Polars and antipodal sets

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Definition(Chen-Nagano)

M : a cpt Riem. sym. sp.

s_x : the geod. symmetry at $x \in M$

$S \subset M$: **antipodal set**

$\Leftrightarrow \forall x, y \in S \ s_x(y) = y$

$\#_2 M = \max\{|A| \mid A : \text{antip. set}\}$

Each comp. of $F(s_o, M)$: **polar**

Investigating polars is a key to investigating antipodal sets.

Polar : totally geodesic submfd

cpt Riem. sym. sp.

$x \in S^n(r)$ $F(s_x, S^n(r)) = \{\pm x\}$

$K = R, C, H$, $x \in P^n(K)$

$F(s_x, P^n(K)) = \{x\} \cup P^{n-1}(K)$

$F(s_{(x_1,x_2)}, M_1 \times M_2)$

$= F(s_{x_1}, M_1) \times F(s_{x_2}, M_2)$

Polars of M :

$$F(s_o, M) = \{o\} \cup \bigcup_{i=1}^{a} \{o_i\} \cup M_1^+$$

In this case, $s_o = s_{o_i}$.

Maximal antip. set (MAS) of M

$\leftrightarrow$ MAS of M_1^+

$\{o, o_1, \ldots, o_a\} \cup A \leftrightarrow A$

(Their congruent classes also correspond.)

$$\tilde{M} = S^{n_1}(r_1) \times S^{n_2}(r_2)$$
$$M = (S^{n_1}(r_1) \times S^{n_2}(r_2))/\{\pm 1\}$$
$$\tilde{M} \to M \; ; (x, y) \mapsto [x, y]$$

Deck transf. and sym. commute.

$$s_{[x,y]}([x_1, y_1]) = [s_x(x_1), s_y(y_1)]$$
$$s_{[x,y]}([x_1, y_1]) = [x_1, y_1]$$
$$\Leftrightarrow \text{“}s_x(x_1) = x_1 \text{ and } s_y(y_1) = y_1\text{”}$$

or

$$\text{“}s_x(x_1) = -x_1 \text{ and } s_y(y_1) = -y_1\text{”}$$

$$F(s_{[x,y]}, M)$$
$$= \{[x, \pm y]\} \cup (S^{n_1-1} \times S^{n_2-1})/\{\pm 1\}$$

MAS of $(S^{n_1}(r_1) \times S^{n_2}(r_2))/\{\pm 1\}$

$\{[x_1, \pm y_1], \ldots, [x_k, \pm y_k]\}$

$x_1, \ldots, x_{n_1+1}$: og. basis of $\mathbb{R}^{n_1+1}$

$y_1, \ldots, y_{n_2+1}$: og. basis of $\mathbb{R}^{n_2+1}$

$k = \min\{n_1, n_2\} + 1$

The above MAS is unique up to congruence.

Compact Lie group

biinvariant Riemannian metric

$\rightarrow$ cpt Riem. sym. sp.

geod. symmetry $s_x(y) = xy^{-1}x$

: algebraic description

By considering a cpt Lie group as a Riem. sym. sp., its algebraic structure can be examined from a geometric viewpoint.

G : a compact Lie group

A : MAS containing the unit e

$\forall x \in A \; x = s_e(x) = x^{-1}, \; x^2 = e$

$\forall y \in A \; y = s_x(y) = xy^{-1}x,$

$xy = yx$

A is commutative. $\quad \forall z \in A$

$$s_z(xy) = z(xy)^{-1}z = zy^{-1}zzx^{-1}z$$
$$= s_z(y)s_z(x) = xy$$

Max. of $A \Rightarrow xy \in A \quad A$: subgr.

The order of each element of A except e is 2.

$$A \cong \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$$

$$\begin{aligned} F(s_e, G) &= \{x \in G \mid x^2 = e\} \\ &= \{e\} \cup \bigcup_{i=1}^{a} \{o_i\} \cup \bigcup_{j=1}^{b} M_j^+ \end{aligned}$$

$M_j^+ = \{g x_j g^{-1} \mid g \in G_0\}$: **polar**

G_0 : **the identity comp. of G**

A : MA subgroup of G

$s_e = s_{o_i}, \ \{e, o_1, \ldots, o_a\} \subset A$

Max. of $A \Rightarrow \exists j \ A \cap M_j^+ \neq \emptyset$

Replace A with a conjugate one.

$x_j \in A \cap M_j^+ \quad A \subset Z_{x_j}(G)$

A : MA subgroup of $Z_{x_j}(G)$

Each MAS of $Z_{x_j}(G)$

: Candidate of MAS of G

The above theory works : G_2

H : quaternions $\quad O := H \times H$

O : octonions $\quad$ product

$$(m, a)(n, b) = (mn - \overline{b}a,\ a\overline{n} + bm)$$

$$((m, a), (n, b) \in O)$$

$$\mathrm{Aut}(O) := \{\alpha \in GL_R(O) \mid$$

$$\alpha(xy) = (\alpha x)(\alpha y)\ (x, y \in O)\}$$

$\mathrm{Aut}(O)$: cpt Lie group of type G_2

$G_2 = \mathrm{Aut}(O)$

$\boldsymbol{\psi : Sp(1) \times Sp(1) \to G_2}$ **def. by**

$$\boldsymbol{\psi(p,q)(m,a) := (qm\overline{q}, pa\overline{q})}$$

$$\boldsymbol{(p, q \in Sp(1),\ (m,a) \in O)}$$

$\boldsymbol{\psi}$ **: homo. of Lie groups**

$$\boldsymbol{\ker \psi = \{\pm(1,1)\}}$$

$$\boldsymbol{\psi(Sp(1)^2) = Z_{\psi(1,-1)}(G_2)}$$

$$\boldsymbol{\cong SO(4)}$$

$G_2 \supset \psi(Sp(1)^2) \cong SO(4)$

Their ranks are 2.

