Antipodal sets of oriented real Grassmann manifolds

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November, 2018

Def.(Chen-Nagano)

M : Riem. symmetric space

s_x : the geod. sym. at $x \in M$

$S \subset M$: antipodal

$\Leftrightarrow \forall x, y \in S \; s_x(y) = y$

$I(M)$: all isometries of M

$I_0(M)$: its identity componet

$S_1, S_2 \subset M$: congruent

$\Leftrightarrow \exists g \in I_0(M)$ s.t. $S_2 = gS_1$

Classification of congruent classes of maximal antipodal sets of M

In this talk

Classification of congruent classes of maximal antipodal sets of oriented real Grassmann manifolds

$G_k(\mathbb{R}^n)$: real Grassmann manifold

$G_k(\mathbb{R}^n)$ is a symmetric R-space

$s_V = 1_V - 1_{V^\perp}$ for $V \in G_k(\mathbb{R}^n)$

$\{e_i\}$: orthonormal basis of $\mathbb{R}^n$

$\{\langle e_{i_1}, \ldots, e_{i_k} \rangle \mid 1 \leq i_1 < \cdots < i_k \leq n\}$

: maximal antipodal set of $G_k(\mathbb{R}^n)$,

unique up to congruence

$\tilde{G}_k(\mathbb{R}^n)$: oriented real Grass. mfd.

$\mathrm{rank}\tilde{G}_k(\mathbb{R}^n) = \min\{k, n-k\} > 2$

$\Rightarrow \tilde{G}_k(\mathbb{R}^n)$: not symmetric R-sp.

Maximal antipodal sets of $\tilde{G}_k(\mathbb{R}^n)$

$\leftrightarrow$ certain families of subsets of

$[n] := \{1, \ldots, n\}$

$\Rightarrow$ a combinatorial problem

$1 \leq k \leq n,\ |\alpha|$: the cardinality of α

$$\binom{[n]}{k} := \{\alpha \subset [n] \mid |\alpha| = k\}$$

For $\alpha, \beta \in \binom{[n]}{k}$, $\alpha \backslash \beta = \{i \in \alpha \mid i \notin \beta\}$

α, β : antipodal $\Leftrightarrow |\alpha \backslash \beta|$: even

$\Leftrightarrow |\alpha \cap \beta| \equiv k \pmod 2$

$A \subset \binom{[n]}{k}$: antipodal

$\Leftrightarrow$ For any $\alpha, \beta \in A$: antipodal.

$A, B \subset \binom{[n]}{k}$: congruent

$\Leftrightarrow \exists g$: permutation s.t. $B = gA$

Thm.(T.2013)

$e_1, \ldots, e_n$: standard basis of $\mathbb{R}^n$

$A \subset \binom{[n]}{k}$: maximal antipodal set

$$\{\pm\langle e_i \mid i \in \alpha\rangle \mid \alpha \in A\}$$

: maximal antipodal set of $\tilde{G}_k(\mathbb{R}^n)$.

$$\left\{\text{cong. classes of max. antip. of } \binom{[n]}{k}\right\}$$

$$\updownarrow \text{ bijection}$$

$$\left\{\text{cong. classes of max. antip. of } \tilde{G}_k(\mathbb{R}^n)\right\}$$

MAS = maximal antipodal set

In the case $k = 1$

$\{\{1\}\}$: **MAS of** $\binom{[n]}{1}$

$\{\pm v\}$: **MAS of** $\tilde{G}_1(\mathbb{R}^n) = S^{n-1}$

for $v \in S^{n-1}$

$\lfloor r \rfloor = \max\{n \in \mathbb{Z} \mid n \leq r\}$ **for** $r \in \mathbb{R}$

In the case $k = 2$

$A(2, 2l) = \{\{1, 2\}, \{3, 4\}, \ldots, \{2l - 1, 2l\}\}$

$A(2, 2l)$: **MAS of** $\binom{[n]}{2}$ $\quad l = \left\lfloor \frac{n}{2} \right\rfloor$

$\{\pm\langle e_1, e_2\rangle, \ldots, \pm\langle e_{2l-1}, e_{2l}\rangle\}$: **MAS of** $\tilde{G}_2(\mathbb{R}^n)$

