The intersection of two real flag manifolds in a complex flag manifold

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Introduction

M : Kähler manifold

$L_0, L_1 \subset M$: real forms

i.e. $\exists \sigma_i$: anti-holomorphic involutive isometry of M $(i = 0, 1)$

s.t. $L_i = \mathrm{Fix}(\sigma_i, M)_0$

totally geodesic Lagrangian submanifold

Problems

1. Is the intersection $L_0 \cap L_1$ discrete? **symmetric triad**
2. If so, count the intersection number $\#(L_0 \cap L_1)$, and describe the geometric meaning of $\#(L_0 \cap L_1)$.

   Floer homology
3. Study the structure of the intersection $L_0 \cap L_1$.

   antipodal set

Antipodal set of a compact symmetric space

M : compact Riemannian symmetric space

s_x : geodesic symmetry at $x \in M$

Definition (Chen-Nagano 1988)

1. $\mathcal{A} \subset M$: **antipodal set**
   $\overset{\text{def}}{\Longleftrightarrow} \quad s_x(y) = y$ for all $x, y \in \mathcal{A}$
2. $\#_2 M := \max\{\#\mathcal{A} \mid \mathcal{A} \subset M$: antipodal set$\}$ **2-number**
3. $\mathcal{A} \subset M$: **great** antipodal set $\overset{\text{def}}{\Longleftrightarrow} \quad \#\mathcal{A} = \#_2 M$

Theorem (Takeuchi 1989)

M : *symmetic R-space* $\Longrightarrow \quad \#_2 M = \dim H_*(M, \mathbb{Z}_2)$

Previous works

Theorem (Tanaka-Tasaki 2012)

M : Hermitian symmetric space of compact type

$L_0, L_1 \subset M$: real forms, $\quad L_0 \pitchfork L_1$

$\Longrightarrow \quad$ *$L_0 \cap L_1$ is an antipodal set of L_0 and L_1.*

Theorem (Iriyeh-S.-Tasaki 2013)

1. *Lagrangian Floer homology of two real forms in irreducible Hermitian symmetric spacecs*
2. *Volume estimate of real forms under Hamiltonian deformations*

Example

$\mathbb{R}P^n \subset \mathbb{C}P^n$

$$\mathcal{A} := \{\mathbb{R}e_1, \dots, \mathbb{R}e_{n+1}\} \subset \mathbb{R}P^n \quad \text{great antipodal set}$$

For $g \in U(n+1)$, $\mathbb{R}P^n \pitchfork g\mathbb{R}P^n$ in $\mathbb{C}P^n$

$$\mathbb{R}P^n \cap g\mathbb{R}P^n \cong \{\mathbb{C}e_1, \dots, \mathbb{C}e_{n+1}\} \subset \mathbb{C}P^n$$

$$\#(\mathbb{R}P^n \cap g\mathbb{R}P^n) = n+1 = \#_2\mathbb{R}P^n = \dim H_*(\mathbb{R}P^n, \mathbb{Z}_2)$$

Aim of our work

Generalizing the results on Hermitian symmetric spaces, study the intersection of two real forms in a complex flag manifold.

Complex flag manifold

G : compact connected semisimple Lie group

$x_0 (\neq 0) \in \mathfrak{g}$

$$\begin{aligned} M &:= \mathrm{Ad}(G)x_0 \subset \mathfrak{g} \quad : \textbf{complex flag manifold} \\ &\cong G/G_{x_0} \cong G^{\mathbb{C}}/P^{\mathbb{C}} \end{aligned}$$

$$G_{x_0} := \{g \in G \mid \mathrm{Ad}(g)x_0 = x_0\}$$

ω : Kirillov-Kostant-Souriau symplectic form on M defined by

$$\omega(X^*_x, Y^*_x) := \langle [X, Y], x \rangle \qquad (x \in M, \quad X, Y \in \mathfrak{g})$$

J : G-invariant complex structure on M compatible with ω

$(\cdot, \cdot) := \omega(\cdot, J\cdot)$: G-invariant Kähler metric

Antipodal set of a complex flag manifold

For $x \in M$

$$G_x := \{g \in G \mid \mathrm{Ad}(g)x = x\}$$
$$Z(G_x) := \{g \in G_x \mid gh = hg \ (\forall h \in G_x)\}$$

Definition

$y \in M$ is **antipodal** to $x \in M$

$\stackrel{\text{def}}{\Longleftrightarrow}$ $\mathrm{Ad}(g)y = y$ for all $g \in Z(G_x)_0$

$\mathcal{A} \subset M$: **antipodal set**

$\stackrel{\text{def}}{\Longleftrightarrow}$ y is antipodal to x for any $x, y \in \mathcal{A}$.

