Lagrangian intersections and antipodal sets

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Lagrangian intersection

(M, J, ω) : Kähler manifold

L : real form of M

i.e. $\exists\sigma$: anti-holomorphic involutive isometry of M

s.t. $L = \mathrm{Fix}(\sigma, M)_0$

totally geodesic Lagrangian submanifold

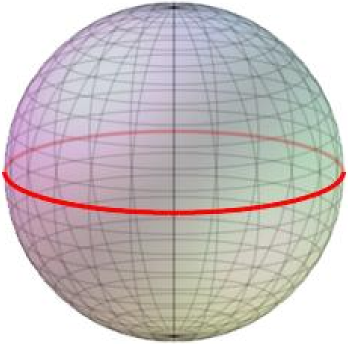
Arnold-Givental Conjecture

Assume $L = \mathrm{Fix}(\sigma, M)_0$ is compact. Then

$$\#(L \cap \phi L) \geq \dim H_*(L, \mathbb{Z}_2)$$

for any $\phi \in \mathrm{Ham}(M, \omega)$ and L intersects ϕL transversely.

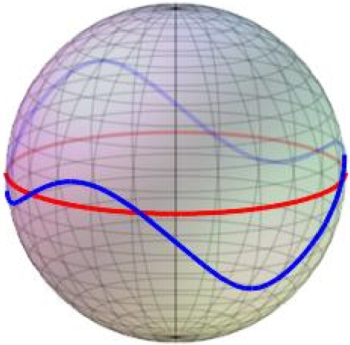
Example



$M = \mathbb{C}P^1$

$L = \mathbb{R}P^1$

Example



$M = \mathbb{C}P^1$

$L = \mathbb{R}P^1$

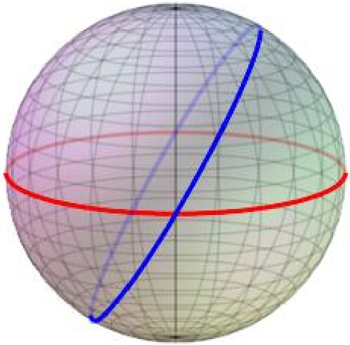
$\forall \phi \in \mathrm{Ham}(M, \omega)$ and $L \pitchfork \phi L$

$\Longrightarrow$

ϕL : area bisecting curve

$\#(L \cap \phi L) \geq 2 = \dim H_*(L, \mathbb{Z}_2)$

Example



$M = \mathbb{C}P^1$

$L = \mathbb{R}P^1$

$\forall g \in \mathrm{Isom}(M)$ and $L \pitchfork gL$

$\Longrightarrow$

$L \cap gL$: antipodal points

$\#(L \cap gL) = 2 = \dim H_*(L, \mathbb{Z}_2)$

Tight Lagrangian submanifolds

G/K : compact Hermitian symmetric space

$L \subset G/K$: closed embedded Lagrangian submanifold

Definition

L : **globally tight** (resp. **locally tight**)

$\overset{\text{def}}{\Longleftrightarrow}$ $$\#(L \cap gL) = \dim H_*(L, \mathbb{Z}_2)$$

for $\forall g \in G$ (resp. $g \in G$ close to the identity)

s.t. L transversely intersects with gL.

Problem 1

Classify all possible locally tight (or globally tight) Lagrangian submanifolds in compact Hermitian symmetric spaces.

Tight Lagrangian submanifolds

Theorem (Y.-G. Oh, 1991)

$L \subset \mathbb{C}P^n$: *locally tight Lagrangian submanifold*

$\Longrightarrow L$ *must be congruent to*

- *totally geodesic* $\mathbb{R}P^n \subset \mathbb{C}P^n$, *when* $n \geq 2$
- *a latitude circle in* $S^2 \cong \mathbb{C}P^1$, *when* $n = 1$

Theorem 1 (Iriyeh-S, 2010)

$L \subset S^2(1) \times S^2(1) \cong Q_2(\mathbb{C})$: locally tight Lagrangian submanifold

$\Longrightarrow L$ must be congruent to one of

1. $\mathbf{M}_0 := \{(x, -x) \mid x \in S^2(1)\} \subset S^2(1) \times S^2(1)$
2. $T_{a,b} := S^1(a) \times S^1(b) \subset S^2(1) \times S^2(1) \qquad (0 < a, b \leq 1)$

International Workshop on

“Fusion of Integrable Systems and Geometry”

April 17–19, 2009, at Tohoku University

A talk on “Tight Lagrangian surfaces in $S^2 \times S^2$”

Lagrangian intersection of two real forms

(M, J, ω) : homogeneous Kähler manifold

L_1, L_2 : real forms of M, **not necessarily congruent** with each other

Problems

1. Is the intersection $L_1 \cap L_2$ discrete?
2. If so, count the intersection number $\#(L_1 \cap L_2)$, and describe the geometric meaning of $\#(L_1 \cap L_2)$.
   Moreover, study the structure of the intersection $L_1 \cap L_2$.

