Antipodal sets in compact Riemannian symmetric spaces

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February 3, 2013

Def.(Chen-Nagano)

M : **Riemannian sym. sp.**

s_x : **the geod. sym. at** $x \in M$

$S \subset M$: **antipodal**

$\Leftrightarrow \forall x, y \in S \ s_x(y) = y$

2-number $\#_2 M$ **of** M

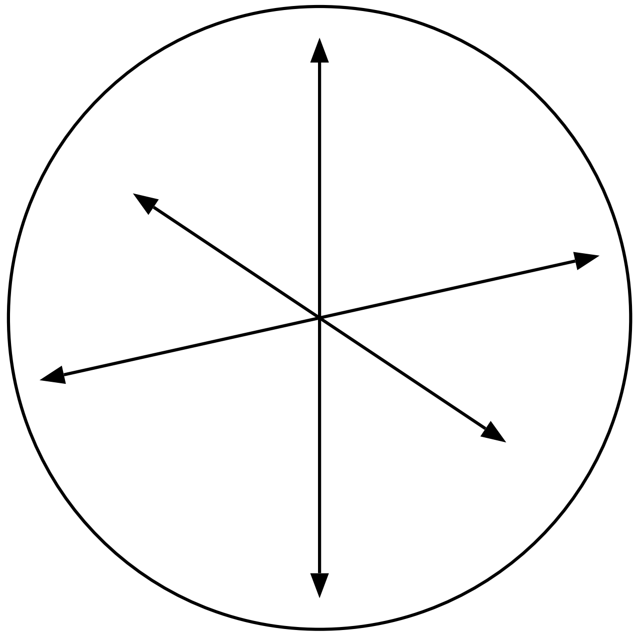
$\#_2 M = \max\{\#S \mid S : \text{antipodal in } M\}$

S : **great** $\Leftrightarrow \#S = \#_2 M$

Examples of antipodal sets

$$\{\pm x\} \text{ in } S^n \quad \#_2 S^n = 2$$

$$\#_2 \mathbb{R}P^2 = 3$$



Antipodal sets in sym. R-spaces

(Takeuchi) M : a sym. R-space

$\Rightarrow \#_2 M = \dim H_*(M, \mathbb{Z}_2)$

(Tanaka-T.) In a sym. R-space

(A) $\forall$ antip. set $\subset \exists$ great a. s.

(B) $\forall$ two great a. s. : congruent

$S_1, S_2 \subset M$: congruent $\Leftrightarrow$

$\exists g \in I_0(M) \quad S_2 = gS_1$

$\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$

$G_k(\mathbb{K}^n)$: Grassmann manifold

$G_k(\mathbb{K}^n)$ is a symmetric R-space

$\mathrm{Inc}_k(n)$: the set of all increasing maps

from $\{1, \ldots, k\}$ to $\{1, \ldots, n\}$

$\{v_i\}$: $\mathbb{K}$-orthonormal basis of $\mathbb{K}^n$

$\Rightarrow \{\langle v_{\alpha(1)}, \ldots, v_{\alpha(k)} \rangle_{\mathbb{K}} \mid \alpha \in \mathrm{Inc}_k(n)\}$

: great antipodal set of $G_k(\mathbb{K}^n)$

$\#_2 G_k(\mathbb{K}^n) = \#\mathrm{Inc}_k(n) = \binom{n}{k}$

$\tilde{G}_k(\mathbb{R}^n)$: **oriented real Grass. mfd.**

$\tilde{G}_k(\mathbb{R}^n) \subset \bigwedge^k \mathbb{R}^n$: **isometric**

$\tilde{G}_k(\mathbb{R}^n) \ni \tilde{V}$: **oriented subspace**

V : **underlying subspace of** $\tilde{V}$

$r_V = 1_V - 1_{V^\perp}$: **reflection w.r.t.** V

$\wedge^k r_V : \bigwedge^k \mathbb{R}^n \to \bigwedge^k \mathbb{R}^n$ **isometric**