Note: This definition is equivalent to the notion of an antipodal set of M defined using k-symmetric structure on M. When M is a Hermitian symmetric space, it is also equivalent to the notion of an antipodal set introduced by Chen-Nagano.

Antipodal set of a complex flag manifold

Proposition

1. For $x, y \in M$
$$y \text{ is antipodal to } x \quad \Longleftrightarrow \quad [x, y] = 0$$
2. $\mathcal{A} \subset M$: maximal antipodal set
$$\Longrightarrow \quad \exists \mathfrak{t} \subset \mathfrak{g} : \text{maximal abelian subalgebra}$$
$$\text{s.t.} \quad \mathcal{A} = M \cap \mathfrak{t}.$$
Hence $\mathcal{A}$ is an orbit of the Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{t}$. In particular, any maximal antipodal sets of M are congruent with each other by G.

Real flag manifolds in a complex flag manifold

(G, K) : symmetric pair of compact type

θ : involution of G s.t. $\mathrm{Fix}(\theta, G)_0 \subset K \subset \mathrm{Fix}(\theta, G)$

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

$x_0 (\neq 0) \in \mathfrak{p}$

$$\begin{array}{cccccc} L & := & \mathrm{Ad}(K)x_0 & \subset & \mathfrak{p} & : \textbf{real flag manifold, } R\textbf{-space} \\ \cap & & \cap & & \cap & \\ M & := & \mathrm{Ad}(G)x_0 & \subset & \mathfrak{g} & : \textbf{complex flag mfd, } C\textbf{-space} \\ & \cong & G/G_{x_0} & \cong & G^{\mathbb{C}}/P^{\mathbb{C}} & \end{array}$$

$d\theta$: $\mathfrak{g} \to \mathfrak{g}$ involutive automorphism

$\sigma := -d\theta$: $M \to M$ anti-holomorphic involutive isometry

$L = M \cap \mathfrak{p} = \mathrm{Fix}(\sigma, M)$ **real form** of M

$(G, K_0), (G, K_1)$: symmetric pairs of compact type
θ_0, θ_1 : involutions of G

$$\mathfrak{g} = \mathfrak{k}_0 + \mathfrak{p}_0 = \mathfrak{k}_1 + \mathfrak{p}_1,$$

$x_0 (\neq 0) \in \mathfrak{p}_0 \cap \mathfrak{p}_1$

$$L_0 := \mathrm{Ad}(K_0)x_0, \quad L_1 := \mathrm{Ad}(K_1)x_0 \ \subset \ M := \mathrm{Ad}(G)x_0$$

For $g \in G$, we consider the intersection of $L_0 \cap \mathrm{Ad}(g)L_1$ in M.

$\mathfrak{a}$: maximal abelian subspace of $\mathfrak{p}_0 \cap \mathfrak{p}_1$ containing x_0
$A := \exp \mathfrak{a} \subset G$: toral subgroup

Then $G = K_0 A K_1$, i.e.
$g = g_0 a g_1 \ (g_0 \in K_0, g_1 \in K_1, a \in A)$

$$L_0 \cap \mathrm{Ad}(g)L_1 = L_0 \cap \mathrm{Ad}(g_0 a g_1)L_1 = \mathrm{Ad}(g_0)(L_0 \cap \mathrm{Ad}(a)L_1)$$

Symmetric triads

Hereafter we assume $\theta_0\theta_1 = \theta_1\theta_0$.

$$\mathfrak{g} = \underline{\underline{(\mathfrak{k}_0 \cap \mathfrak{k}_1) + (\mathfrak{p}_0 \cap \mathfrak{p}_1)}} + \underline{\underline{(\mathfrak{k}_0 \cap \mathfrak{p}_1) + (\mathfrak{p}_0 \cap \mathfrak{k}_1)}}$$

