Maximal antipodal sets of oriented real Grassmann manifolds

Hiroyuki Tasaki

University of Tsukuba

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Def.(Chen-Nagano)

M : Riemannian sym. sp.

s_x : the geod. sym. at $x \in M$

$S \subset M$: **antipodal**

$\Leftrightarrow \forall x, y \in S \; s_x(y) = y$

2-number $\#_2 M$ of M

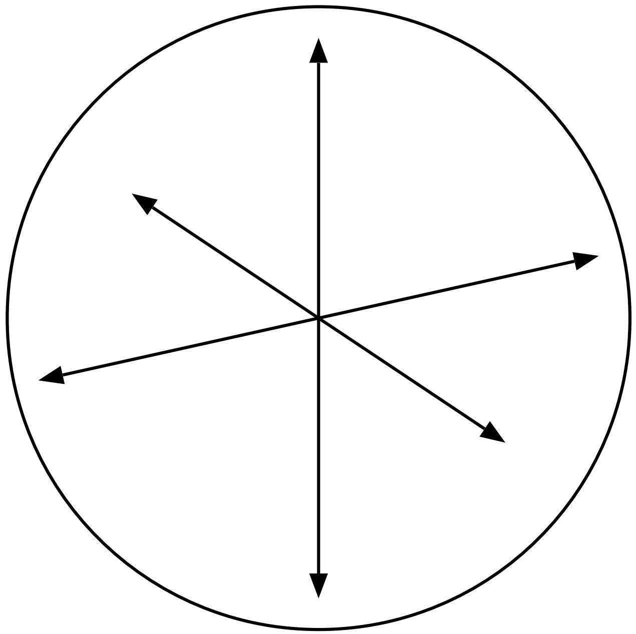
$\#_2 M = \max\{|S| \mid S : \text{antipodal in } M\}$

S : **great** $\Leftrightarrow |S| = \#_2 M$

Examples of great antipodal sets

$\{\pm x\}$ in S^n $\#_2 S^n = 2$

$\#_2 \mathbb{R}P^2 = 3$



Antipodal sets in sym. R-spaces

(Tanaka-T.) **In a sym. R-space**

(A) $\forall$ antip. set $\subset$ $\exists$ great a. s.

(B) $\forall$ two great a. s. : congruent

$S_1, S_2 \subset M$: **congruent** $\Leftrightarrow$

$\exists g \in I_0(M) \quad S_2 = gS_1$

$\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$

$G_k(\mathbb{K}^n)$: **Grassmann manifold**

$G_k(\mathbb{K}^n)$ **is a symmetric** R**-space**

$\{e_i\}$: $\mathbb{K}$**-orthonormal basis of** $\mathbb{K}^n$

$\{\langle e_{i_1}, \ldots, e_{i_k}\rangle_{\mathbb{K}} \mid 1 \leq i_1 < \cdots < i_k \leq n\}$

: **great antipodal set of** $G_k(\mathbb{K}^n)$

$$\#_2 G_k(\mathbb{K}^n) = \binom{n}{k}$$

$\tilde{G}_k(\mathbb{R}^n)$: oriented real Grass. mfd.

$\text{rank}\tilde{G}_k(\mathbb{R}^n) = \min\{k, n-k\} > 2$
$\Rightarrow \tilde{G}_k(\mathbb{R}^n)$: not symmetric R-sp.

Maximal antipodal sets of $\tilde{G}_k(\mathbb{R}^n)$
$\leftrightarrow$ certain families of subsets of
$[n] := \{1, \ldots, n\}$

$\Rightarrow$ a combinatorial problem

$1 \leq k \leq n,\ |\alpha|$: the card. of a fin. set α

$$\binom{[n]}{k} := \{\alpha \subset [n] \mid |\alpha| = k\}$$

$\alpha, \beta \in \binom{[n]}{k}$: antipodal

$\Leftrightarrow \alpha \backslash \beta = \{i \in \alpha \mid i \notin \beta\}$ has even cardinality. $\Leftrightarrow |\alpha \cap \beta| \equiv k \pmod 2$

$A \subset \binom{[n]}{k}$: antipodal

$\Leftrightarrow$ any two elements in A are antipodal.