$\wedge^k r_V(v_1 \wedge \cdots \wedge v_k) = r_V(v_1) \wedge \cdots \wedge r_V(v_k)$

$s_V = \wedge^k r_V|_{\tilde{G}_k(\mathbb{R}^n)}$: **geod. symmetry at** $\tilde{V}$

The double covering

$p : \tilde{G}_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n); \tilde{V} \mapsto V$

$\tilde{G}_k(\mathbb{R}^n) \supset S$ **: antipodal**

$\Rightarrow G_k(\mathbb{R}^n) \supset p(S)$ **: antipodal**

$\Rightarrow \exists\{v_i\}$ **: orthonormal basis of** $\mathbb{R}^n$

$p(S) \subset \{\langle v_{\alpha(1)}, \ldots, v_{\alpha(k)} \rangle_{\mathbb{R}} \mid \alpha \in \mathrm{Inc}_k(n)\}$

Hence

$S \subset \{\pm v_{\alpha(1)} \wedge \cdots \wedge v_{\alpha(k)} \mid \alpha \in \mathrm{Inc}_k(n)\}$

We obtain the following lemma.

Lemma Any max. antip. set in $\tilde{G}_k(\mathbb{R}^n)$ is described as follows:

$\exists v = \{v_i\}$: orthonaomal basis of $\mathbb{R}^n$

$\exists A \subset \mathrm{Inc}_k(n)$

$\mathcal{A}_v(A) := \{\pm v_{\alpha(1)} \wedge \cdots \wedge v_{\alpha(k)} \mid \alpha \in A\}$

We have to characterize A.

We identify $\mathrm{Inc}_k(n) \to P_k(n); \alpha \mapsto \{\alpha(i)\}$

The geod. sym. at $\vec{v}_\alpha := v_{\alpha(1)} \wedge \cdots \wedge v_{\alpha(k)}$

$s_{\vec{v}_\alpha}(\vec{v}_\beta) = (-1)^{\#(\beta-\alpha)} \vec{v}_\beta \ (\alpha, \beta \in P_k(n))$

Lemma For $A \subset P_k(n)$

$\mathcal{A}_v(A)$: antipodal in $\tilde{G}_k(\mathbb{R}^n)$

$\Leftrightarrow \#(\beta - \alpha)$: even $(\alpha, \beta \in A)$

Def. $A \subset P_k(n)$: antipodal

$\Leftrightarrow \#(\beta - \alpha)$: even $(\alpha, \beta \in A)$

Thm. Any max. antip. set in $\tilde{G}_k(\mathbb{R}^n)$ is described as follows:

