quad (i = 1, 2, 3)$

$u_i \perp u_j, \quad v_i \perp v_j \quad (i \neq j)$

$B := \{[u_1, \pm v_1], [u_2, \pm v_2], [u_3, \pm v_3]\}$ **is a unique maximal antipodal set of** $(S^2 \times S^2)/\mathbb{Z}_2$ **up to congruence.**

$B \;\leftrightarrow\; B_{1,1} \subset M^+_{1,1}$

<u>Theorem 4</u> A **is a maximal antipodal subgroup of** G_2 **iff** A **is conjugate to** $\{e, o\} \cup B_{1,1}$.

$\#_2 G_2 = |\{e, o\} \cup B_{1,1}| = 8$