$A, B \subset \binom{[n]}{k}$: congruent

$\Leftrightarrow \exists g \in \mathfrak{S}_n\ B = gA$

Thm.(T.) $e_1, \ldots, e_n$ **: o. n. b. of** $\mathbb{R}^n$

$A \subset \binom{[n]}{k}$ **: max. antip.**

$$\{\pm\langle e_i \mid i \in \alpha\rangle_{\mathbb{R}} \mid \alpha \in A\}$$

: max. antip. of $\tilde{G}_k(\mathbb{R}^n)$.

$$\left\{\text{cong. classes of max. antip. of } \binom{[n]}{k}\right\}$$

$$\updownarrow \text{ bijection}$$

$$\left\{\text{cong. classes of max. antip. of } \tilde{G}_k(\mathbb{R}^n)\right\}$$

MAS = maximal antipodal set

In the case $k = 1$

$\{\{1\}\}$: MAS of $\binom{[n]}{1}$

$\{\pm v\}$: MAS of $\tilde{G}_1(\mathbb{R}^n) = S^{n-1}$

$\lfloor r \rfloor = \max\{n \in \mathbb{Z} \mid n \leq r\}$

In the case $k = 2$

$$A(2, 2l) = \{\{1, 2\}, \{3, 4\}, \dots, \{2l - 1, 2l\}\}$$

$A\left(2, 2\left\lfloor \frac{n}{2} \right\rfloor\right)$: MAS of $\binom{[n]}{2}$

Kähler form $\theta^1 \wedge \theta^2 + \cdots + \theta^{2l-1} \wedge \theta^{2l}$

: $U(l)$-invariant 2-form

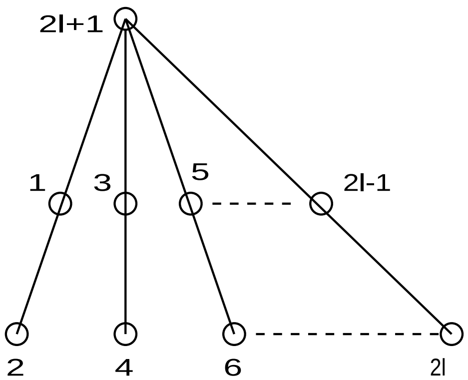
In the case $k = 3$

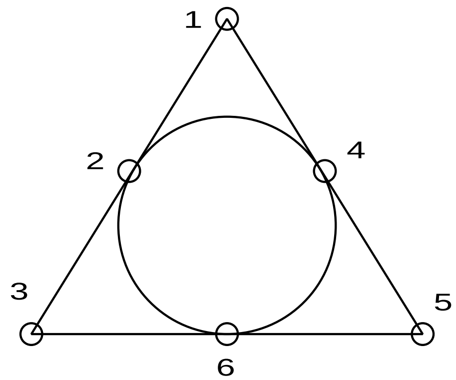
Antipodal sets of $\binom{[n]}{3}$

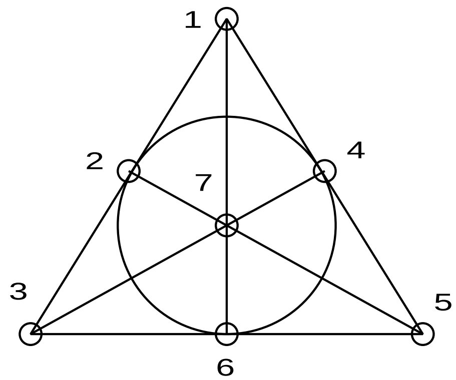
$$A(3, 2l+1) = \{\alpha \cup \{2l+1\} \mid \alpha \in A(2, 2l)\}$$

$$B(3,6) = \{\{1,2,3\},\{1,4,5\},\{2,4,6\},\{3,5,6\}\}$$

$$B(3,7) = B(3,6) \cup \{\{1,6,7\},\{2,5,7\},\{3,4,7\}\}$$







Thm.(T.)