$\exists A$: max. antip. subset in $P_k(n)$

$\exists v$: orthonormal basis of $\mathbb{R}^n$ $\quad \mathcal{A}_v(A)$

Max. antip. sets in $\tilde{G}_k(\mathbb{R}^n)$

$\leftrightarrow$ **MAS's in $P_k(n)$**

$A, B \subset P_k(n)$: **congruent**

$\Leftrightarrow \exists g \in \mathrm{Sym}(n) \quad gA = B$

In the case $k = 1$

$\{\{1\}\}$: **MAS in $P_1(n)$**

$\{\pm v\}$: **max. antip. in S^n**

In the case $k = 2$

$\alpha \neq \beta \in P_2(n)$: **antipodal**

$\Leftrightarrow \alpha \cap \beta = \emptyset$

$l = [n/2]$

$\{\{1, 2\}, \{3, 4\}, \ldots, \{2l - 1, 2l\}\}$

: **MAS in** $P_2(n)$

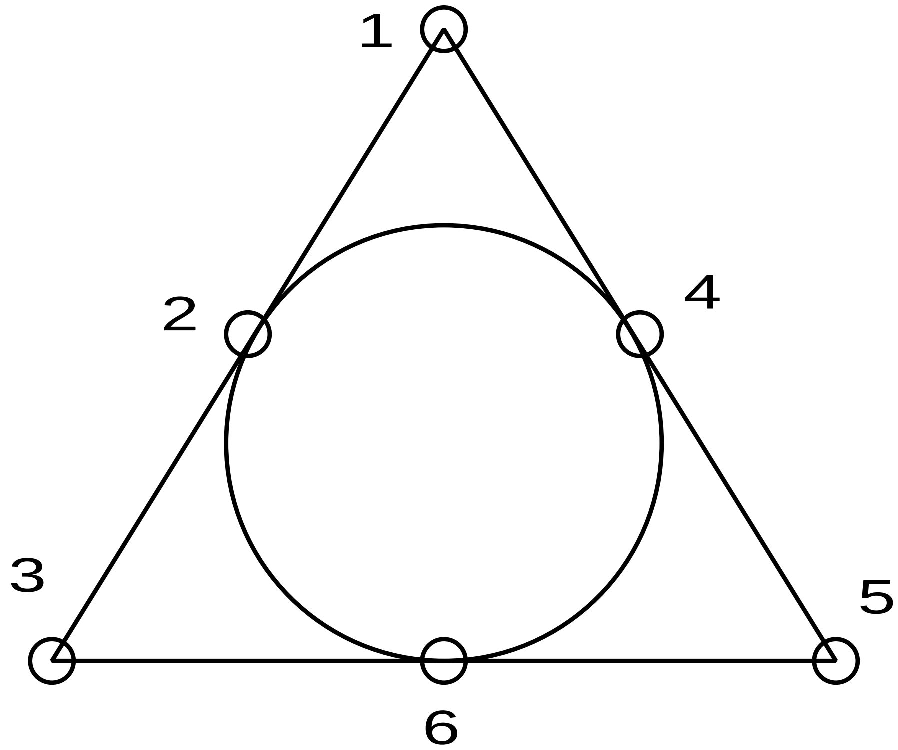
$\{\pm v_1 \wedge v_2, \ldots, \pm v_{2l-1} \wedge v_{2l}\}$