$\mathrm{ad}(\mathfrak{a})$-invariant $\mathrm{ad}(\mathfrak{a})$-invariant

Then $\big((\mathfrak{k}_0 \cap \mathfrak{k}_1) + (\mathfrak{p}_0 \cap \mathfrak{p}_1),\ (\mathfrak{k}_0 \cap \mathfrak{k}_1),\ d\theta_0 = d\theta_1\big)$ is an orthogonal symmetric Lie algebra. For $\lambda \in \mathfrak{a} \subset \mathfrak{p}_0 \cap \mathfrak{p}_1$

$$\mathfrak{p}_\lambda := \{X \in \mathfrak{p}_0 \cap \mathfrak{p}_1 \mid [H, [H, X]] = -\langle \lambda, H \rangle^2 X \ (H \in \mathfrak{a})\}$$

$$V_\lambda := \{X \in \mathfrak{p}_0 \cap \mathfrak{k}_1 \mid [H, [H, X]] = -\langle \lambda, H \rangle^2 X \ (H \in \mathfrak{a})\}$$

$$\Sigma := \{\lambda \in \mathfrak{a} \setminus \{0\} \mid \mathfrak{p}_\lambda \neq \{0\}\}$$

$$W := \{\lambda \in \mathfrak{a} \setminus \{0\} \mid V_\lambda \neq \{0\}\}$$

$$\tilde{\Sigma} := \Sigma \cup W$$

$(\tilde{\Sigma}, \Sigma, W)$: **symmetric triad**

The structure of the intersection

$$\mathfrak{a}_{\mathrm{reg}} := \bigcap_{\substack{\lambda\in\Sigma\\ \alpha\in W}} \left\{ H \in \mathfrak{a} \;\middle|\; \langle \lambda, H\rangle \notin \pi\mathbb{Z}, \langle \alpha, H\rangle \notin \frac{\pi}{2} + \pi\mathbb{Z} \right\}$$

$W(\widetilde{\Sigma})$: Weyl group of the root system $\widetilde{\Sigma}$ of $\mathfrak{a}$

Theorem (Ikawa-Iriyeh-Okuda-S.-Tasaki)

For $a = \exp H$ $(H \in \mathfrak{a})$, the intersection $L_0 \cap \mathrm{Ad}(a)L_1$ is discrete if and only if $H \in \mathfrak{a}_{\mathrm{reg}}$. Moreover, if $L_0 \cap \mathrm{Ad}(a)L_1$ is discrete, then

$$L_0 \cap \mathrm{Ad}(a)L_1 = M \cap \mathfrak{a} = W(\widetilde{\Sigma})x_0.$$

In particular, $L_0 \cap \mathrm{Ad}(a)L_1$ is an antipodal set of M.

Lagrangian Floer homology

(M, ω) : closed symplectic manifold

$J = \{J_t\}_{0 \le t \le 1}$: family of almost complex structures on M compatible with ω

L_0, L_1 : closed Lagrangian submanifolds, $L_0 \pitchfork L_1$

Definition

For $p, q \in L_0 \cap L_1$,

$u : \mathbb{R} \times [0,1] \longrightarrow M$: ***J-holomorphic strip*** from p to q

$$\overset{\text{def}}{\Longleftrightarrow} \begin{cases} \bar{\partial}_J u := \frac{\partial u}{\partial s} + J_t(u)\frac{\partial u}{\partial t} = 0 \\ u(s,0) \in L_0, \quad u(s,1) \in L_1 \\ u(-\infty, t) = p, \quad u(+\infty, t) = q \end{cases}$$

$\mathcal{M}_J(L_0, L_1 : p, q) := \{u : J\text{-holomorphic strips from } p \text{ to } q\}$

Lagrangian Floer homology

$\mathcal{M}_J(L_0, L_1 : p, q) := \{u : J\text{-holomorphic strips from } p \text{ to } q\}$

$$CF(L_0, L_1) := \bigoplus_{p \in L_0 \cap L_1} \mathbb{Z}_2\, p$$

$\partial : CF(L_0, L_1) \longrightarrow CF(L_0, L_1)$

$$\partial(p) = \sum_{q \in L_0 \cap L_1} n(p, q) \cdot q$$

$n(p, q) := \#\{\text{isolated } J\text{-holomorphic strips from } p \text{ to } q\} \pmod 2$

$$\partial \circ \partial = 0 \quad \Longrightarrow \quad HF(L_0, L_1 : \mathbb{Z}_2) := \ker \partial / \mathrm{im} \partial$$

- $HF(\phi L_0, \psi L_1 : \mathbb{Z}_2) \cong HF(L_0, L_1 : \mathbb{Z}_2)$ for $\forall \phi, \psi \in \mathrm{Ham}(M, \omega)$ with $\phi L_0 \pitchfork \psi L_1$.

Lagrangian Floer homology for two real forms

Theorem (Ikawa-Iriyeh-Okuda-S.-Tasaki)

M : complex flag manifold with a Kähler-Einstein metric

$L_0, L_1 \subset M$: real flag manifolds, $\theta_0\theta_1 = \theta_1\theta_0$

minimal Maslov numbers $\Sigma_{L_0}, \Sigma_{L_1} \geq 3$.