$T = \{\psi(e^{is}, e^{it}) \mid s, t \in \mathbb{R}\}$

: max. torus of $\psi(Sp(1)^2)$ **and** G_2

$$G_2 = \bigcup_{g \in G_2} gTg^{-1}$$

$$F(s_e, G_2) = \bigcup_{g \in G_2} gF(s_e, T)g^{-1}$$

$$F(s_e, T) = \{\psi(1, \pm 1), \psi(i, \pm i)\}$$

$\psi(1,-1), \psi(i, \pm i)$: conjugate

$F(s_e, G_2) \backslash \{e\}$

$= \{g\psi(1,-1)g^{-1} \mid g \in G_2\}$

: unique polar except e

A : MAS of G_2

We can suppose $\psi(1,-1) \in A$.

$A \subset Z_{\psi(1,-1)}(G_2) = \psi(Sp(1)^2)$

A is conj. with $\Psi = \{\psi(p, \pm p) \mid$

$p = 1, i, j, k\}$: unique MAS of G_2

$\boldsymbol{\Psi}$: unique MAS of $\boldsymbol{Z_{\psi(1,-1)}(G_2)}$

$\boldsymbol{\Psi = \{\psi(p, \pm p) \mid p = 1, i, j, k\}}$

$\boldsymbol{\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2}$: rank 3

$\boldsymbol{\#_2 G_2 = |\Psi| = 2^3}$

$\boldsymbol{\mathrm{rank}\Psi = 3 > 2 = \mathrm{rank} G_2}$

$\boldsymbol{\Psi}$ cannot be included in any max. torus of $\boldsymbol{G_2}$.

$$F(s_e, G_2) = \{e\} \cup M_1^+,$$

$$M_1^+ = \{g\psi(1,-1)g^{-1} \mid g \in G_2\}$$

$$\cong G_2/SO(4)$$

MAS of M_1^+

$$\Psi_1 = \{\psi(1,-1)\}$$

$$\cup \{\psi(p, \pm p) \mid p = i, j, k\}$$

$$\#_2 M_1^+ = |\Psi_1| = 7$$

$\tilde{G}_{\text{ass}}$: associative Grassmann mfd

$\tilde{G}_{\text{ass}} \subset \tilde{G}_3(R^7)$

: totally geodesic submanifold

G_2 acts transitively on $\tilde{G}_{\text{ass}}$.

$\tilde{G}_{\text{ass}} \cong G_2/S_{\text{Im}H\times\{0\}}(G_2)$

$S_{\text{Im}H\times\{0\}}(G_2) = Z_{\psi(1,-1)}(G_2)$

$M_1^+ \cong G_2/Z_{\psi(1,-1)}(G_2)$

$\Psi_1 \subset M_1^+ \to \tilde{\Psi}_1 \subset \tilde{G}_{\text{ass}} \subset \tilde{G}_3(R^7)$

$\tilde{\Psi}_1$ is antipodal in $\tilde{G}_3(R^7)$, $|\tilde{\Psi}_1| = 7$

MAS of $\tilde{G}_k(R^n)$

$[n] = \{1, 2, 3, \ldots, n\}$

$\binom{[n]}{k}$: all subsets of card. k in $[n]$

$A \subset \binom{[n]}{k}$: antipodal set

$\Leftrightarrow \forall \alpha, \beta \in A \ |\alpha \backslash \beta|$: even

congr. classes of MAS in $\binom{[n]}{k}$

congr. classes of MAS in $\tilde{G}_k(R^n)$

: one-to-one correspondence

$\boldsymbol{e_1, \ldots, e_n}$: o. n. basis of $\boldsymbol{R^n}$

$\binom{[n]}{k} \ni \alpha = \{\alpha_1, \ldots, \alpha_k\}$

$A \subset \binom{[n]}{k}$: antipodal set

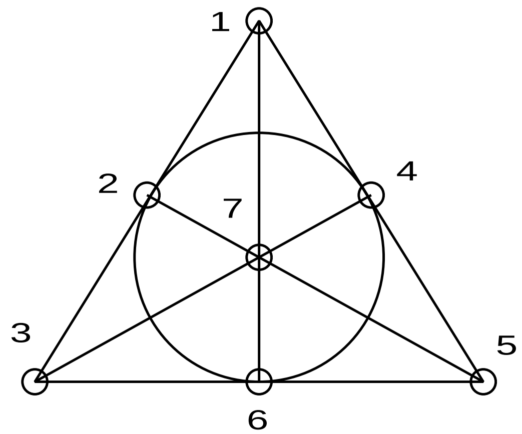
$\mathcal{A}(A) = \{\pm\langle e_{\alpha_1}, \ldots, e_{\alpha_k}\rangle \mid \alpha \in A\}$

: antipodal set of $\tilde{G}_k(R^n)$

MAS of $\binom{[n]}{k} \leftrightarrow$ MAS of $\tilde{G}_k(R^n)$

$$A \leftrightarrow \mathcal{A}(A)$$

MAS of $\binom{[7]}{3}$: Fano plane (7-pt. set)



$$\Psi_1 \subset M_1^+ \cong \tilde{G}_{\text{ass}} \subset \tilde{G}_3(\boldsymbol{R}^7)$$

$\{\pm\xi \mid \xi \in \Psi_1\}$: MAS of $\tilde{G}_3(\boldsymbol{R}^7)$

Ψ_1 determ. $\boldsymbol{O}$ prod. op. table by comb. of lines of Fano plane.