: **max. antipodal set in** $\tilde{G}_2(\mathbb{R}^n)$

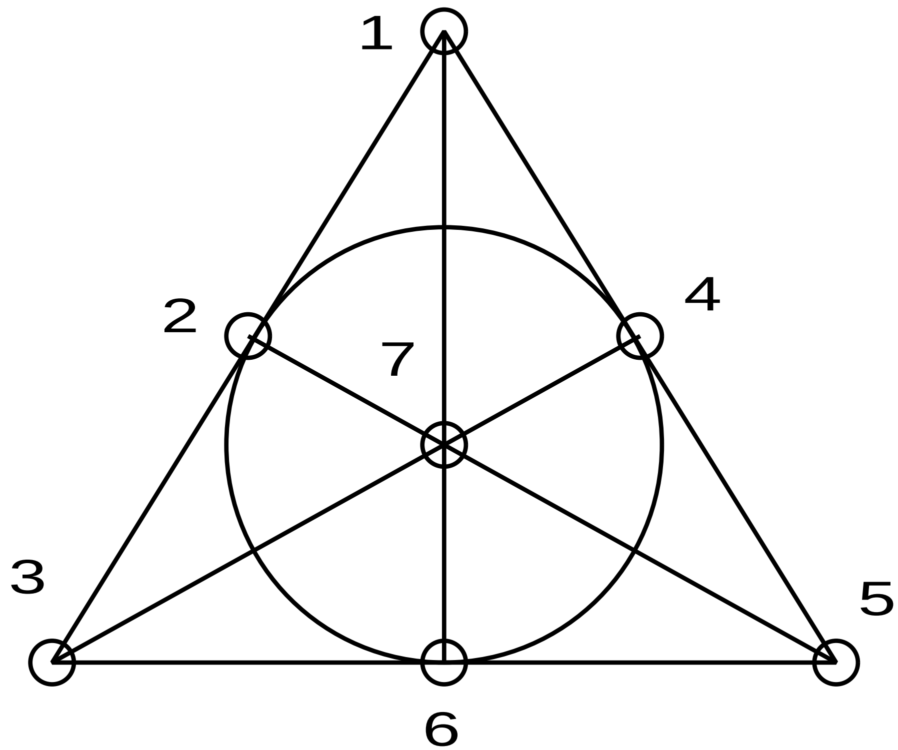
In the case $k = 3$

$\{\{1,2,3\},\{1,4,5\},\{2,4,6\},\{3,5,6\}\}$

: MAS in $P_3(6)$

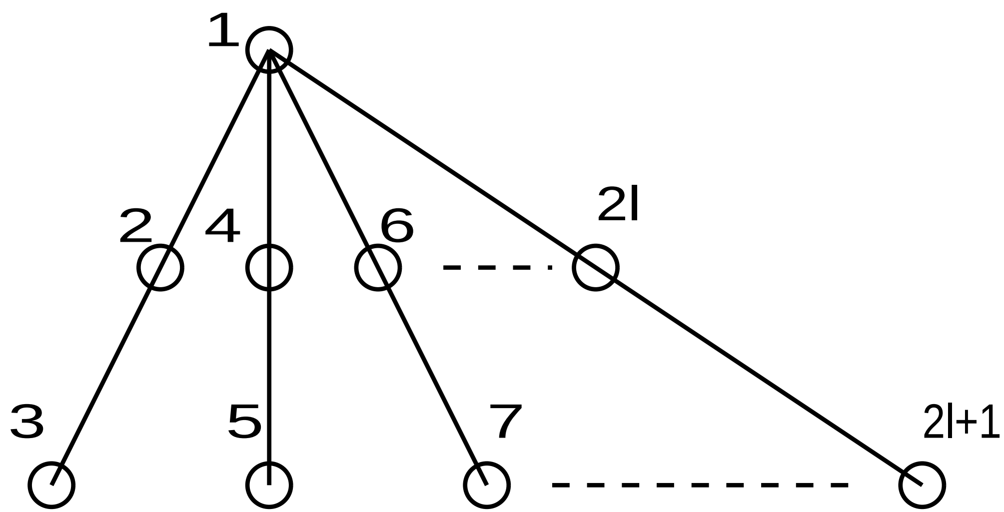


$$\{\{1,2,3\},\{1,4,5\},\{2,4,6\},\{1,6,7\},$$
$$\{2,5,7\},\{3,4,7\},\{3,5,6\}\}$$
$$\text{: MAS in } P_3(n)\ (n \geq 7)$$

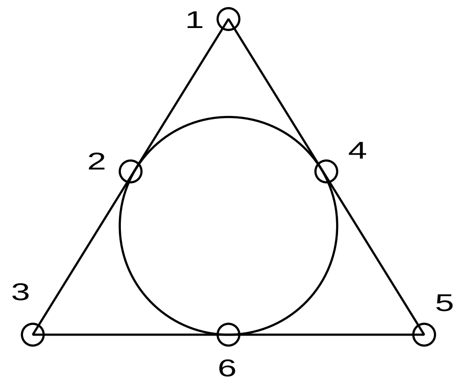


$$\{\{1,2,3\},\{1,4,5\},\ldots,\{1,2l,2l+1\}\}$$

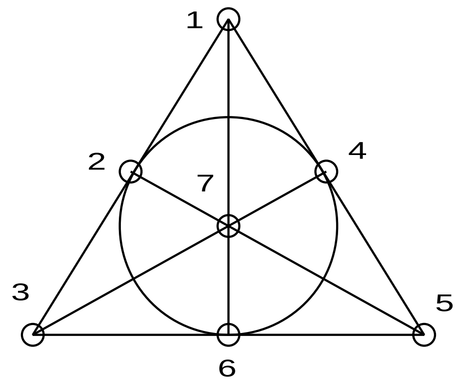
: MAS in $P_3(n)$ ($n \geq 9$)