MAS of $\binom{[n]}{3}$:

n	$3, 4$	5	6	$7, 8$
	$A(3,3)$	$A(3,5)$	$B(3,6)$	$B(3,7)$

n	more than 8
	$A\left(3, 2\lfloor\frac{n-1}{2}\rfloor + 1\right), B(3,7)$

$B(3,6)$ is congruent to

$\{\{2,4,5\},\{1,4,6\},\{2,3,6\},\{1,3,5\}\}$

$\mathrm{Re}(dz^1 \wedge dz^2 \wedge dz^3),\ dz^j = \theta^{2j-1} + \sqrt{-1}\theta^{2j}$

: $SU(3)$-invariant 3-form

special Lagrangian 3-form

$B(3,7)$ determines a 3-form on $\mathbb{R}^7$

: invariant under the action of G_2 on

$\mathrm{Im}\mathbb{O} \cong \mathbb{R}^7$

($G_2 = \mathrm{Aut}(\mathbb{O})$: of exceptional type)

constructed by Harvey-Lawson.

In the case $k = 4$

Antipodal sets of $\binom{[n]}{4}$

$$A(4, 2l) = \{\alpha \cup \beta \mid \alpha, \beta \in A(2, 2l),\ \alpha \neq \beta\}$$

$$B(4, 7) = \{[7] \backslash \alpha \mid \alpha \in B(3, 7)\}$$

$$B(4, 8) = B(4, 7) \cup \{\alpha \cup \{8\} \mid \alpha \in B(3, 7)\}$$

In general if A is an antipodal set of $\binom{[n]}{k}$, then

$$\{[n] \backslash \alpha \mid \alpha \in A\}$$

is an antipodal set of $\binom{[n]}{n-k}$

Thm.(T.)

MAS of $\binom{[n]}{4}$ **: Certain unions of copies of**

$$\boldsymbol{A}(4, 2\boldsymbol{l})\ (\boldsymbol{l} \geq \mathbf{2}, \neq 4), \quad \boldsymbol{B}(4, \mathbf{7}), \quad \boldsymbol{B}(4, 8)$$

Remark. $\boldsymbol{A}(4, 8) \subsetneq \boldsymbol{B}(4, 8) \Rightarrow \boldsymbol{A}(4, 8)$ **is not maximal.**

$A(4,2l)$ **corresponds to** ω^2
(ω **: Kähler form on** $\mathbb{C}^l$**)**
: $U(l)$**-invariant 4-form**

$B(4,7)$ **corresponds to the Hodge star of the** G_2**-invarinat 3-form on** $\mathbb{R}^7$

$B(4,8)$ **corresponds to** $\omega_I^2 + \omega_J^2 + \omega_K^2$ **on**
$\mathbb{H}^2 \cong \mathbb{R}^8$ **:** $Sp(2)Sp(1)$**-invarinat 4-form constructed by Krains**
$\omega_I, \omega_J, \omega_K$ **: Kähler forms w.r.t.** I, J, K

$$a(k,n) = \max\left\{ |A| \;\middle|\; A : \text{antip of } \binom{[n]}{k} \right\}$$

The existences of $A(2k, 2l)$,
$A(2k+1, 2l+1)$ **imply**

$$a(2k, n) \geq \binom{\lfloor \frac{n}{2} \rfloor}{k},$$

$$a(2k+1, n) \geq \binom{\lfloor \frac{n-1}{2} \rfloor}{k}.$$

The results on the classifications of MAS of $\binom{[n]}{k}$ in the cases $k \le 4$ show

$$a(1,n) = 1, \quad a(2,n) = \left\lfloor \frac{n}{2} \right\rfloor$$

n	4	5	6	$7,\ldots,16$	more than 16
$a(3,n)$	1	2	4	7	$\lfloor \frac{n-1}{2} \rfloor$

n	5	6	7	$8,\ldots,11$	more than 11
$a(4,n)$	1	3	7	14	$\binom{\lfloor \frac{n}{2} \rfloor}{2}$

In the case $k = 5$

Thm(T.) **If $n \geq 87$,**

$$a(5,n) = \left| A\left(5, 2\left\lfloor \frac{n-1}{2} \right\rfloor + 1\right) \right| = \binom{\left\lfloor \frac{n-1}{2} \right\rfloor}{2}.$$