$\Longrightarrow$ $\exists$ a real flag manifold $L_1' \cong L_1$ s.t. $L_0 \pitchfork L_1'$ and

$$HF(L_0, L_1' : \mathbb{Z}_2) \cong \bigoplus_{p \in L_0 \cap L_1'} \mathbb{Z}_2[p]$$

Corollary

$$\#(\phi L_0 \cap \psi L_1) \geq \#(L_0 \cap L_1') = \dim HF(L_0, L_1' : \mathbb{Z}_2)$$

for any $\phi, \psi \in \mathrm{Ham}(M, \omega)$ with $\phi L_0 \pitchfork \psi L_1$.

Example

$$(G, K_0, K_1) = (SU(2n), SO(2n), Sp(n))$$

$$\theta_0(g) = \bar{g}, \quad \theta_1(g) = J_n \bar{g} J_n^{-1} (g \in G) \quad \text{where} \quad J_n := \begin{bmatrix} O & I_n \\ -I_n & O \end{bmatrix}$$

$$\mathfrak{p}_0 \cap \mathfrak{p}_1 = \left\{ \begin{bmatrix} \sqrt{-1}X & \sqrt{-1}Y \\ -\sqrt{-1}Y & \sqrt{-1}X \end{bmatrix} \middle| \begin{array}{l} X, Y \in M_n(\mathbb{R}) \\ \mathrm{trace} X = 0 \\ {}^tX = X, \ {}^tY = -Y \end{array} \right\}$$

Fix a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}_0 \cap \mathfrak{p}_1$ as

$$\mathfrak{a} = \left\{ \begin{bmatrix} \sqrt{-1}X & O \\ O & \sqrt{-1}X \end{bmatrix} \middle| \begin{array}{l} X = \mathrm{diag}(t_1, \dots, t_n), \\ t_1, \dots, t_n \in \mathbb{R}, \ t_1 + \cdots + t_n = 0 \end{array} \right\}$$

$$\widetilde{\Sigma} = \Sigma = W = \{\pm(e_i - e_j) \mid 1 \leq i < j \leq n\}$$

where $e_i - e_j \in \mathfrak{a}$ $(i \neq j)$ is defined by $\langle e_i - e_j, H \rangle = t_i - t_j$.

$$x_0 = \begin{bmatrix} \sqrt{-1}X & O \\ O & \sqrt{-1}X \end{bmatrix} \in \mathfrak{a}$$

where $X = \mathrm{diag}(x_1 I_{n_1}, \ldots, x_{r+1} I_{n_{r+1}})$ and x_i are distinct real numbers satisfying $n_1 x_1 + \cdots + n_{r+1} x_{r+1} = 0$.

$$L_0 = \mathrm{Ad}(K_0)x_0 \cong F^{\mathbb{R}}_{2n_1,\ldots,2n_r}(\mathbb{R}^{2n})$$
$$L_1 = \mathrm{Ad}(K_1)x_0 \cong F^{\mathbb{H}}_{n_1,\ldots,n_r}(\mathbb{H}^{n})$$
$$M = \mathrm{Ad}(G)x_0 \cong F^{\mathbb{C}}_{2n_1,\ldots,2n_r}(\mathbb{C}^{2n})$$

$\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$

$n, n_1, \ldots, n_r$ satisfying $n_{r+1} := n - (n_1 + \cdots + n_r) > 0$

$$F^{\mathbb{K}}_{n_1,\ldots,n_r}(\mathbb{K}^n) = \left\{ (V_1, \ldots, V_r) \;\middle|\; \begin{array}{l} V_j \text{ is a } \mathbb{K}\text{-subspace of } \mathbb{K}^n, \\ \dim_{\mathbb{K}} V_j = n_1 + \cdots + n_j, \\ V_1 \subset V_2 \subset \cdots \subset V_r \subset \mathbb{K}^n \end{array} \right\}$$

$$a = \exp H, \quad H = \begin{bmatrix} \sqrt{-1}Y & O \\ O & \sqrt{-1}Y \end{bmatrix} \in \mathfrak{a}$$

where $Y = \mathrm{diag}(t_1, \ldots, t_n)$ and $t_1, \ldots, t_n \in \mathbb{R}$ which satisfy $t_1 + \cdots + t_n = 0$. Then

$L_0 \cap \mathrm{Ad}(a)L_1$ is discrete

$$\iff H \in \mathfrak{a}_{\mathrm{reg}} = \left\{ H \in \mathfrak{a} \;\middle|\; \langle e_i - e_j, H \rangle \notin \frac{\pi}{2}\mathbb{Z} \ (1 \leq i < j \leq n) \right\}$$

$$L_0 \cap \mathrm{Ad}(a)L_1 = M \cap \mathfrak{a} = W(\widetilde{\Sigma})x_0.$$

In this case, a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}_0 \cap \mathfrak{p}_1$ is also a maximal abelian subspace in $\mathfrak{p}_1$.