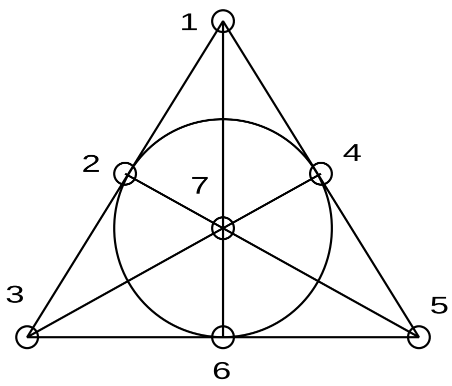
Thm. **MAS in $P_3(6)$ is cong. with**



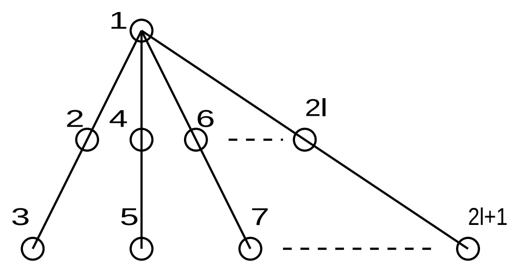
MAS in $P_3(n)$ $(n = 7, 8)$ is cong. with



MAS in $P_3(n)$ $(n \geq 9)$ is cong. with



or



In the case $k = 4$

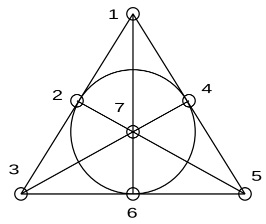
$$\{\alpha \cup \beta \mid \alpha, \beta \in \{\{1,2\},\{3,4\},\{5,6\},\{7,8\}\}, \alpha \neq \beta\},$$

$$\cup \{\{i_1, i_2, i_3, i_4\} \mid i_j \in \{2j-1, 2j\}, \text{the number of even numbers is even}\}$$

$$=: A_8$$

: MAS in $P_4(8)$

$\boldsymbol{A_7}$ **: compl. of all elements of MAS**



in $\boldsymbol{P_3(7)}$

$$\boldsymbol{A'_{2l} = \{\alpha \cup \beta \mid \alpha, \beta = \{2j-1, 2j\}}$$
$$\boldsymbol{(1 \le j \le l), \alpha \ne \beta\} \subset P_4(2l)}$$

Thm. MAS in $\boldsymbol{P_4(n)}$ **is cong. with certain disjoint unions of copies of** $\boldsymbol{A_7}$**,** $\boldsymbol{A_8}$ **and at most one** $\boldsymbol{A'_{2l}}$ $\boldsymbol{(l \ne 4)}$

Examples

$n = 4, 5 \quad A'_4 \qquad n = 6 \quad A'_6$

$n = 7 \quad A_7 \qquad n = 8, 9 \quad A_8$

$n = 10 \quad A_8, A'_{10}$

$n = 11 \quad A_8, A'_{10}, A_7 \cup (A'_4 + 7)$

$n = 12 \quad A_8 \cup (A'_4 + 8), A'_{12}$

$n = 13 \quad A_7 \cup (A'_6 + 7), A_8 \cup (A'_4 + 8), A'_{12}$

$n = 14 \quad A_7 \cup (A_7 + 7), A_8 \cup (A'_6 + 8), A'_{14}$

$n = 15 \quad A_7 \cup (A_8 + 7), A'_{14}$

$n = 16 \quad A_8 \cup (A_8 + 8), A'_{16}$

$n = 17 \quad A_7 \cup (A'_{10} + 7), A_8 \cup (A_8 + 8), A'_{16}$