Moreover an antipodal set of $\binom{[n]}{5}$ which attains $a(5,n)$ is congrunet to

$$A\left(5, 2\left\lfloor \frac{n-1}{2} \right\rfloor + 1\right).$$

Thm.(Frankl-Tokushige)

$$a(2k,n) = \binom{\lfloor \frac{n}{2} \rfloor}{k},$$

$$a(2k+1,n) = \binom{\lfloor \frac{n-1}{2} \rfloor}{k}.$$

for $n \geq \exists n_k$. Moreover antipodal sets of $\binom{[n]}{2k}$, $\binom{[n]}{2k+1}$ which attain $a(2k,n)$, $a(2k+1,n)$ are congruent to

$$A\left(2k, 2\left\lfloor \frac{n}{2} \right\rfloor\right), \quad A\left(2k+1, 2\left\lfloor \frac{n-1}{2} \right\rfloor + 1\right).$$

MAS of $Spin(n)$

$$\mathrm{Fix}(s_e, Spin(n)) = \bigcup_{k=0}^{\lfloor \frac{n}{4} \rfloor} M_k^+,$$

$$M_k^+ \cong \tilde{G}_{4k}(\mathbb{R}^n)$$

A : **antipodal subgroup of** $Spin(n)$

$\Rightarrow A \subset \mathrm{Fix}(s_e, Spin(n))$

$$A = \bigcup_{k=0}^{\lfloor \frac{n}{4} \rfloor} A \cap M_k^+$$

$$F(n) = \bigcup_{k=0}^{\lfloor \frac{n}{4} \rfloor} \binom{[n]}{4k}$$

$\alpha, \beta \in F(n)$: **antipodal**

$\Leftrightarrow \alpha \backslash \beta = \{i \in \alpha \mid i \notin \beta\}$ has even cardinality. $\Leftrightarrow |\alpha \cap \beta|$: even

$A \subset F(n)$: **antipodal**

$\Leftrightarrow$ any two elements in A are antipodal.

Thm.(Suzuki, Arai-T.)

$e_1, \ldots, e_n$: o. n. b. of $\mathbb{R}^n$

$A \subset F(n)$: MAS

$$\{\pm e_{\alpha_1} \cdots e_{\alpha_{4k}} \mid \alpha = \{\alpha_1, \ldots, \alpha_{4k}\} \in A\}$$

: MAS of $Spin(n)$.

{cong. classes of MAS of $F(n)$}

$\updownarrow$ bijection

{cong. classes of MAS of $Spin(n)$}

In the cases $n = 1, 2, 3$

MAS of $F(n)$: $\{\emptyset\}$, $1 = 2^0$

MAS of $Spin(n)$: $\{\pm 1\}$

Thm.(Arai-T.)

MAS of $F(n)$

for $n = 4, 5$: $\{\emptyset, [4]\}$, 2

for $n = 6$: $\{\emptyset\} \cup A(4, 6)$, $4 = 2^2$

for $n = 7$: $\{\emptyset\} \cup B(4, 7)$, $8 = 2^3$

for $n = 8, 9$: $\{\emptyset\} \cup B(4, 8) \cup \{[8]\}$, 2^4

$$\text{for } n = 10: \{\emptyset\} \cup A(4,10) \cup A(8,10),\ 2^4$$

$$\{\emptyset\} \cup B(4,8) \cup \{[8]\},\ 2^4$$

$$\text{for } n = 11: \{\emptyset\} \cup A(4,10) \cup A(8,10),\ 2^4$$

$$\{\emptyset\} \cup B(4,8) \cup \{[8]\},\ 2^4$$

$$\{\emptyset\} \cup B(4,7) \cup ([4]+7) \cup$$

$$B(4,7) \times ([4]+7),\ 2^4$$

$$\text{for } n = 12: \{\emptyset\} \cup A(4,12) \cup A(8,12) \cup \{[12]\},\ 2^5$$

$$\{\emptyset\} \cup B(4,8) \cup \{[4]+8, [8], [12]\} \cup$$

$$B(4,8) \times \{[4]+8\},\ 2^5$$

$$X + n = \{x + n \mid x \in X\}$$

$$A \times B = \{\alpha \cup \beta \mid \alpha \in A,\ \beta \in B\}$$