The intersection of two real forms in Hermitian symmetric spaces of compact type

Hiroyuki Tasaki

University of Tsukuba, Japan

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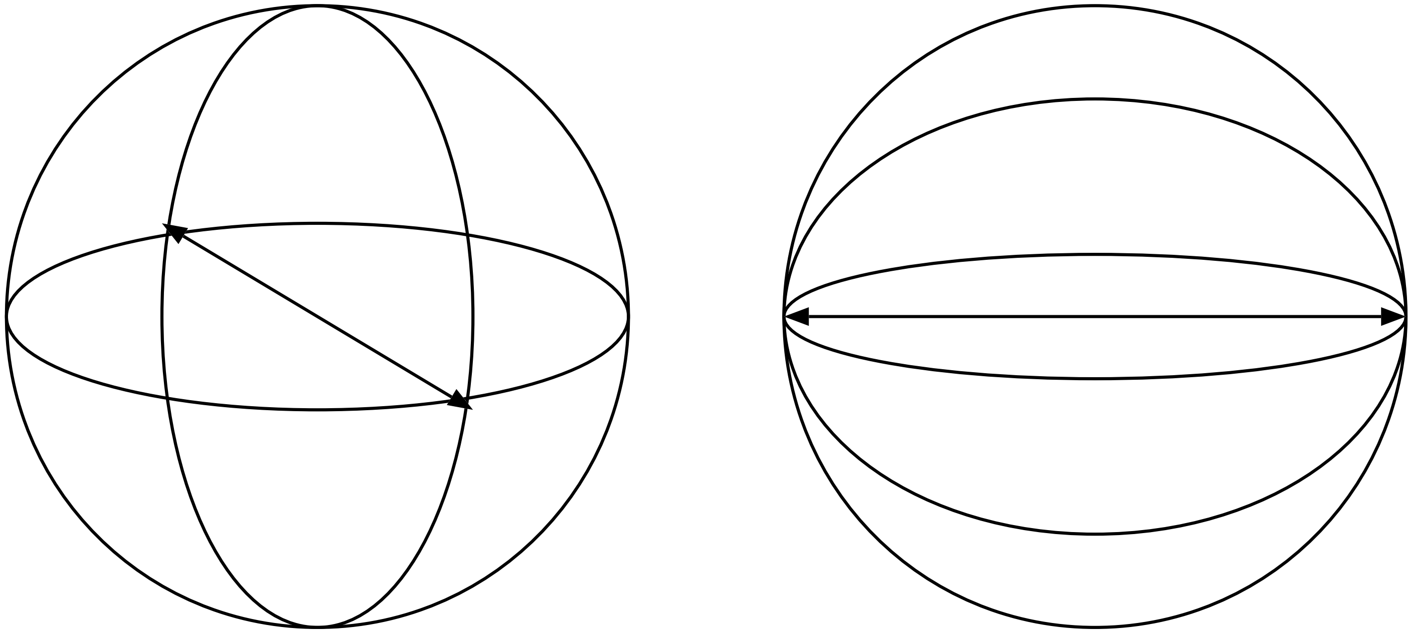
Great circles



Pair of antipodal points

Complex hyperquadric : T.

General case : joint work with Tanaka

Fundamental items :

real form : Leung, Takeuchi

antipodal set : Chen-Nagano

2-number and topology : Takeuchi

1. Main results and related topics

$\bar{M}$: Hermitian symmetric space

M : real form of $\bar{M}$

$\exists$ involutive anti-holomorphic isometry

$\sigma : \bar{M} \longrightarrow \bar{M}$

$$M = \{x \in \bar{M} \mid \sigma(x) = x\}$$

real form

: totally geodesic Lagrangian submanifold

Examples of real forms

$\mathbb{R}^{n} \subset \mathbb{C}^{n}$,

Lagrangian subspace,

$\mathbb{R}P^{n} \subset \mathbb{C}P^{n}$,

$\mathbb{R}P^{1} \subset \mathbb{C}P^{1}$ **: the first example**

Hermitian symmetric spaces of compact type classification of real forms

: Leung, Takeuchi

$\mathbb{C}P^n$: $\mathbb{R}P^n$

$G_r^{\mathbb{C}}(\mathbb{C}^{n+r})$: $G_r^{\mathbb{R}}(\mathbb{R}^{n+r})$,

$G_q^{\mathbb{H}}(\mathbb{H}^{m+q})$ $(n = 2m, r = 2q)$,

$U(n)$ $(n = r)$

M : **Riemannian symmetric space**

$S \subset M$: **antipodal set**

$\Leftrightarrow \forall x, y \in S \quad s_x y = y$

$\#_2 M$: **2-number** **of** M

$= \sup\{\#S \mid S \text{ is antipodal}\}$

S : **great antipodal set**

$\Leftrightarrow \#_2 M = \#S$

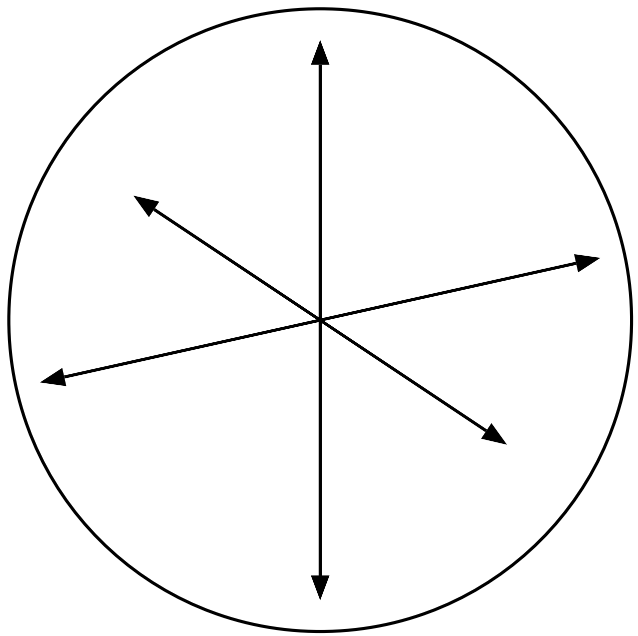
(These are introduced by Chen-Nagano)