The intersection of two real forms in complex hyperqudrics

$$Q_n(\mathbb{C}) = \{[z] \in \mathbb{C}P^{n+1} \mid z_1^2 + \cdots + z_{n+2}^2 = 0\}$$

$$\begin{array}{ccc} Q_n(\mathbb{C}) & \cong & \widetilde{G_2}(\mathbb{R}^{n+2}) \subset \wedge^2\mathbb{R}^{n+2} \\ [x+\sqrt{-1}y] & \longleftrightarrow & x \wedge y \end{array}$$

$u_1, u_2, e_1, \ldots, e_n$: o.n.b. of $\mathbb{R}^{n+2}$. For $0 \leq k \leq [n/2]$

$$\begin{aligned} S^{k,n-k} &= S^k(\langle u_1, e_1, \ldots, e_k\rangle_{\mathbb{R}}) \wedge S^{n-k}(\langle u_2, e_{k+1}, \ldots, e_n\rangle_{\mathbb{R}}) \\ &= S^k \times S^{n-k}/\mathbb{Z}_2 \subset Q_n(\mathbb{C}) \end{aligned}$$

Theorem (Tasaki 2010)

Let $L_1 \cong S^{k,n-k}$ and $L_2 \cong S^{l,n-l}$ are real forms in $Q^n(\mathbb{C})$ where $0 \leq k \leq l \leq [n/2]$. If L_1 and L_2 intersect transversely, then

$$L_1 \cap L_2 \cong \{\pm u_1 \wedge u_2, \pm e_1 \wedge e_2, \ldots, \pm e_{2k-1} \wedge e_{2k}\},$$

which is an antipodal set of L_1 and L_2. In particular, $L_1 \cap L_2$ is an great antipodal set of L_1.

Antipodal sets of a compact symmetric space

M : compact Riemannian symmetric space

s_x : geodesic symmetry at $x \in M$

Definition (Chen-Nagano 1988)

1. $\mathcal{A} \subset M$: **antipodal set** $\overset{\text{def}}{\iff}$ $s_x(y) = y$ for all $x, y \in \mathcal{A}$
2. $\#_2 M := \max\{\#\mathcal{A} \mid \mathcal{A} \subset M$: antipodal set$\}$ **2-number**
3. $\mathcal{A} \subset M$: **great** antipodal set $\overset{\text{def}}{\iff}$ $\#\mathcal{A} = \#_2 M$

Theorem (Takeuchi 1989)

M : *symmetric R-space* $\implies$ $\#_2 M = \dim H_*(M, \mathbb{Z}_2)$

Example

$\mathbb{R}P^n \subset \mathbb{C}P^n$

$$\mathcal{A} := \{\mathbb{R}e_1, \dots, \mathbb{R}e_{n+1}\} \subset \mathbb{R}P^n \quad \text{great antipodal set}$$

For $u \in U(n+1)$, $\mathbb{R}P^n \pitchfork u\mathbb{R}P^n$ in $\mathbb{C}P^n$

$$\mathbb{R}P^n \cap u\mathbb{R}P^n \cong \{\mathbb{C}e_1, \dots, \mathbb{C}e_{n+1}\} \subset \mathbb{C}P^n$$

$$\#(\mathbb{R}P^n \cap u\mathbb{R}P^n) = n + 1 = \#_2\mathbb{R}P^n = \dim H_*(\mathbb{R}P^n, \mathbb{Z}_2)$$

The intersection of two real forms in Herm. symm. sp.

Theorem (Tanaka-Tasaki 2012)

M : *Hermitian symmetric space of compact type*

$L_1, L_2 \subset M$: *real forms,* $L_1 \pitchfork L_2$

$\Longrightarrow$ $L_1 \cap L_2$ *is an antipodal set of* L_1 *and* L_2.

In addition, if L_1 *and* L_2 *are congruent to each other,*

$\Longrightarrow$ $L_1 \cap L_2$ *is a great antipodal set of* L_1 *and* L_2.