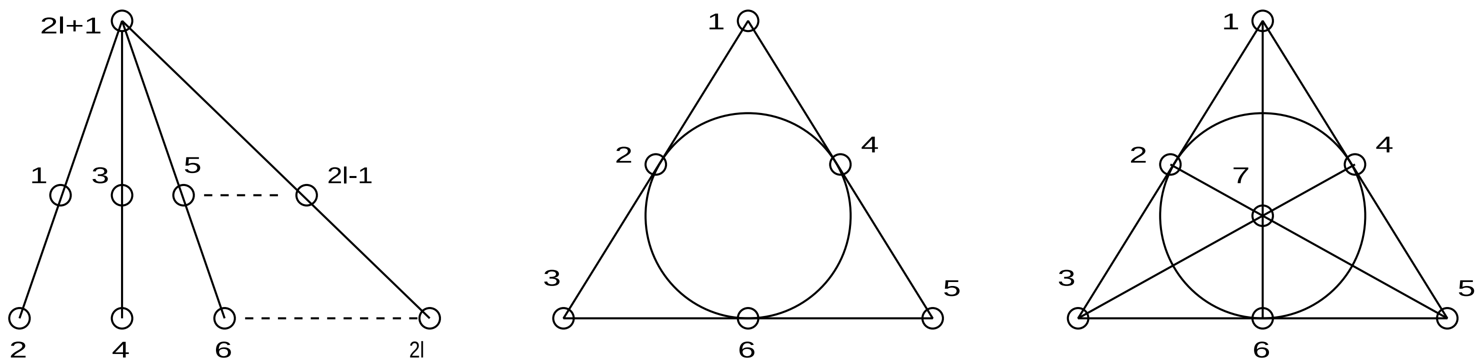
In the case $k = 3$

Antipodal sets of $\binom{[n]}{3}$

$$A(3, 2l+1) = \{\alpha \cup \{2l+1\} \mid \alpha \in A(2, 2l)\}$$

$$B(3,6) = \{\{1,2,3\},\{1,4,5\},\{2,4,6\},\{3,5,6\}\}$$

$$B(3,7) = B(3,6) \cup \{\{1,6,7\},\{2,5,7\},\{3,4,7\}\}$$



Fano plane = projective plane over $\mathbb{F}_2$

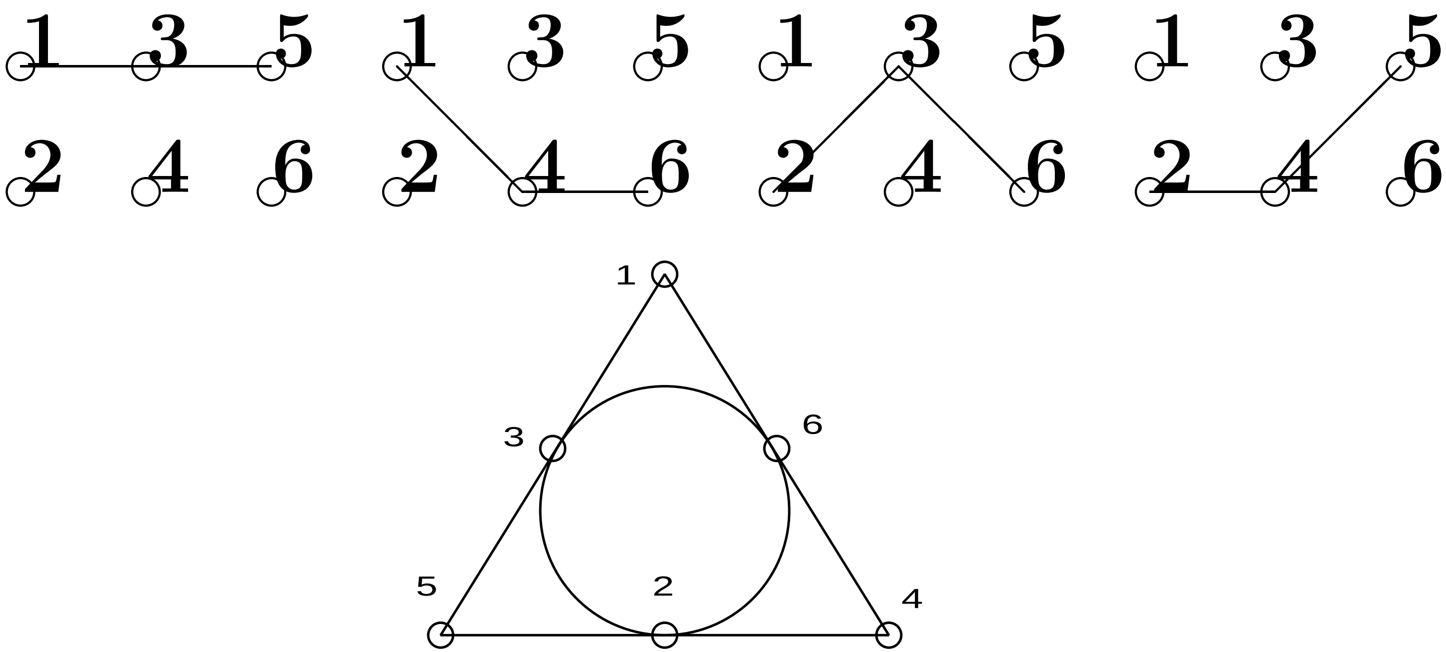
Thm.(T.2013)

MAS of $\binom{[n]}{3}$:

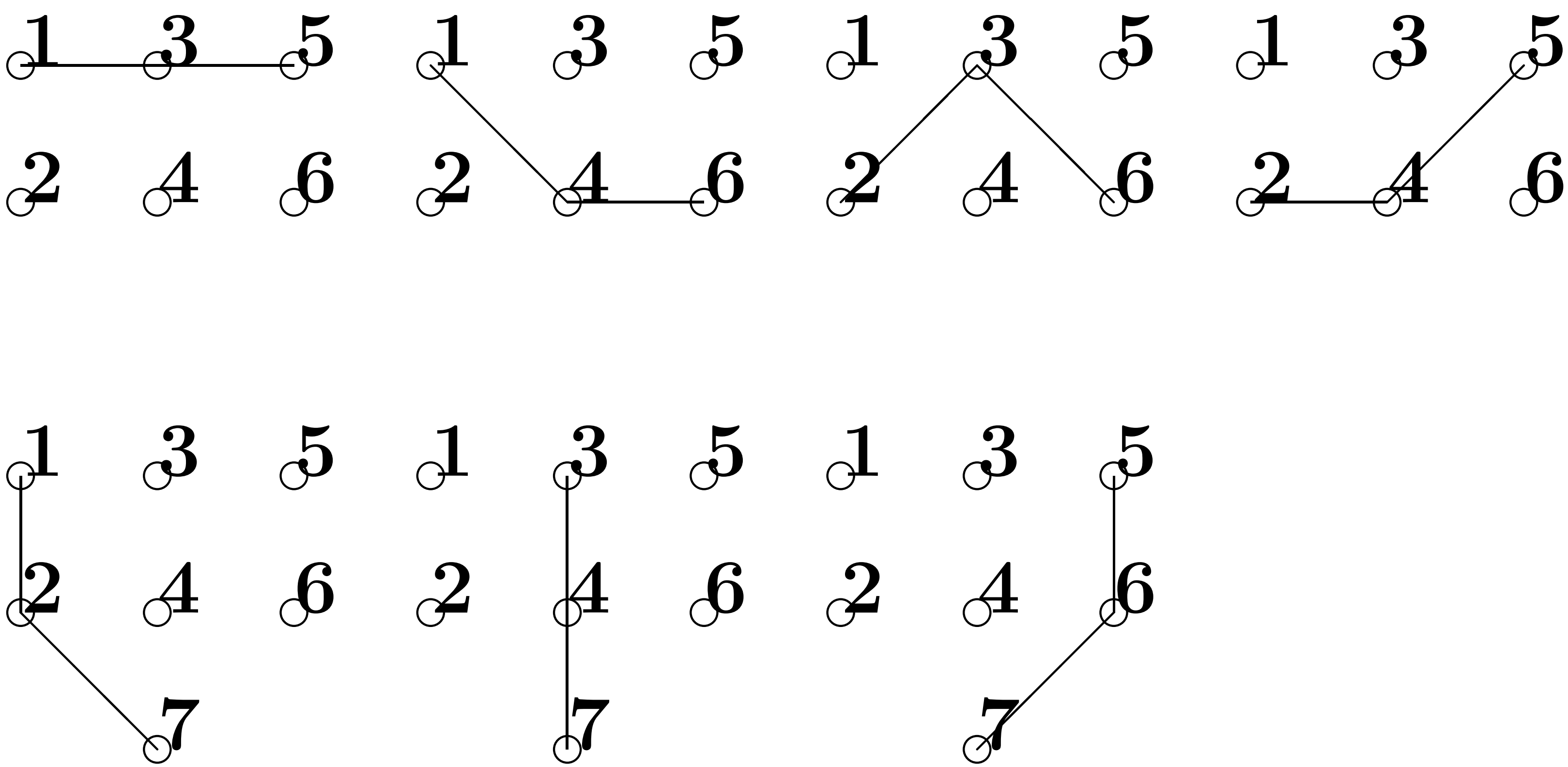
n	$3, 4$	5	6	$7, 8$
	$A(3,3)$	$A(3,5)$	$B(3,6)$	$B(3,7)$

n	more than 8
	$A\left(3, 2\lfloor\frac{n-1}{2}\rfloor + 1\right), B(3,7)$

$B(3,6)$ is congruent to $Ev_6 =$ $\{\{1,3,5\},\{1,4,6\},\{2,3,6\},\{2,4,5\}\}$



$B(3,7)$ is congruent to $Ev_6 \cup A(3,7)$



In the case $k = 4$

Antipodal sets of $\binom{[n]}{4}$

$$A(4, 2l) = \{\alpha \cup \beta \mid \alpha, \beta \in A(2, 2l),\ \alpha \neq \beta\}$$

$$B(4, 7) = \{[7] \backslash \alpha \mid \alpha \in B(3, 7)\}$$

$$B(4, 8) = B(4, 7) \cup \{\alpha \cup \{8\} \mid \alpha \in B(3, 7)\}$$

In general if A is an antipodal set of $\binom{[n]}{k}$, then

$$\{[n] \backslash \alpha \mid \alpha \in A\}$$

is an antipodal set of $\binom{[n]}{n-k}$.

