The intersection of two real forms in Hermitian symmetric spaces of compact type

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(joint work with Makiko Sumi Tanaka)

This is based on [12] and a joint paper [11] with Tanaka.

The 1-dimensional Hermitian symmetric space of compact type is the complex projecive line $\mathbb{C}P^1$. If we regard $\mathbb{C}P^1$ as the 2-dimensional sphere, then its real form is a great circle. Two different great circles intersect at just two points and their intersection is always a pair of antipodal points. The purpose of this talk is to generalize this phenomenon to the intersection of two real forms in any Hermitian symmetric space of compact type. This study has not yet reached to any result of integral geometry, however I think an exact information on the intersection of fundamental submanifolds is important for formulation of several integral formulas in integral geometry. This is one of my motivations of this study.

1. Main results

Let $\bar{M}$ be a Hermitian symmetric space. A submanifold M is called a *real form* of $\bar{M}$, if there exists an involutive anti-holomorphic isometry σ of $\bar{M}$ satisfying

$$M = \{x \in \bar{M} \mid \sigma(x) = x\}.$$

Any real form M is a totally geodesic Lagrangian submanifold of $\bar{M}$. Leung [4] and Takeuchi [9] classified real forms of Hermitian symmetric spaces of compact type.

A subset S in a Riemannian symmetric space M is called an *antipodal set*, if $s_x y = y$ for any points x and y in S, where s_x is the geodesic symmetry with respect to x. The 2-*number* $\#_2 M$ of M is the supremum of the cardinalities of antipodal sets of M. We call an antipodal set in M *great* if its cardinality attains $\#_2 M$. These were introduced by Chen and Nagano [2]. Takeuchi [10] proved

$$\#_2 M = \dim H_*(M, \mathbb{Z}_2)$$

for any symmetric R-space M, where $H_*(M, \mathbb{Z}_2)$ is the homology group of M with coefficient $\mathbb{Z}_2$. A compact Riemannian symmetric space is called a *symmetric R-space*, if its maximal torus has an orthonormal basis of the lattice for a suitable invariant metric. He also showed that any real form of Hermitian symmetric spaces of compact type is a symmetric R-space in [9].

Now se can state our main results.

Theorem 1.1 ([11]). *Let M be a Hermitian symmetric space of compact type. If two real forms L_1 and L_2 of M transversally intersect, then $L_1 \cap L_2$ is an antipodal set of L_1 and L_2.*

Two submanifolds in a Hermitian symmetric space are *congruent*, if one is transformed to another by a holomorphic isometry.

Theorem 1.2 ([11])**.** *Let M be a Hermitian symmetric space of compact type and let L_1 and L_2 be two real forms of M which are congruent and transversally intersect. Then $L_1 \cap L_2$ is a great antipodal set of L_1 and L_2. That is, $\#(L_1 \cap L_2) = \#_2 L_1 = \#_2 L_2$.*

Theorem 1.3 ([11])**.** *Let M be an irreducible Hermitian symmetric space of compact type and let L_1 and L_2 be two real forms of M which transversally intersect.*

(1) *If $M = G^{\mathbb{C}}_{2m}(\mathbb{C}^{4m})$ $(m \geq 2)$, L_1 is congruent to $G^{\mathbb{H}}_m(\mathbb{H}^{2m})$ and L_2 is congruent to $U(2m)$, then*

$$\#(L_1 \cap L_2) = 2^m < \binom{2m}{m} = \#_2 L_1 < 2^{2m} = \#_2 L_2.$$

(2) *Otherwise, $L_1 \cap L_2$ is a great antipodal set of one of L_i's whose 2-number is less than or equal to another and we have*

$$\#(L_1 \cap L_2) = \min\{\#_2 L_1, \#_2 L_2\}.$$

We call a Lagrangian submanifold L of a Hermitian symmetric space M *globally tight*, if L satisfies

$$\#(L \cap g \cdot L) = \dim H_*(L, \mathbb{Z}_2)$$

for any holomorphic isometry g of M with the property that L transversally intersects with $g \cdot L$ (Oh [6]). We obtain the following corollary from Theorem 1.2.

Corollary 1.4. *Any real form of a Hermitian symmetric space of compact type is a globally tight Lagrangian submanifold.*

2. Outline of the proofs

We need the following lemma to start the proofs.

Lemma 2.1 ([12])**.** *Let M be a compact Kähler manifold with positive holomorphic sectional curvature. If L_1 and L_2 are totally geodesic compact Lagrangian submanifolds in M, then $L_1 \cap L_2 \neq \emptyset$.*

The proof of this lemma is similar to that of a result of Frankel [3] concerning the intersection of two totally geodesic submanifolds in a Riemannian manifold with positive sectional curvature.

Since Hermitian symmetric spaces of compact type have positive holomorphic sectional curvature, we can apply Lemma 2.1 to real forms of Hermitian symmetric spaces of compact type. Hence two real forms of them always intersect.

According to a result by Takeuchi [8] on maximal tori of compact symmetric spaces and a result by Sakai [7] on cut loci of compact symmetric spaces, we can prove Theorem 1.1.

Let M be a compact connected Riemannian symmetric space. We decompose the fixed point set $F(s_o, M)$ of the geodesic symmetry s_o at the origin o to the disjoint union of its connected components:

$$F(s_o, M) = \bigcup_{j=0}^{r} M_j^+.$$

We call each connected component M_j^+ a *polar* of M. The notion of polar was introduced and investigated by Chen and Nagano [1], [5].

If M is a Hermitian symmetric space of compact type, then each polar M^+ is also a Hermitian symmetric space of compact type. If L is a real form through o and if $L \cap M^+$ is not empty, then the intersection $L \cap M^+$ is a real form of M^+.

We assume that M, L_1, L_2 are manifolds stated in Theorem 1.2 or 1.3. We can suppose $o \in L_1 \cap L_2$. By Theorem 1.1, $L_1 \cap L_2$ is an antipodal set of L_1 and L_2, so $L_1 \cap L_2$ is an antipodal set of M, too. Hence $L_1 \cap L_2 \subset F(s_o, M)$. Therefore we have the equality:

$$L_1 \cap L_2 = \bigcup_{j=0}^{r} \{(L_1 \cap M_j^+) \cap (L_2 \cap M_j^+)\}.$$

The intersection of two real forms in M is reduced to that of two real forms in M_j^+. We can prove Theorem 1.2 by induction of the polars. In order to prove Theorem 1.3 we use the classification of irreducible Hermitian symmetric spaces of compact type and their real forms.

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