Theorem (Ikawa-Tanaka-Tasaki 2015)

A necessary and sufficient condition for two real forms in a compact Hermitian symmetric space to intersect transversally is given in terms of the symmetric triad $(\tilde{\Sigma}, \Sigma, W)$.

Lagrangian Floer homology of two real forms in HSS

Theorem (Iriyeh-S.-Tasaki 2013)

Let M be a Hermitian symmetric space of compact type which is Kähler-Einstein. Let L_1 and L_2 are real forms of M s.t. $L_1 \pitchfork L_2$ with minimal Maslov numbers $\Sigma_{L_1}, \Sigma_{L_2} \geq 3$. Then

$$HF(L_1, L_2; \mathbb{Z}_2) \cong \bigoplus_{p \in L_1 \cap L_2} \mathbb{Z}_2[p].$$

Generalized Arnold-Givental inequality

$$\#(L_1 \cap \phi L_2) \geq \min\{\dim H_*(L_1, \mathbb{Z}_2), \dim H_*(L_2, \mathbb{Z}_2)\}$$

for any $\phi \in \mathrm{Ham}(M, \omega)$ and $L_1 \pitchfork \phi L_2$ except for the case where $L_1 \cong G_m^{\mathbb{H}}(\mathbb{H}^{2m})$ and $L_2 \cong U(2m)$ in $M = G_{2m}^{\mathbb{C}}(\mathbb{C}^{4m})$.

Volume estimate of real forms under Hamiltonian deformations

Complex flag manifolds

G : compact, connected semisimple Lie group

$x_0 (\neq 0) \in \mathfrak{g}$

$$\begin{aligned} M &:= \mathrm{Ad}(G)x_0 \subset \mathfrak{g} \quad : \textbf{complex flag manifold} \\ &\cong G/G_{x_0} \cong G^{\mathbb{C}}/P^{\mathbb{C}} \end{aligned}$$

$$G_{x_0} := \{g \in G \mid \mathrm{Ad}(g)x_0 = x_0\}$$

$$\mathfrak{g}_{x_0} = \{X \in \mathfrak{g} \mid [x_0, X] = 0\}$$

ω : Kirillov-Kostant-Souriau symplectic form on M defined by

$$\omega(X^*_x, Y^*_x) := \langle [X, Y], x \rangle \qquad (x \in M,\ X, Y \in \mathfrak{g})$$

J : G-invariant complex structure on M compatible with ω

$(\cdot, \cdot) := \omega(\cdot, J\cdot)$: G-invariant Kähler metric

Antipodal set of a complex flag manifold

For $x \in M$ and $g \in Z(G_{x_0})$, define $s_{x,g} : M \to M$ by

$$s_{x,g}(y) := \mathrm{Ad}(g_x g g_x^{-1})y \qquad (y \in M),$$

where $g_x \in G$ satisfying $\mathrm{Ad}(g_x)x_0 = x$.

$$\mathrm{Fix}(s_x, M) := \{y \in M \mid s_{x,g}(y) = y \ (\forall g \in Z(G_{x_0}))\}$$

Definition

$\mathcal{A} \subset M$: **antipodal set** $\overset{\text{def}}{\Longleftrightarrow}$ $y \in \mathrm{Fix}(s_x, M)$ for all $x, y \in \mathcal{A}$

Theorem (Iriyeh-S-Tasaki)

$\mathcal{A} \subset M$: *maximal antipodal set*

$\Longrightarrow$ $\exists \mathfrak{t} \subset \mathfrak{g}$: *maximal abelian subalgebra* *s.t.* $\mathcal{A} = M \cap \mathfrak{t}$.

Hence $\mathcal{A}$ *is an orbit of the Weyl group of* $\mathfrak{g}$ *with respect to* $\mathfrak{t}$.

Maximal antipodal sets of M *are congruent to each other by* G.