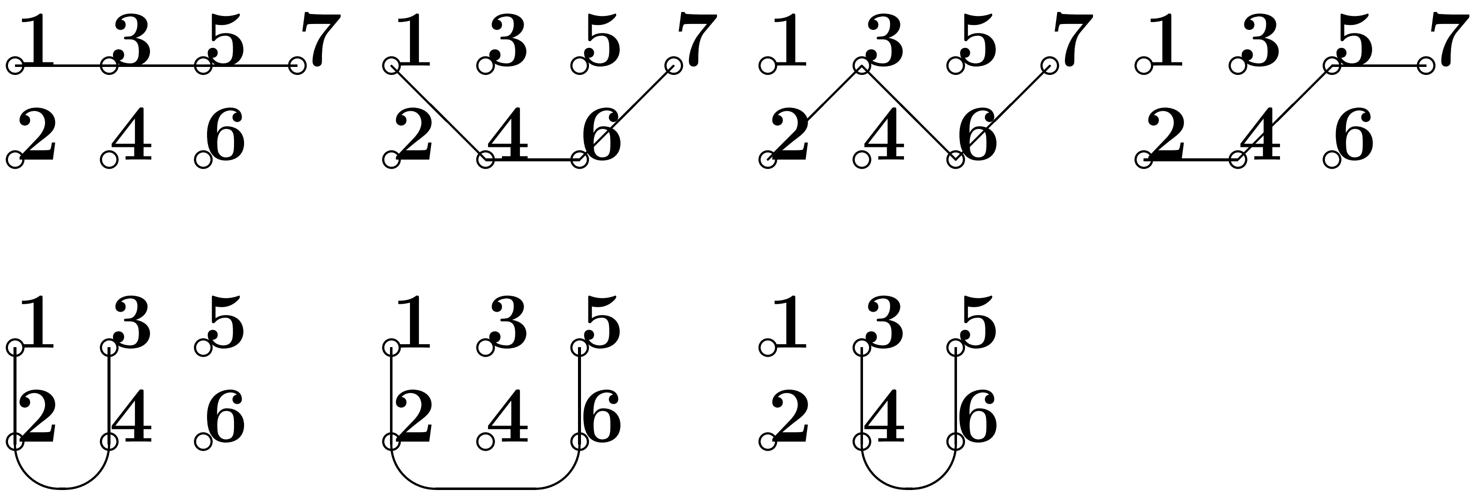
Thm.(T.2013)
MAS of $\binom{[n]}{4}$: Certain unions of copies of

$$\boldsymbol{A}(4, \mathbf{2}\boldsymbol{l})\ (\boldsymbol{l} \geq \mathbf{2}, \neq 4), \quad \boldsymbol{B}(4, \mathbf{7}), \quad \boldsymbol{B}(4, \mathbf{8})$$

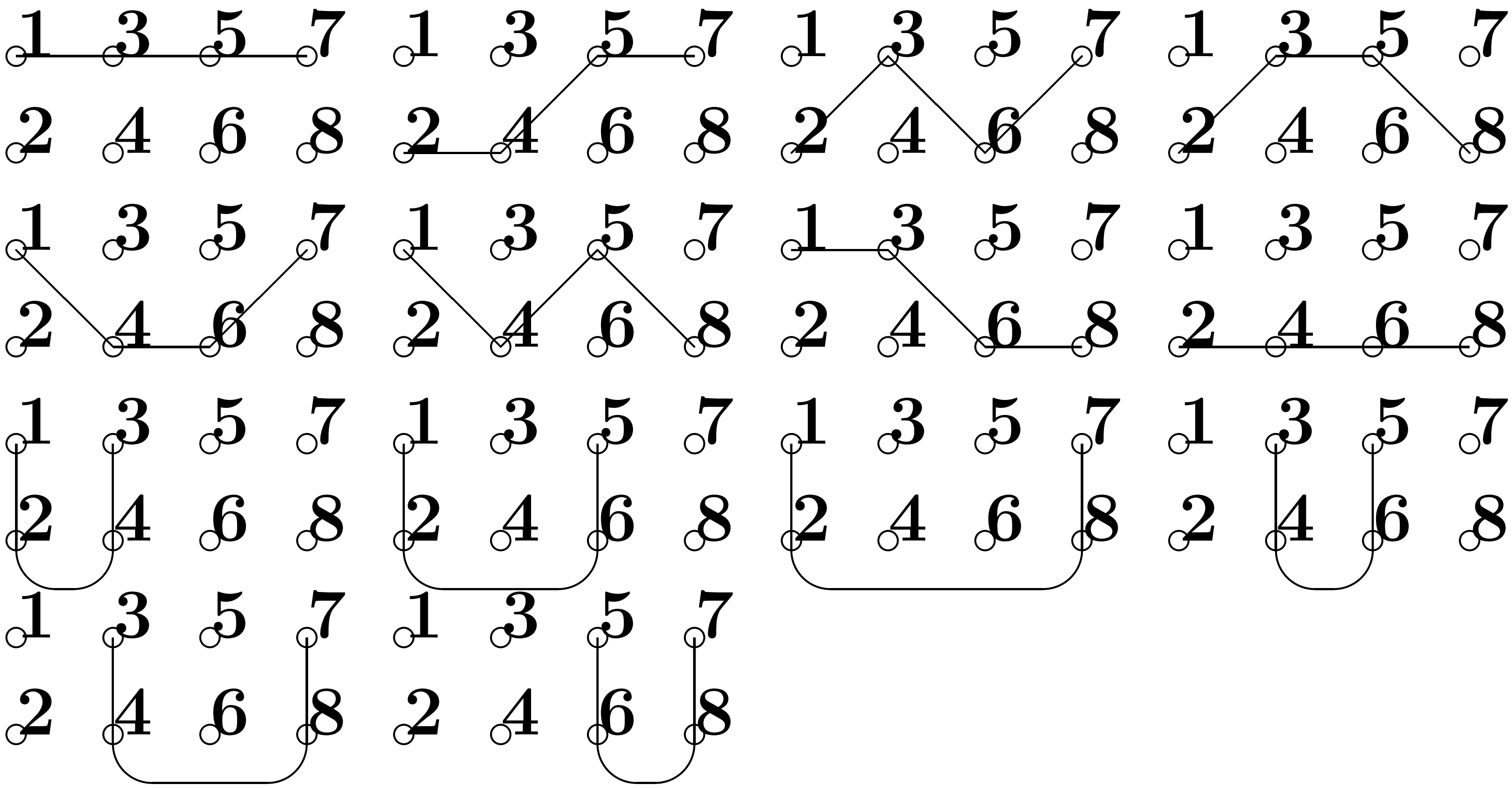
Remark. $\boldsymbol{A}(4, \mathbf{8}) \subsetneq \boldsymbol{B}(4, \mathbf{8}) \Rightarrow \boldsymbol{A}(4, \mathbf{8})$ **is not maximal.**