Examples of antipodal sets

pair of antipodal points in S^n

$$\#_2 S^n = 2$$

$$\#_2 \mathbb{R}P^2 = 3$$



Chen-Nagano : $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$

$$\#_2 G_r^{\mathbb{K}}(\mathbb{K}^{n+r}) = \binom{n+r}{r}$$

Takeuchi :

M **is a symmetric** R**-space**

$\Rightarrow \#_2 M = \dim H_*(M, \mathbb{Z}_2)$

real form of Hermitian symmetric space of compact type

: symmetric R**-space**

Thorem 1.1

M **: Hermitian symmetric space of compact type**

L_1, L_2 **: real forms of** M **transversally intersect**

$\Rightarrow$

$L_1 \cap L_2$

: antipodal set of L_1 **and** L_2

Thorem 1.2

M : Hermitian symmetric space

of compact type

L_1, L_2 : congruent real forms of M

transversally intersect

$\Rightarrow$

$L_1 \cap L_2$

: great antipodal set of L_1 and L_2

Theorem 1.3

M : irreducible Hermitian symmetric space of compact type

L_1, L_2 : real forms of M

transversally intersect, $\#_2 L_1 \leq \#_2 L_2$

(1) $(M, L_1, L_2) \ncong$

$(G_{2m}^{\mathbb{C}}(\mathbb{C}^{4m}), G_m^{\mathbb{H}}(\mathbb{H}^{2m}), U(2m))$ $(m \geq 2)$

$\Rightarrow L_1 \cap L_2$: great antipodal set of L_1

$$(2)\ (M, L_1, L_2) \cong$$

$$(G^{\mathbb{C}}_{2m}(\mathbb{C}^{4m}), G^{\mathbb{H}}_{m}(\mathbb{H}^{2m}), U(2m))\ (m \geq 2)$$

$$\Rightarrow$$

$$\begin{aligned} \#(L_1 \cap L_2) &= 2^m \\ &< \binom{2m}{m} = \#_2 G^{\mathbb{H}}_{m}(\mathbb{H}^{2m}) \\ &< 2^{2m} = \#_2 U(2m) \end{aligned}$$

M : **Hermitian symmetric space**

L : **Lagrangian submanifold in** M

L : **globally tight(Y.-G. Oh)**

$\Leftrightarrow L, g \cdot L$ **transversally intersect**

(g : **holomorphic isometry)**

$$\#(L \cap g \cdot L) = \dim H_*(L, \mathbb{Z}_2)$$

Corollary 1.4

Any real form of a Hermitian symmetric space of compact type is globally tight.

2. Outline of the proofs

Lemma 2.1

M : compact Kähler manifold

holom. sect. curvature > 0

L_1, L_2 : compact totally geodesic

Lagrangian submanifolds of M

$\Rightarrow L_1 \cap L_2 \neq \emptyset$

Results of

maximal torus by Takeuchi

cut locus by Sakai

Lemma 2.2

The intersection of two maximal tori of a compact symmetric space has a special shape.

Lemma 2.2 $\Rightarrow$ Theorem 1.1

M : compact symmetric space

Fixed point set of s_o

$$F(s_o, M) = \bigcup_{j=0}^{r} M_j^+$$

each connected component M_j^+

: polar

polar : totally geodesic submanifold

compact symmetric space

(This is introduced by Chen-Nagano)

Examples of polars

$$F(s_x, S^n) = \{x, -x\}$$

$$\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$$

$$F(s_o, \mathbb{K}P^n) = \{o\} \cup \mathbb{K}P^{n-1}$$

$$F(s_o, G_r^{\mathbb{K}}(\mathbb{K}^{n+r}))$$

$$= \bigcup_{j=0}^{r} G_j^{\mathbb{K}}(\mathbb{K}^r) \times G_{r-j}^{\mathbb{K}}(\mathbb{K}^n)$$

M **: Hermitian symmetric space**

of compact type

$\Rightarrow$ **each polar** M^+ **: same as above**

L **: real form of** M $L \cap M^+ \neq \emptyset$

$\Rightarrow$ $L \cap M^+$ **: real form of** M^+

M, L_1, L_2 : in Thms. 1.2 and 1.3

$F(s_o, M) = \bigcup_{j=0}^{r} M_j^+$

We can suppose $o \in L_1 \cap L_2$

$L_1 \cap L_2$: antipodal by Th. 1.1

$L_1 \cap L_2 \subset F(s_o, M)$

$L_1 \cap L_2 =$

$\bigcup_{j=0}^{r} \{(L_1 \cap M_j^+) \cap (L_2 \cap M_j^+)\}$

Proofs : induction on polars

$M = G_r^{\mathbb{C}}(\mathbb{C}^{n+r})$

$L_1 = G_r^{\mathbb{R}}(\mathbb{R}^{n+r}), \;\; u \in U(n+r)$

L_1, uL_1 **transversally intersect in** M

$\Rightarrow \exists$ **an orthonormal basis of** $\mathbb{R}^{n+r}$

$$v_1, \ldots, v_{n+r}$$

$$L_1 \cap uL_1 = \{\{v_{i_1}, \ldots, v_{i_r}\}_{\mathbb{C}} \mid 1 \leq i_1 < \cdots < i_r \leq n+r\}.$$