Real flag manifolds in a complex flag manifold

(G,K) : symmetric pair of compact type

θ : involution of G s.t. $\mathrm{Fix}(\theta,G)_0 \subset K \subset \mathrm{Fix}(\theta,G)$

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

$x_0 (\neq 0) \in \mathfrak{p}$

$$\begin{array}{ccccccl} L & := & \mathrm{Ad}(K)x_0 & \subset & \mathfrak{p} & & : \textbf{real flag manifold, } R\textbf{-space} \\ \cap & & \cap & & \cap & & \\ M & := & \mathrm{Ad}(G)x_0 & \subset & \mathfrak{g} & & : \textbf{complex flag manifold, } C\textbf{-space} \\ & \cong & G/G_{x_0} & \cong & G^{\mathbb{C}}/P^{\mathbb{C}} & & \end{array}$$

$\mathfrak{g}' := \mathfrak{k} + \sqrt{-1}\mathfrak{p}$ non-compact real form of $\mathfrak{g}^{\mathbb{C}}$

σ : complex conjugation of $\mathfrak{g}^{\mathbb{C}}$ w.r.t. $\mathfrak{g}'$

$\tilde{\sigma}$: anti-holomorphic involution on M.

$$L = M \cap \mathfrak{p} \cong K/K_{x_0} \cong G'/(G' \cap P^{\mathbb{C}})$$

The intersection of real flag manifolds

$(G, K_1), (G, K_2)$: symmetric pairs of compact type

θ_1, θ_2 : involutions of G

$$\mathfrak{g} = \mathfrak{k}_1 + \mathfrak{p}_1 = \mathfrak{k}_2 + \mathfrak{p}_2,$$

$x_0 (\neq 0) \in \mathfrak{p}_1 \cap \mathfrak{p}_2$

$$L_1 := \mathrm{Ad}(K_1)x_0, \quad L_2 := \mathrm{Ad}(K_2)x_0 \ \subset \ M := \mathrm{Ad}(G)x_0$$

For $g \in G$, we consider the intersection of $L_1 \cap \mathrm{Ad}(g)L_2$ in M.

$\mathfrak{a}$: maximal abelian subspace of $\mathfrak{p}_1 \cap \mathfrak{p}_2$ containing x_0

$A := \exp \mathfrak{a} \subset G$: toral subgroup

Then $G = K_1 A K_2$, i.e. $g = g_1 a g_2 \ (g_1 \in K_1, g_2 \in K_2, a \in A)$

$$L_1 \cap \mathrm{Ad}(g)L_2 = L_1 \cap \mathrm{Ad}(g_1 a g_2)L_2 = \mathrm{Ad}(g_1)\Big(L_1 \cap \mathrm{Ad}(a)L_2\Big)$$

Symmetric triads

Hereafter we assume $\theta_1\theta_2 = \theta_2\theta_1$.

$$\mathfrak{g} = (\mathfrak{k}_1 \cap \mathfrak{k}_2) + (\mathfrak{p}_1 \cap \mathfrak{p}_2) + (\mathfrak{k}_1 \cap \mathfrak{p}_2) + (\mathfrak{p}_1 \cap \mathfrak{k}_2)$$

Then $\big((\mathfrak{k}_1 \cap \mathfrak{k}_2) + (\mathfrak{p}_1 \cap \mathfrak{p}_2),\ (\mathfrak{k}_1 \cap \mathfrak{k}_2),\ d\theta_1 = d\theta_2\big)$
is an orthogonal symmetric Lie algebra.

For $\lambda \in \mathfrak{a} \subset \mathfrak{p}_1 \cap \mathfrak{p}_2$

$$\mathfrak{p}_\lambda := \{X \in \mathfrak{p}_1 \cap \mathfrak{p}_2 \mid [H,[H,X]] = -\langle \lambda, H\rangle^2 X\ (H \in \mathfrak{a})\}$$
$$V_\lambda := \{X \in \mathfrak{p}_1 \cap \mathfrak{k}_2 \mid [H,[H,X]] = -\langle \lambda, H\rangle^2 X\ (H \in \mathfrak{a})\}$$

$$\Sigma := \{\lambda \in \mathfrak{a} \setminus \{0\} \mid \mathfrak{p}_\lambda \neq \{0\}\}$$
$$W := \{\lambda \in \mathfrak{a} \setminus \{0\} \mid V_\lambda \neq \{0\}\}$$
$$\widetilde{\Sigma} := \Sigma \cup W$$

$(\widetilde{\Sigma}, \Sigma, W)$: **symmetric triad** with multiplicities (Ikawa)

The structure of the intersection

$$\mathfrak{a}_{\mathrm{reg}} := \bigcap_{\substack{\lambda\in\Sigma \\ \alpha\in W}} \left\{ H \in \mathfrak{a} \;\middle|\; \langle \lambda, H\rangle \notin \pi\mathbb{Z}, \langle \alpha, H\rangle \notin \frac{\pi}{2} + \pi\mathbb{Z} \right\}$$

$W(\tilde{\Sigma})$: Weyl group of the root system $\tilde{\Sigma}$ of $\mathfrak{a}$