$B(4,7)$ is congruent to

$\{\alpha \cup \{7\} \mid \alpha \in Ev_6\} \cup A(4,6)$.



$B(4,8)$ is congruent to $Ev_8 \cup A(4,8)$.



Warp

$$\boldsymbol{A(2,2l) = \{\{1,2\},\dots,\{2l-1,2l\}\} \subset \binom{[2l]}{2}}$$

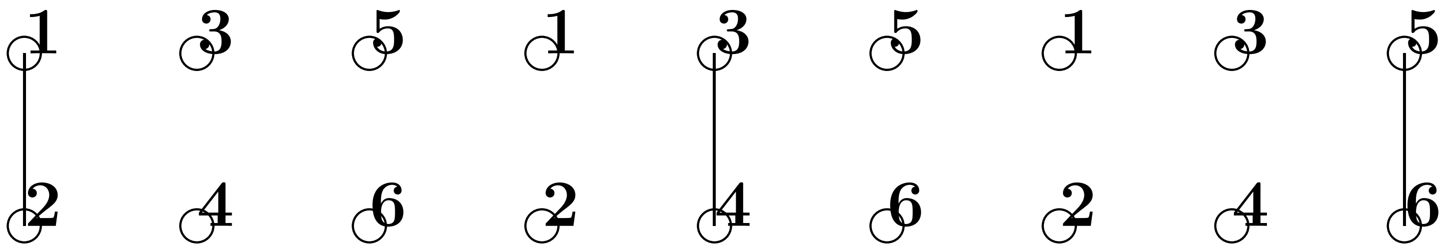
$$\boldsymbol{A(2k,2l)}$$

$$\boldsymbol{= \left\{ \alpha_1 \cup \cdots \cup \alpha_k \,\middle|\, \begin{matrix} \alpha_i \in A(2,2l) \\ \alpha_i \neq \alpha_j \end{matrix} \right\} \subset \binom{[2l]}{2k}}$$

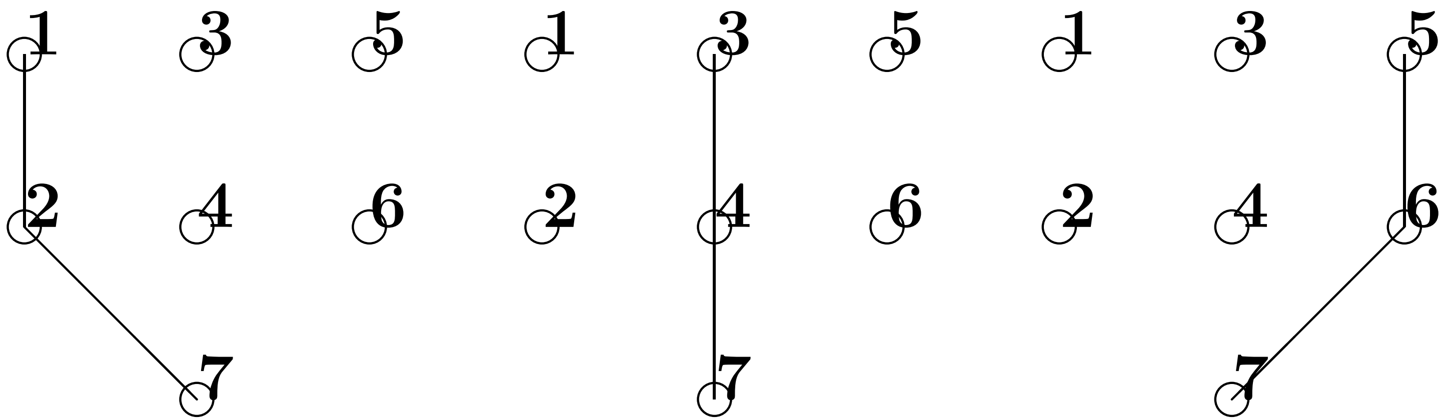
$$\boldsymbol{A(2k+1,2l+1)}$$

$$\boldsymbol{= \{\alpha \cup \{2l+1\} \mid \alpha \in A(2k,2l)\} \subset \binom{[2l+1]}{2k+1}}$$