We express the intersection in the flag model $F^{\mathbb{C}}_{2n_1,\dots,2n_r}(\mathbb{C}^{2n})$.

$v_1, \dots, v_{2n}$: standard basis of $\mathbb{C}^{2n}$

$W_i := \langle v_i, v_{n+i}\rangle_{\mathbb{C}} = \langle v_i\rangle_{\mathbb{H}} \quad (1 \le i \le n)$

Proposition

For $a = \exp H \ (H \in \mathfrak{a}_{\mathrm{reg}})$,

$$
\begin{aligned}
&F^{\mathbb{R}}_{2n_1,\dots,2n_r}(\mathbb{R}^{2n}) \cap aF^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n) \\
&= \{(W_{i_1} \oplus \cdots \oplus W_{i_{n_1}}, W_{i_1} \oplus \cdots \oplus W_{i_{n_1+n_2}}, \dots \\
&\qquad \dots, W_{i_1} \oplus \cdots \oplus W_{i_{n_1+\cdots+n_r}}) \\
&\quad \mid 1 \le i_1 < \cdots < i_{n_1} \le n,\ 1 \le i_{n_1+1} < \cdots < i_{n_1+n_2} \le n, \dots, \\
&\quad\ 1 \le i_{n_1+\cdots+n_{r-1}+1} < \cdots < i_{n_1+\cdots+n_r} \le n, \\
&\quad\ \#\{i_1, \dots, i_{n_1+\cdots+n_r}\} = n_1 + \cdots + n_r\},
\end{aligned}
$$

which is an antipodal set of $F^{\mathbb{C}}_{2n_1,\dots,2n_r}(\mathbb{C}^{2n})$.

Example

$$\begin{aligned}
&\dim HF(F^{\mathbb{R}}_{2n_1,\ldots,2n_r}(\mathbb{R}^{2n}), F^{\mathbb{H}}_{n_1,\ldots,n_r}(\mathbb{H}^n) : \mathbb{Z}_2) \\
&= \#\left(F^{\mathbb{R}}_{2n_1,\ldots,2n_r}(\mathbb{R}^{2n}) \cap aF^{\mathbb{H}}_{n_1,\ldots,n_r}(\mathbb{H}^n)\right) \\
&= \#_I(F^{\mathbb{H}}_{n_1,\ldots,n_r}(\mathbb{H}^n)) = \dim H_*(F^{\mathbb{H}}_{n_1,\ldots,n_r}(\mathbb{H}^n) : \mathbb{Z}_2) \\
&= \frac{n!}{n_1!n_2!\cdots n_{r+1}!} \\
&< \#_I(F^{\mathbb{R}}_{2n_1,\ldots,2n_r}(\mathbb{R}^{2n})) = \dim H_*(F^{\mathbb{R}}_{2n_1,\ldots,2n_r}(\mathbb{R}^{2n}) : \mathbb{Z}_2) \\
&= \#_k(F^{\mathbb{C}}_{2n_1,\ldots,2n_r}(\mathbb{C}^{2n})) = \dim H_*(F^{\mathbb{C}}_{2n_1,\ldots,2n_r}(\mathbb{C}^{2n}) : \mathbb{Z}_2) \\
&= \frac{(2n)!}{(2n_1)!(2n_2)!\cdots(2n_{r+1})!}
\end{aligned}$$

for $a = \exp H \ \ (H \in \mathfrak{a}_{\mathrm{reg}})$

Further problems

1. Study the intersection of two real flag manifolds in the case where $\theta_0\theta_1 \neq \theta_1\theta_0$.
2. Determine Hamiltonian volume minimizing properties of all real forms in irreducible Hermitian symmetric spaces, more generally, in complex flag manifolds.

Thank you very much for your attention

Further problems

1. Study the intersection of two real flag manifolds in the case where $\theta_0\theta_1 \neq \theta_1\theta_0$.
2. Determine Hamiltonian volume minimizing properties of all real forms in irreducible Hermitian symmetric spaces, more generally, in complex flag manifolds.

Thank you very much for your attention

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