$\mathfrak{a}_i$: maximal abelian subspace of $\mathfrak{p}_i$ containing $\mathfrak{a}$ $(i = 1, 2)$

$W(R_i)$: Weyl group of the restricted root system R_i of $(\mathfrak{g}, \mathfrak{k}_i)$ w.r.t. $\mathfrak{a}_i$

Theorem (Ikawa-Iriyeh-Okuda-S.-Tasaki)

For $a = \exp H$ $(H \in \mathfrak{a})$, the intersection $L_1 \cap \mathrm{Ad}(a)L_2$ is discrete if and only if $H \in \mathfrak{a}_{\mathrm{reg}}$. Moreover, if $L_1 \cap \mathrm{Ad}(a)L_2$ is discrete, then

$$L_1 \cap \mathrm{Ad}(a)L_2 = W(\tilde{\Sigma})x_0 = W(R_1)x_0 \cap \mathfrak{a} = W(R_2)x_0 \cap \mathfrak{a},$$

in particular, $L_1 \cap \mathrm{Ad}(a)L_2$ is an antipodal set of M.

Outline of the proof

$$\mathfrak{p}_1 \cap \mathrm{Ad}(a)\mathfrak{p}_2 = \mathfrak{a} + \sum_{\substack{\lambda \in \Sigma^+ \\ \langle \lambda, H \rangle \in \pi\mathbb{Z}}} \mathfrak{p}_\lambda + \sum_{\substack{\alpha \in W^+ \\ \langle \alpha, H \rangle \in \frac{\pi}{2} + \pi\mathbb{Z}}} \mathfrak{p}_\alpha$$

If $x_0 \in \mathfrak{a}_{\mathrm{reg}}$, then $\mathfrak{p}_1 \cap \mathrm{Ad}(a)\mathfrak{p}_2 = \mathfrak{a}$. Therefore

$$\begin{aligned} L_1 \cap \mathrm{Ad}(a)L_2 &= (M \cap \mathfrak{p}_1) \cap (M \cap \mathrm{Ad}(a)\mathfrak{p}_2) \\ &= M \cap (\mathfrak{p}_1 \cap \mathrm{Ad}(a)\mathfrak{p}_2) \\ &= M \cap \mathfrak{a} \\ &= W(\tilde{\Sigma})x_0 = W(R_1)x_0 \cap \mathfrak{a} = W(R_2)x_0 \cap \mathfrak{a} \end{aligned}$$

Hermann actions

$$\mathfrak{a}_{\mathrm{reg}} := \bigcap_{\substack{\lambda\in\Sigma \\ \alpha\in W}} \left\{ H \in \mathfrak{a} \;\middle|\; \langle \lambda, H\rangle \notin \pi\mathbb{Z}, \langle \alpha, H\rangle \notin \frac{\pi}{2} + \pi\mathbb{Z} \right\}$$

P : cell, a connected component of $\mathfrak{a}_{\mathrm{reg}}$

$$\begin{array}{ccccc} & & K_2 \times K_1 \curvearrowright G & & \\ & \swarrow^{\pi_1} & & \searrow^{\pi_2} & \\ K_2 \curvearrowright G/K_1 & & & & K_2\backslash G \curvearrowleft K_1 \\ & \searrow & & \swarrow & \\ & & K_2\backslash G/K_1 \cong \overline{P} & & \end{array}$$

Hermann actions

Proposition (Ikawa)

For $a = \exp H$ $(H \in \mathfrak{a})$, orbits $K_2 a K_1 \subset G$, $K_2 \pi_1(a) \subset G/K_1$, $\pi_2(a) K_1 \subset K_2\backslash G$ are regular if and only if $H \in \mathfrak{a}_{\mathrm{reg}}$.

Further problems

1. Study the intersection of two real flag manifolds in the case where $\theta_1\theta_2 \neq \theta_2\theta_1$.
2. Calculate Lagrangian Floer homologies of pairs of real flag manifolds in complex flag manifolds.
3. Determine Hamiltonian volume minimizing properties of all real forms in irreducible Hermitian symmetric spaces, more generally, in complex flag manifolds.
4. Understand the relationship between antipodal sets and the topology of a compact symmetric space.

Conjecture (Iriyeh-Ma-Miyaoka-Ohnita, H. Ono)

Any compact connected minimal Lagrangian submanifold in an irreducible Hermitian symmetric space of compact type is Hamiltonian non-displaceable.