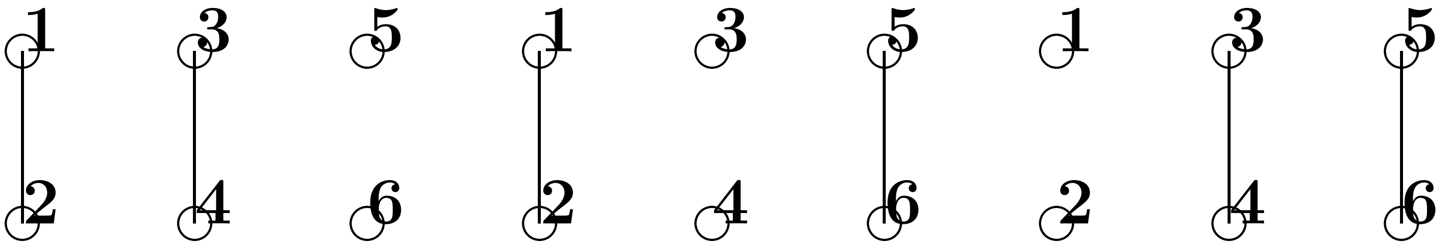
$$\boldsymbol{A(2,6)}$$



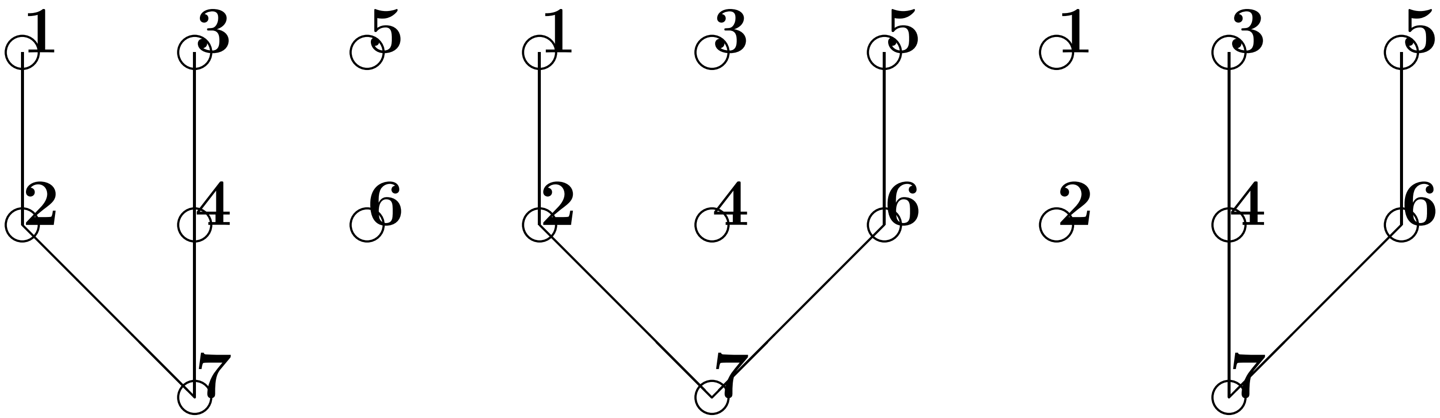
$$\boldsymbol{A(3,7) = \{\alpha \cup \{7\} \mid \alpha \in A(2,6)\}}$$



$\boldsymbol{A(4,6)}$



$\boldsymbol{A(5,7) = \{\alpha \cup \{7\} \mid \alpha \in A(4,6)\}}$



$$\mathbf{AS = antipodal\ set}$$

$$a(k,n) = \max\left\{ |A| \;\middle|\; A : \text{ AS in } \binom{[n]}{k} \right\}$$

$$a(1,n) = 1 \qquad a(2,n) = \left\lfloor \frac{n}{2} \right\rfloor$$

$$a(2k,n) \geq \left| A\left(2k, 2\left\lfloor \frac{n}{2} \right\rfloor\right) \right| = \binom{\lfloor n/2 \rfloor}{k},$$

$$a(2k+1,n) \geq \left| A\left(2k+1, 2\left\lfloor \frac{n-1}{2} \right\rfloor + 1\right) \right| = \binom{\lfloor (n-1)/2 \rfloor}{k}.$$

Thm(T.2014)

$\boldsymbol{A(2k,2l), A(2k+1,2l+1)}$ **: AS**

If $\boldsymbol{l \geq 3k+1}$**,**

$\boldsymbol{A(2k,2l)}$ **: MAS in** $\boldsymbol{\binom{[2l]}{2k}, \binom{[2l+1]}{2k}}$

$\boldsymbol{A(2k+1,2l+1)}$

: MAS in $\boldsymbol{\binom{[2l+1]}{2k+1}, \binom{[2l+2]}{2k+1}}$

For sufficiently large $\boldsymbol{l}$**,** $\boldsymbol{A(2k,2l)}$ **and** $\boldsymbol{A(2k+1,2l+1)}$ **are MAS.**

The results on the classifications of MAS of $\binom{[n]}{k}$ in the cases $k \leq 4$ show

$$a(1,n) = 1, \quad a(2,n) = \left\lfloor \frac{n}{2} \right\rfloor$$

n	4	5	6	$7,\ldots,16$	more than 16
$a(3,n)$	1	2	4	7	$\lfloor \frac{n-1}{2} \rfloor$

n	5	6	7	$8,\ldots,11$	more than 11
$a(4,n)$	1	3	7	14	$\binom{\lfloor \frac{n}{2} \rfloor}{2}$

In the case $k = 5$

Thm(T.2015) **If $n \geq 87$,**

$$a(5,n) = \left| A\left(5, 2\left\lfloor \frac{n-1}{2} \right\rfloor + 1\right) \right| = \binom{\left\lfloor \frac{n-1}{2} \right\rfloor}{2}.$$

Moreover an antipodal set of $\binom{[n]}{5}$ which attains $a(5,n)$ is congrunet to

$$A\left(5, 2\left\lfloor \frac{n-1}{2} \right\rfloor + 1\right).$$

Thm.(Frankl-Tokushige 2016)

$$\boldsymbol{a(2k, n) = \binom{\lfloor \frac{n}{2} \rfloor}{k},}$$

$$\boldsymbol{a(2k+1, n) = \binom{\lfloor \frac{n-1}{2} \rfloor}{k}.}$$

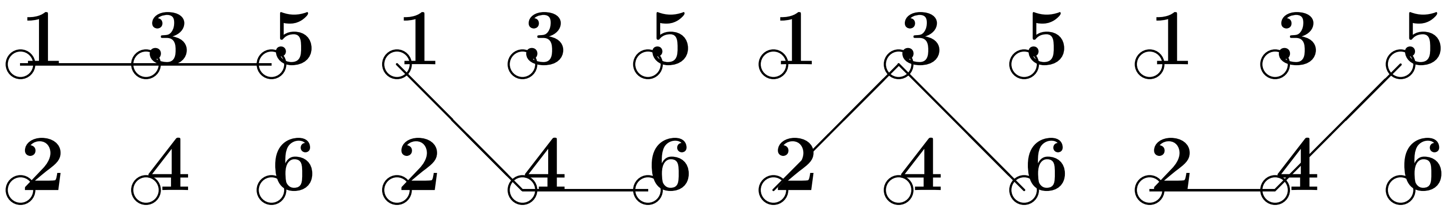
for large $\boldsymbol{n}$. Moreover antipodal sets of $\boldsymbol{\binom{[n]}{2k}}$, $\boldsymbol{\binom{[n]}{2k+1}}$ which attain $\boldsymbol{a(2k, n)}$, $\boldsymbol{a(2k+1, n)}$ are congruent to

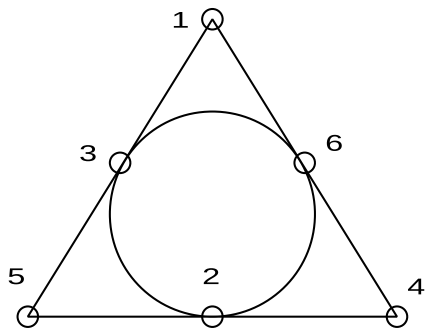
$$\boldsymbol{A\left(2k, 2\left\lfloor \frac{n}{2} \right\rfloor\right), \quad A\left(2k+1, 2\left\lfloor \frac{n-1}{2} \right\rfloor + 1\right).}$$

Weft

$Ev_{2m} = \{\{\alpha(1), \ldots, \alpha(m)\} \mid \quad \alpha(i) \in \{2i-1, 2i\}, |\{i \mid \alpha(i) : \text{even}\}| : \text{even}\}$

$Ev_{2m} \subset \binom{[2m]}{m}$. Ev_6 :





Thm(T.2014)

(1) If $m \equiv 1, 2, 3 (\mathrm{mod} 4)$,

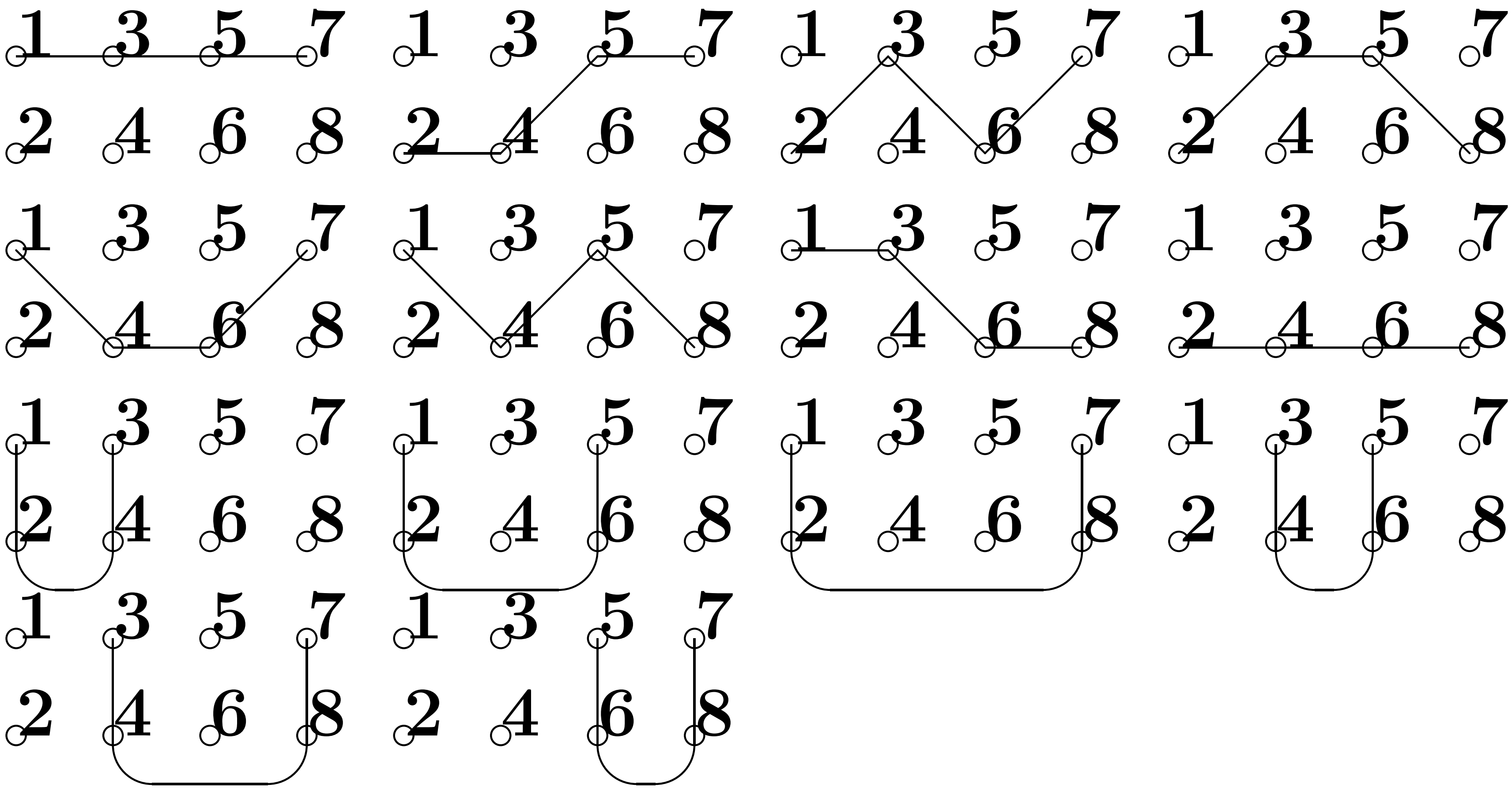
$$Ev_{2m} : \text{MAS of } \binom{[2m]}{m}$$

(2) Ev_{8m} : not MAS of $\binom{[8m]}{4m}$

$$Ev_{8m} \cup A(4m, 8m) : \text{MAS of } \binom{[8m]}{4m}$$

Is there any relation between this result and representations of $SO(2m)$?

$\boldsymbol{Ev_8} \cup \boldsymbol{A}(4,8)$: MAS of $\binom{[8]}{4}$



Union of weft $\boldsymbol{Ev_*}$ and warp $\boldsymbol{A(*,**)}$

$$\boldsymbol{Ev^+_{8m} = Ev_{8m} \cup A(4m, 8m),}$$

$$\boldsymbol{Ev^+_{8m+2} = Ev_{8m+2} \cup \{\alpha \cup \{8m+3, 8m+4, 8m+5\} \mid \alpha \in A(4m-2, 8m+2)\},}$$

$$\boldsymbol{Ev^+_{8m+4} = Ev_{8m+4} \cup \{\alpha \cup \{8m+5, 8m+6\} \mid \alpha \in A(4m, 8m+4)\},}$$

$$\boldsymbol{Ev^+_{8m+6} = Ev_{8m+6} \cup \{\alpha \cup \{8m+7\} \mid \alpha \in A(4m+2, 8m+6)\}.}$$

Thm(T.2017) MAS of $\binom{[n]}{k}$:

$k \backslash n$	$8m$	$8m+1$	$8m+2$	$8m+3$
$4m$	Ev^+_{8m}	Ev^+_{8m}	Ev^+_{8m}	Ev^+_{8m}
$4m+1$			Ev_{8m+2}	Ev_{8m+2}

$k \backslash n$	$8m+4$	$8m+5$	$8m+6$	$8m+7$
$4m+1$	Ev_{8m+2}	Ev^+_{8m+2}		
$4m+2$	Ev_{8m+4}	Ev_{8m+4}	Ev^+_{8m+4}	
$4m+3$			Ev_{8m+6}	Ev^+_{8m+6}